MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Otto Forster

WS 2004/2005
Dec. 14, 2004

# Algebraic Number Theory

## Problem Sheet #8

### Problem 29

a) Prove that $K = \mathbb{Q}(\sqrt{-43})$ has class number 1.

b) Determine a generator of the ideal

$$\mathfrak{a} = \Big(17, \frac{29 + \sqrt{-43}}{2}\Big)_{\mathbb{Z}} \subset \mathfrak{o}_K$$

### Problem 30

Prove that $K := \mathbb{Q}(\sqrt{-23})$ has class number 3 and determine three ideals $\mathfrak{a}_i \subset \mathfrak{o}_K$, $i = 1, 2, 3$, which represent all ideal classes.

### Problem 31

Let $K$ be a quadratic number field with discriminant $D$. Consider two ideals

$$\mathfrak{a}_i = \Big(a_i, \frac{b_i + \sqrt{D}}{2}\Big)_{\mathbb{Z}} \subset \mathfrak{o}_K, \qquad i = 1, 2,$$

where $a_i, b_i \in \mathbb{Z}$, $a_i > 0$ and $4a_i \mid b_i^2 - D$.

Suppose $\gcd(a_1, a_2) = 1$ and $ua_1 + va_2 = 1$, $u, v \in \mathbb{Z}$. Show that

$$\mathfrak{a}_1\mathfrak{a}_2 = \Big(a_1a_2, \frac{(ua_1b_2 + va_2b_1) + \sqrt{D}}{2}\Big)_{\mathbb{Z}}$$

### Problem 32

Let $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{o}_K$ be two ideals as in the previous problem.
In the general case, let $\alpha := \gcd(a_1, a_2, (b_1 + b_2)/2)$. Why is $b_1 + b_2$ even?

Prove that

$$\mathfrak{a}_1\mathfrak{a}_2 = \alpha\Big(a_3, \frac{b_3 + \sqrt{D}}{2}\Big)_{\mathbb{Z}}$$

for some integers $a_3, b_3$ with $4a_3 \mid b_3^2 - D$. Determine these integers.

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**Due:** Tuesday, December 21, 2004, 14:10 h