MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Otto Forster

WS 2004/2005
Dec. 6, 2004

Algebraic Number Theory

Problem Sheet #7

Problem 25

Let d be a non-zero integer $\not\equiv 1 \bmod 4$, p a prime not dividing d, and x an integer with $x^2 \equiv d \bmod p$. Show that the ideal generated by the elements p and $x + \sqrt{d}$ in the ring $\mathbb{Z}[\sqrt{d}]$ is equal to the lattice generated by p and $x + \sqrt{d}$, i.e.

$$\mathbb{Z}[\sqrt{d}\,] \cdot p \;+\; \mathbb{Z}[\sqrt{d}\,] \cdot (x + \sqrt{d}) \;=\; \mathbb{Z} \cdot p \;+\; \mathbb{Z} \cdot (x + \sqrt{d}).$$

Problem 26

In the ring $\mathbb{Z}[\sqrt{10}\,]$ consider the ideals $\mathfrak{a} := (2, 4 + \sqrt{10})$ and $\mathfrak{b} := (3, 4 + \sqrt{10})$.

a) Prove that $\mathfrak{a}$ and $\mathfrak{b}$ are not principal ideals.

b) Calculate $\mathfrak{a}^2$ and $\mathfrak{ab}$ and show that they are principal ideals.

c) Prove that $\mathfrak{a}$ and $\mathfrak{b}$ belong to the same ideal class and determine a $\lambda \in \mathbb{Q}(\sqrt{10})$ such that $\mathfrak{b} = \lambda\mathfrak{a}$.

Problem 27

Let Λ be a lattice in the quadratic number field $K = \mathbb{Q}(\sqrt{d}\,)$, i.e. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1, \omega_2 \in K$ are linearly independent over $\mathbb{Q}$. Define

$$R := \{z \in K : z\Lambda \subset \Lambda\}.$$

a) Show that R is a subring of the ring $\mathfrak{o}_K$ and that Λ is a fractional ideal of R.

b) Give an example for each of the cases $R = \mathfrak{o}_K$ and $R \neq \mathfrak{o}_K$.

Problem 28

A lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, ($\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$), is said to have *complex multiplication* if there exists a non-real complex number $z \in \mathbb{C} \setminus \mathbb{R}$ with $z\Lambda \subset \Lambda$.

Show that Λ has complex multiplication if and only if $\tau := \omega_1/\omega_2$ belongs to an imaginary quadratic number field.

Due: Tuesday, December 14, 2004, 14:10 h