MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Otto Forster

WS 2004/2005
Nov. 18, 2004

Algebraic Number Theory

Problem Sheet #5

Problem 17

a) Let $A := \mathbb{R}[T]$. Show that the maximal spectrum $X := \mathrm{Specm}(A)$ is a disjoint union $X = X_1 \,\dot\cup\, X_2$, where $X_1 = \{\mathfrak{m} \in X : A/\mathfrak{m} \cong \mathbb{R}\}$ and $X_2 = \{\mathfrak{m} \in X : A/\mathfrak{m} \cong \mathbb{C}\}$.

b) Let $B := \mathbb{R}[T]\left[\sqrt{T^2+1}\right]$ and $Y := \mathrm{Specm}(B)$. Show that Y can be decomposed as $Y = Y_1 \,\dot\cup\, Y_2$, where Y_k is defined analogously to X_k of part a).

c) Consider the map

$$\pi : Y \longrightarrow X$$

induced by the inclusion $A \subset B$. For every $\mathfrak{m} \in X$ determine the fiber $\pi^{-1}(\mathfrak{m})$, i.e. the maximal ideals of B lying over $\mathfrak{m}$.

Problem 18

Let A be the ring of integers in an algebraic number field. Show that the natural map

$$A \to \prod_{\mathfrak{m} \in \mathrm{Specm}(A)} A/\mathfrak{m}$$

is an injective ring homomorphism.

Problem 19

a) Let $A \subset B$ be two integral domains and $s \in A \smallsetminus \{0\}$. Prove

$$B \text{ integral over } A \quad \Longrightarrow \quad B[1/s] \text{ integral over } A[1/s]$$

($A[1/s] \subset B[1/s]$ can be regarded as subrings of the quotient field of B.)

b) Show that the converse implication "$\Longleftarrow$" is not true in general.

c) Let $s_1, s_2 \in A \smallsetminus \{0\}$ with $As_1 + As_2 = A$. Prove

$$B[1/s_k] \text{ integral over } A[1/s_k] \text{ for } k = 1, 2 \quad \Longrightarrow \quad B \text{ integral over } A$$

Problem 20

Calculate the continued fraction expansions of $\sqrt{17}$ and $\sqrt{19}$.

Due: Tuesday, November 30, 2004, 14:10 h