MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2004/2005
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Algebraic Number Theory

Problem Sheet #4

Problem 13

Prove that every element of the ring $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is associated to an element of the subring $\mathbb{Z}[\sqrt{-3}]$.

Problem 14

Prove: A natural number n can be represented as

$$n = x^2 + 2y^2, \qquad x, y \in \mathbb{Z},$$

if and only if

$$n = 2^k a^2 p_1 p_2 \cdot \ldots \cdot p_r$$

with $a \in \mathbb{Z}$ and (not necessarily distinct) prime numbers $p_j \equiv 1, 3 \bmod 8$, $k \geq 0$, $r \geq 0$.

Problem 15

a) Prove that the ring $\mathbb{Z}[\frac{1}{3}\sqrt{2}]$ is a principal ideal ring.

b) Show that $\mathbb{Z}[\frac{1}{3}\sqrt{2}]$ is not an integral algebraic extension of $\mathbb{Z}$ and the induced map

$$\mathrm{Spec}(\mathbb{Z}[\tfrac{1}{3}\sqrt{2}]) \to \mathrm{Spec}(\mathbb{Z})$$

is not surjective. Which prime ideals lie above $(2), (3), (5), (7), (11) \in \mathrm{Spec}(\mathbb{Z})$?

Problem 16

Let X be a compact topological space (e.g. a closed bounded set in $\mathbb{R}^n$) and $A := \mathcal{C}(X)$ the ring of all continuous functions $f : X \to \mathbb{R}$. For every $x \in X$, define an ideal $\mathfrak{m}_x \subset A$ by

$$\mathfrak{m}_x := \{f \in A : f(x) = 0\},$$

Show that $\mathfrak{m}_x$ is a maximal ideal of A, and the map

$$X \longrightarrow \mathrm{Specm}(A), \qquad x \mapsto \mathfrak{m}_x,$$

is bijective.

Hint: Prove that if functions $f_1, \ldots, f_m \in A$ have no common zero in X, they generate the unit ideal in A.

Due: Tuesday, November 23, 2004, 14:10 h