MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2004/2005
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# Algebraic Number Theory

## Problem Sheet #2

### Problem 5

Show that the ring $\mathbb{Z}[\sqrt{-5}]$ is not a UFD (unique factorization domain) by decomposing the element 21 in two essentially different ways as a product of irreducible factors.

### Problem 6

The ring $\mathbb{Z}[\sqrt{6}]$ is euclidean, hence a UFD. The decompositions

$$6 = 2 \cdot 3 = \sqrt{6} \cdot \sqrt{6}$$

seem to contradict unique factorization.

Explain why this is not a counter example to unique factorization.

What is a correct factorization of 6 as a product of prime elements?

### Problem 7

Show that every unit $u$ in the ring $\mathbb{Z}[\sqrt{2}]$ can be written as

$$u = \pm(1 + \sqrt{2})^n$$

for some $n \in \mathbb{Z}$.

### Problem 8

Let $A = \mathfrak{o}_K$ be the ring of integers in a quadratic number field $K = \mathbb{Q}(\sqrt{d})$.

Show that for every pair of associated elements $x, y \in A$ one has $|N(x)| = |N(y)|$.

Is the converse true, i.e. does $|N(x)| = |N(y)|$ imply that $x$ and $y$ are associated? (Proof or counter example).

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**Due:** Tuesday, November 9, 2004, 14:10 h