A Theorem on Zero Schemes of Sections in Two-Bundles over Affine Schemes with Applications to Set Theoretic Intersections[1]

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We consider the following problem. Let E be a rank 2 vector bundle over an affine scheme X and f a section of E with zero scheme $Z \subset X$. If $\operatorname{codim} Z = 2$ and there exists a reasonable theory of Chern classes on X, then Z represents the second Chern class $c_2(E)$. Since the second Chern class of a vector bundle and of its dual coincide, one may ask whether E^* admits a section φ with the same zero scheme Z.

We prove that this is true if X is an affine algebraic surface over an algebraically closed field (Proposition 1.3). The proof uses Serre's extension theory for codimension 2 ideals and the cancellation theorem of Murthy-Swan. In an elementary way we then prove the existence of φ in a more general situation: X is an arbitrary affine scheme and the only condition is that $\det(E) \mid Z$ be trivial (Proposition 1.5).

We apply these results to prove generalizations of the theorem of Storch [St] and Eisenbud-Evans [EE] on the minimal number of equations for the set theoretical description of closed subschemes of an affine scheme. By other methods, similar results have been obtained by Boratyński [B], Lyubeznik [L], and Mandal [M]. In Theorem 2.6 we prove: Let $Y \subset X = \operatorname{Spec} R$ be a subscheme. If Y is defined by a locally principal ideal $I \subset R$ such that the conormal module I/I^2 is generated by m elements ($m \geqslant 2$), then Y can be set theoretically defined by m functions. For arbitrary codimension we derive the following result: Y can be set theoretically defined by $n := \dim X$ functions if Y is a locally complete intersection without zero-dimensional components. In fact n functions suffice in a more general case. The conditions on the ideal I are as follows. For $k \geqslant 1$ let Y_k the set of points $y \in Y$ such that I_y requires at least k generators. We suppose $\dim Y_k \leqslant n-k$ for $1 \leqslant k \leqslant n-1$ and $Y_n = \emptyset$. Then Y can be set theoretically defined by n functions (cf. Theorem 3.6).

1. Zero schemes of sections in 2-bundles

1.1. Let E be a *vector bundle* over a locally ringed space $(X, \mathcal{O}_X)$. By this we mean a locally free $\mathcal{O}_X$-module of finite type. We denote its dual bundle by E^*. A section $f \in \Gamma(X, E)$ defines a morphism of $\mathcal{O}_X$-modules

$$E^* \longrightarrow \mathcal{O}_X, \quad \varphi \mapsto \langle \varphi, f \rangle,$$

which we identify with f. The ringed subspace Z with structure sheaf

$$\mathcal{O}_Z := \operatorname{Coker}(E^* \xrightarrow{f} \mathcal{O}_X)$$

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is called the *zero scheme* of f and denoted by $\mathrm{Sch}_E(f)$ or briefly by $\mathrm{Sch}(f)$. Its underlying topological space is

$$V(f) = V_E(f) := \{x \in X : f(x) = 0\}.$$

Here $f(x)$ denotes the element induced by f in the vector space $E(x) := E_x/\mathfrak{m}_x E_x$.

1.2. Suppose now that the vector bundle E on X has constant rank 2 and that the zero scheme $Z = \mathrm{Sch}(f)$ of a section f in E has codimension 2. If for example X is a non-singular variety over an algebraically closed field, Z represents the Chern class $c_2(E)$, which is equal to $c_2(E^*)$. So the question arises if the dual bundle E^* admits a section with the same zero scheme Z.

Of course, this is not always true. Assume for instance that X is Cohen-Macaulay in every point of Z. Then a simple necessary condition can be formulated as follows: If both E and E^* admit sections with zero scheme Z, then $\det(E)^2 \mid Z$ is trivial. To see this, we consider the conormal bundle $\nu_Z := \mathcal{I}_Z/\mathcal{I}_Z^2$ of Z, where $\mathcal{I}_Z$ is the ideal sheaf defining Z. The epimorphism

$$E^* \xrightarrow{f} \mathcal{I}_Z \to 0$$

induces an isomorphism $(E^* \mid Z) \xrightarrow{\sim} \nu_Z$. Analogously, we have an isomorphism $(E \mid Z) \xrightarrow{\sim} \nu_Z$. This implies $\det(E)^2 \mid Z \cong \mathcal{O}_Z$. This necessary condition is evidently fulfilled if Z consists of finitely many points. This assumption is sufficient, as the following proposition shows.

1.3. Proposition. *Let X be an affine algebraic surface over an algebraically closed field and E an algebraic vector bundle of rank 2 over X. Let $f \in \Gamma(X, E)$ be a section such that* $\mathrm{Sch}(f)$ *is zero-dimensional and consists of Cohen-Macaulay points of X. Then there exists a section $\varphi \in \Gamma(X, E^*)$ of the dual bundle with* $\mathrm{Sch}(\varphi) = \mathrm{Sch}(f)$.

Remark. Later we will prove a theorem which contains Proposition 1.3 as a special case. Nevertheless we will bring a separate proof of 1.3, because it is of independent interest.

Proof. Let $Z = \mathrm{Sch}(f)$ and $\mathcal{I}_Z := \mathrm{Im}(f : E^* \to \mathcal{O}_X)$ the ideal sheaf of Z. Since X is Cohen-Macaulay in every $x \in Z$, we have an exact sequence (Koszul complex)

$$0 \longrightarrow L^* \longrightarrow E^* \xrightarrow{f} \mathcal{I}_Z \longrightarrow 0,$$

where $L = \det(E)$. This exact sequence defines an element $\xi \in \mathrm{Ext}^1(\mathcal{I}_Z, L^*) = \Gamma(X, \mathcal{E}xt^1(\mathcal{I}_Z, L^*))$. Now

$$\mathcal{E}xt^1(\mathcal{I}_Z, L^*) \cong \mathcal{E}xt^2(\mathcal{O}_Z, L^*) \cong \det(\nu_Z) \otimes L^* \cong \det(E) \otimes L^* \otimes \mathcal{O}_Z \cong \mathcal{O}_Z.$$

Since E^* is locally free, we have by Serre theory: ξ_x is a generator of $\mathcal{E}xt^1(\mathcal{I}_Z, L^*)_x$ for all $x \in X$. On the other hand,

$$\mathcal{E}xt^1(\mathcal{I}_Z, L) \cong \det(E) \otimes L \otimes \mathcal{O}_Z.$$

Since Z is zero-dimensional, we have a (non-canonical) isomorphism $\mathcal{E}xt^1(\mathcal{I}_Z, L) \cong \mathcal{E}xt^1(\mathcal{I}_Z, L^*)$. Let $\tilde{\xi} \in \operatorname{Ext}^1(\mathcal{I}_Z, L)$ be the element which corresponds to ξ under this isomorphism and let

$$0 \longrightarrow L \longrightarrow V \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

be the extension correspondimg to $\tilde{\xi}$. Again by Serre, V is locally free of rank 2. We will prove $V \cong E$. First, by Schanuel's lemma,

$$V \oplus L^* \cong E^* \oplus L.$$

We have to use the following

1.4. Lemma. *Let W be a vector bundle over a two-dimensional affine scheme X with* $\det(W) \cong \mathcal{O}_X$. *Then $W \cong W^*$.*

Proof of the lemma. We may assume that W has constant rank m. The assertion is clear for $m = 1$ and also for $m = 2$, since for a vector bundle E of constant rank 2 one has

$$E^* \cong E \otimes \det E^*.$$

If $m > 2$, by a well known theorem of Serre, we can write $W \cong W' \oplus \mathcal{O}_X^{m-2}$, where W' is a vector bundle of rank 2, and the assertion follows.

We return to the proof of Proposition 1.3. Applying Lemma 1.4 we obtain

$$V \oplus L^* \cong E^* \oplus L \cong E \oplus L^*.$$

By the cancellation theorem of Murthy and Swan [MS] this implies $V \cong E$, and we have an exact sequence

$$0 \longrightarrow L \longrightarrow E \xrightarrow{\varphi} \mathcal{I}_Z \longrightarrow 0,$$

which proves Proposition 1.3.

Remark. For the application of Murthy-Swan's cancellation theorem we had to suppose that X is an affine algebraic surface over an algebraically closed field. Actually the assertion holds in a much more general situation.

1.5. Theorem. *Let X be an affine scheme, E a vector bundle of rank 2 over X and $f \in \Gamma(X, E)$ a section with zero scheme $Z := \operatorname{Sch}(f)$. Suppose that the restriction of the line bundle $L := \det(E)$ to Z is trivial. Then there exists a section $\varphi \in \Gamma(X, E^*)$ with zero scheme Z.*

Note that we do not require that X is Cohen-Macaulay in the points of Z nor that Z is of codimension 2. The condition that $\det(E) \mid Z$ is trivial is automatically fulfilled if Z consists of finitely many points.

Proof. Since $L \mid Z$ is trivial there exists a section $h \in \Gamma(X, L)$ such that $h \mid Z$ has no zeros. Therefore $(f, h) \in \Gamma(X, E \oplus L)$ is *unimodular* (i.e. a section without zeros). Hence there exists a section $(\psi, \lambda) \in \Gamma(X, E^* \oplus L^*)$ such that

$$(*) \qquad \langle \psi, f \rangle + \langle \lambda, h \rangle = 1.$$

Define

$$\Phi := \psi \otimes \psi + i(\lambda) : E \longrightarrow E^*,$$

where $i(\lambda) : E \to E^*$ is defined by

$$\langle i(\lambda)v, w \rangle := \langle \lambda, v \wedge w \rangle$$

for sections v, w of E. Let $\varphi := f \circ \Phi \in \Gamma(X, E^*)$ be the composition of the maps

$$E \xrightarrow{\Phi} E^* \xrightarrow{f} \mathcal{O}_X,$$

i.e.

$$\langle \varphi, v \rangle = \langle \Phi(v), f \rangle = \langle \psi, v \rangle \langle \psi, f \rangle + \langle \lambda, v \wedge f \rangle.$$

It remains to show that

$$\mathrm{Im}(E \xrightarrow{\varphi} \mathcal{O}_X) = \mathrm{Im}(E^* \xrightarrow{f} \mathcal{O}_X) =: \mathcal{I}_Z.$$

i) We prove the equality $\mathrm{Im}\,\varphi_x = \mathcal{I}_{Z,x}$ first for $x \in V(\lambda)$. By definition, $\mathrm{Im}\,\varphi \subset \mathcal{I}_Z$. From $(*)$ it follows that $\langle \varphi, f \rangle(x) = 1$. Now

$$\langle \varphi, f \rangle = \langle \psi, f \rangle^2,$$

hence $\varphi_x(f)$ is invertible, so $\mathrm{Im}\,\varphi_x = \mathcal{O}_{X,x} \supset \mathcal{I}_{Z,x}$.

ii) The equality $\mathrm{Im}\,\varphi_x = \mathcal{I}_{Z,x}$ for $x \notin V(\lambda)$ follows immediately from the fact that $\Phi \mid X \smallsetminus V(\lambda)$ is an isomorphism. This will be shown using the following funny formula.

1.6. Proposition. *Let E be a rank 2 vector bundle and let $S, A : E \to E^*$ be morphisms, S symmetric and A antisymmetric. Then*

$$\det(S + A) = \det(S) + \det(A).$$

Remark. These determinants are sections of the line bundle $\det(E^*)^2$.

Proof. Since the assertion is local, the formula can be verified by simple matrix calculus.

Now we can complete the proof of Theorem 1.5. We apply the proposition to Φ and get

$$\det \Phi = \det(\psi \otimes \psi) + \det(i(\lambda)) = 0 + \lambda^2,$$

hence $\det \Phi$ is invertible on $X \smallsetminus V(\lambda)$, q.e.d.

2. Set theoretic description of hypersurfaces

For the proof of our theorem on the set theoretic description of hypersurfaces in affine schemes we need some preparations

2.1. Let $X = \mathrm{Spec}(R)$ be the spectrum of a ring R and $\Omega = \mathrm{Specm}(R) \subset X$ its maximal spectrum. For subsets $Z \subset Y \subset X$, where Z is closed in Y, we have the notion of combinatorial *(Krull) dimension* $\dim Y$ and $\mathrm{codim}_Y Z$. We will also use the following notations:

$$\begin{aligned} \mathrm{dimm}\, Y &:= \dim (Y \cap \Omega), \\ \mathrm{Codim}_Y Z &:= \min \{\mathrm{codim}_Y Z,\ \mathrm{codim}_{Y\cap\Omega}(Z \cap \Omega)\}. \end{aligned}$$

While always $\dim(Y \cap \Omega) \leqslant \dim Y$, examples show that $\mathrm{codim}_{Y\cap\Omega}(Z\cap\Omega)$ may be less, equal or bigger than $\mathrm{codim}_Y Z$.

2.2. Lemma. *Let Y be an affine scheme whose underlying topological space is noetherian. Let $L_1, \ldots, L_r$ be line bundles on Y such that $L_1 \oplus \ldots \oplus L_r$ admits a unimodular section. Then there exists a unimodular section $(f_1, \ldots, f_r) \in \Gamma(Y, L_1 \oplus \ldots \oplus L_r)$ such that*

$$\mathrm{Codim}_Y V(f_1, \ldots, f_k) \geqslant k$$

for all $k = 1, \ldots, r$.

Proof. Let $(g_1, \ldots, g_r) \in \Gamma(Y, L_1 \oplus \ldots \oplus L_r)$ be unimodular. Then $f_1, \ldots, f_r$ are constructed by induction in such a way that $(f_1, \ldots, f_k, g_{k+1}, \ldots, g_r)$ is unimodular and the above inequalities hold.

2.3. Proposition. *Let L be a line bundle on an affine scheme X and $\varphi \in \Gamma(X, L^*)$. Set $Y := \mathrm{Sch}(\varphi)$. Suppose that $L \mid Y$ is generated by m global sections, $m \geqslant 2$. Then there exist $f_1, \ldots, f_m \in \Gamma(X, L)$ such that*

$$\mathrm{Sch}(f_1, \ldots, f_m) \subset Y.$$

If Y has noetherian topology, the sections $f_1, \ldots, f_m$ may be chosen in such a way that in addition

$$\mathrm{Codim}_Y \, \mathrm{Sch}(f_1, \ldots, f_m) \geqslant m - 1.$$

Proof. Choose $g_1, \ldots, g_m \in \Gamma(X, L)$ that generate $L \mid Y$. Then g_1 has no zeros on $V(g_2, \ldots, g_m) \cap Y$. Therefore there exists also a $\varphi_1 \in \Gamma(X, L^*)$ which has no zeros on $V(g_2, \ldots, g_m) \cap Y$. Then $(\varphi_1, g_2, \ldots, g_m) \mid Y$ is a unimodular section of $L^* \oplus L^{\oplus(m-1)} \mid Y$. If Y is a noetherian topological space, we may assume by Lemma 2.2 that

$$\mathrm{Codim}_Y V(\varphi_1, g_2, \ldots, g_{m-1}) \cap Y \geqslant m - 1.$$

Set

$$Z := \mathrm{Sch}(\varphi, \varphi_1, g_2, \ldots, g_{m-1}) \subset Y$$

and

$$X' := \mathrm{Sch}(g_2, \ldots, g_{m-1}).$$

Since $g_m \mid Z$ has no zeros, $L \mid Z$ is trivial. Application of Theorem 1.5 to the bundle $L^* \oplus L^* \mid X'$ and its section $(\varphi, \varphi_1) \mid X'$ yields $(f_1, f_2) \in \Gamma(X, L \oplus L)$ such that

$$Z = \mathrm{Sch}(f_1, f_2) \cap X' = \mathrm{Sch}(f_1, f_2, g_2, \ldots, g_{m-1}).$$

Now

$$(f_1, f_2, \ldots, f_m) := (f_1, f_2, g_2, \ldots, g_{m-1})$$

satisfies the assertion of the proposition.

2.4. In the sequel we will use the following notations. For a module M over a ring R we denote by $\mu(M)$ its minimal number of generators. We say that an ideal $I \subset R$ is *generated up to radical* by m elements, if there exists an ideal $J \subset I$ with $\sqrt{J} = \sqrt{I}$ and $\mu(J) \leqslant m$.

2.5. We will need the following fact: If $\mathcal{F}$ is a finitely generated $\mathcal{O}_Y$-module over a reduced scheme Y such that $\mu(\mathcal{F}_y)$ is constant, then $\mathcal{F}$ is locally free.

The following theorem gives a bound on the number of generators up to radical of a hypersurface ideal I by the number of generators of the conormal bundle I/I^2.

2.6. Theorem. *Let R be a ring and $I \subset R$ a finitely generated locally principal ideal with $\mu(I/I^2) \leqslant m$ for some $m \geqslant 2$. Then I is generated up to radical by m elements.*
If $\mathrm{Supp}(I/I^2)$ *is noetherian, the following more precise statement holds: There exists an ideal $J \subset I$ with $\sqrt{J} = \sqrt{I}$, $\mu(J) \leqslant m$ and*

$$\mathrm{Codim}_{\mathrm{Supp}(I/I^2)} \mathrm{Supp}(I/J) \geqslant m - 1.$$

Proof. Set $\mathfrak{a} := \sqrt{\mathrm{Ann}\, I}$, $R' := R/\mathfrak{a}$ and let $X' := \mathrm{Spec}\, R'$ be the affine scheme of R'. The underlying topological space of X' is $V(\mathfrak{a}) = \mathrm{Supp}\, I$. Since $\mu((I/\mathfrak{a}I)_x) = 1$ for all $x \in X'$, and R' is reduced, the R'-module $I/\mathfrak{a}I$ is locally free of rank 1 by (2.5). We denote by L the line bundle associated to $I/\mathfrak{a}I$. The inclusion $I \to R$ induces a morphism $\varphi : L \to \mathcal{O}_{X'}$ with

$$V_{L^*}(\varphi) = V(f) \cap X' = \mathrm{Supp}(I/I^2);$$

this is the underlying topological space of $Y := \mathrm{Sch}(\varphi)$. We have

$$\Gamma(Y, L \mid Y) = I/(I + \mathfrak{a})I,$$

hence $\mu(\Gamma(Y, L \mid Y)) \leqslant \mu(I/I^2) \leqslant m$. By Proposition 2.3 there exist sections

$$f_1, \ldots, f_m \in \Gamma(X', L) = I/\mathfrak{a}I$$

with

$$Z := V(f_1, \ldots, f_m) \subset Y,$$

and, if $\mathrm{Supp}(I/I^2)$ is noetherian, $\mathrm{Codim}_Y Z \leqslant m - 1$. Let $F_1, \ldots, F_m \in I$ be representatives of $f_1, \ldots, f_m$, and $J \subset R$ the ideal generated by $F_1, \ldots, F_m$. By construction

$$\mathrm{Supp}(I/J) = Z \subset \mathrm{Supp}(I/I^2),$$

hence $V(J) = V(I)$. This proves Theorem 2.6.

2.7. Corollary. *Let I be a finitely generated, locally principal ideal in a ring R such that* $\mathrm{Specm}(R/I)$ *is noetherian and satisfies*

$$\dim \mathrm{Specm}(R/I) \leqslant n - 1$$

for some $n \geqslant 2$. Then I is generated up to radical by n elements.

Proof. Since $Y = \mathrm{Specm}(R/I)$ has dimension $\leqslant n-1$ and $\mu((I/I^2)_y) \leqslant 1$ for all $y \in Y$, it follows that I/I^2 is generated by n elements ([F],[Sw]).

Remark. Corollary 2.7 says in particular: Let R be an n-dimensional noetherian ring, $n \geqslant 2$. Then every locally principal ideal can be generated up to radical by n elements. This has been proved by Boratyński [B] for R a 2-dimensional affine algebra over an algebraically closed field and by Murthy for n-dimensional regular affine algebras over algebraically closed fields (mentioned in [L]). Mandal proved it for arbitrary n-dimensional noetherian Cohen-Macaulay rings [M].

2.8. Corollary. *Let $Y \subset X$ be an effective Cartier divisor on an n-dimensional Stein space X, $n \geqslant 3$. Then the ideal $I(Y)$ of Y is generated up to radical by $\lfloor \frac{n+1}{2} \rfloor$ holomorphic functions.*

Remark. On an n-dimensional Stein space any vector bundle of rank d can be generated by $d + \lfloor n/2 \rfloor$ global sections. (In [FR] this is proved over Stein manifolds; the proof is valid for arbitrary Stein spaces by the results of Hamm ([H1], [H2]) on the topology of Stein spaces with singularities.) This implies that $I(Y)$ can be generated by $1 + \lfloor n/2 \rfloor$ holomorphic functions (without restriction on n).

Proof of Corollary 2.8. By the above remark, $I(Y)/I(Y)^2$ can be generated by $1 + \lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor$ elements.

3. Set theoretic description of subschemes

3.1. Lemma. *Let M be a finitely generated module over a ring R. We denote by X the affine scheme of R and by $\mathcal{M}$ the $\mathcal{O}_X$-module associated to M. Suppose that $Y_0 := \operatorname{Supp}(\mathcal{M})$ is noetherian. Then there exist $\alpha_1, \dots, \alpha_m \in R$ such that for*

$$Y_j := V(\alpha_1, \dots, \alpha_j) \cap Y_0$$

we have

i) $\mathcal{M} \mid (Y_{j-1} \smallsetminus Y_j)$ *is free for* $j = 1, \dots, m$,

ii) $Y_m = \emptyset$.

Here $Y_{j-1} \smallsetminus Y_j$ is considered as a reduced subscheme of X. For any locally closed subscheme $Z \subset X$ the restriction $\mathcal{M} \mid Z$ denotes the sheaf $\mathcal{M} \otimes \mathcal{O}_Z$ on Z.

Proof. The α_j are constructed by induction. To find α_{j+1}, let $y \in Y_j$ be a point such that $\mu(\mathcal{M}_y)$ is minimal in Y_j. Then by (2.5) the sheaf $\mathcal{M} \mid Y_j$ is free in some neighbourhood of y in Y_j, which can be chosen as $Y_j \smallsetminus V(\alpha_{j+1})$.

3.2. Lemma. *Let P be a module over a ring R, and $\alpha \in R$ such that P_α is a free R_α-module of rank r and $D(\alpha) := \operatorname{Spec}(R) \smallsetminus V(\alpha)$ is a noetherian topological space. Then for every $g \in P$ there exists $f \in P$ such that*

i) $f \equiv g \bmod \alpha P$,

ii) $\operatorname{Codim}_{D(\alpha)} V(f \mid D(\alpha)) \geqslant r$.

Proof. There exist $e_1, \dots, e_r \in \alpha P$ such that their images $\overline{e}_j := e_j \mid D(\alpha) \in P_\alpha$ form a basis of P_α. Define $g_j \in R_\alpha$ by

$$g \mid D(\alpha) = \sum_{j=1}^{r} g_j \overline{e}_j.$$

By induction on j choose $a_j \in R$ such that the sets $Y_0 := D(\alpha)$ and

$$Y_j := \{x \in Y_{j-1} : g_j(e) = a_j(x)\}$$

satisfy

$$\operatorname{Codim}_{Y_{j-1}} Y_j \geqslant 1 \qquad \text{for} \quad j = 1, \dots, r.$$

For this it suffices that $g_j(x_\mu) \neq a_j(x_\mu)$, $\mu = 1, \dots, m$, where $\{x_1, \dots, x_m\}$ meets all irreducible components of Y_{j-1} and of $Y_{j-1} \cap \operatorname{Specm}(R)$. For

$$f := g - \sum_{j=1}^{r} a_j e_j$$

we have $V(f \mid V(\alpha)) = Y_r$, which implies the assertion.

3.3. Let M be a finitely generated module over a ring R. For $k \in \mathbb{N}$ we define subsets $X_k(M)$ of $X := \mathrm{Spec}(R)$ as

$$X_k(M) := \{x \in X : \mu(M_x) \geqslant k\}.$$

All $X_k(M)$ are closed sets. We have $X_0(M) = X$, $X_1(M) := \mathrm{Supp}(M)$, and $X_k(M) = \emptyset$ for large k. We will apply this concept especially to the conormal module I/I^2 of a finitely generated ideal I. Note that $X_1(I/I^2) = \mathrm{Supp}(I) \cap V(I)$ and $X_k(I/I^2) = X_k(I)$ for $k \geqslant 2$.

To estimate the minimal number of generators of a module M over R we define the invariant

$$b(M) := \begin{cases} \sup\{k + \dim\mathrm{m}\, X_k(M) : k \geqslant 1 \text{ and } X_k(M) \neq \emptyset\}, & \text{if } M \neq 0, \\ 0, \quad \text{if } M = 0. \end{cases}$$

If $\mathrm{Specm}(R)$ is noetherian, we have $\mu(M) \leqslant b(M)$, (cf. [F], [Sw]).

3.4. Proposition. *Let M be a finitely generated R-module such that* $\mathrm{Supp}(M)$ *is noetherian. For $k \in \mathbb{N}$ let $X'_k := X_k(M) \smallsetminus X_{k+1}(M)$. There exists an $f \in M$ such that*

$$\mathrm{Codim}_{X'_k} V(f \mid X'_k) \geqslant k \quad \textit{for all } k.$$

(Note that, by definition, the empty subset of any topological space has codimension $+\infty$.)

Proof. Let $\mathrm{Supp}(M) = Y_0 \supset Y_1 \supset \ldots \supset Y_m = \emptyset$ be a stratification as in Lemma 3.1. We find f by constructing $f_j = f \mid Y_j$ for $j = m, m-1, \ldots, 0$ inductively with the aid of Lemma 3.2.

Remark. Proposition 3.4 contains as a special case the following well known result [S]: Let P be a finitely generated projective module of rank r over a ring with noetherian spectrum. Then there exists an $f \in P$ such that $\mathrm{Codim} V(f) \geqslant r$. If, in particular, $\dim \mathrm{Specm}(R) < r$, the module P has a direct summand isomorphic to R.

3.5. Corollary. *Let M be a finitely generated R-module such that* $\mathrm{Supp}(M)$ *is noetherian. Suppose that for some $m \geqslant 2$ we have*

$$b(M) \leqslant m, \qquad X_m(M) = \emptyset.$$

Then there exist elements $f_1, \ldots, f_{m-2} \in M$ such that for $j = 1, \ldots, m-2$ the module $M_j := M/(f_1, \ldots, f_{m-2})$ satisfies

$$b(M_j) \leqslant m - j, \qquad X_{m-j}(M_j) = \emptyset.$$

Proof by induction on j, using Proposition 3.4.

In particular, M_{m-2} has a support $Y := \mathrm{Supp}(M_{m-2})$ with $\mathrm{dimm}\, Y \leqslant 1$, and M_{m-2} induces by (2.5) a line bundle on the reduced subscheme Y of $\mathrm{Spec}\, R$.

3.6. Theorem. *Let I be a finitely generated ideal of a ring R such that* $\mathrm{Supp}(I/I^2)$ *is noetherian. Suppose that for some positive integer m we have*

$$b(I/I^2) \leqslant m \quad \textit{and} \quad X_m(I/I^2) = \emptyset.$$

Then there exists an ideal $J \subset I$ with $\mu(J) \leqslant m$, $\sqrt{J} = \sqrt{I}$ and $\mathrm{dimm}\,\mathrm{Supp}(I/J) \leqslant 0$.

Proof. For $m = 1$ we have $I/I^2 = 0$ and the assertion is trivial. Therefore suppose $m \geqslant 2$. By Corollary 3.5 there exist $f_1, \dots, f_{m-2} \in I$ such that the ideal

$$I' := I/(f_1, \dots, f_{m-2})$$

of the ring

$$R' := R/(f_1, \dots, f_{m-2})$$

satisfies

$$b(I'/I'^2) \leqslant 2 \quad \text{and} \quad X_2(I'/I'^2) = \emptyset.$$

Identifying $\mathrm{Spec}(R')$ with $V(f_1, \dots, f_{m-2}) \subset \mathrm{Spec}(R)$ we have $V(I') = V(I)$. By Theorem 2.6 there exists an ideal $J' \subset I'$ generated by two elements f'_{m-1}, f'_m, such that

$$V(J') = V(I') \quad \text{and} \quad \mathrm{dimm}\,\mathrm{Supp}(I'/J') \leqslant 0.$$

Let $f_{m-1}, f_m \in I$ be representatives of f'_{m-1}, f'_m and $J := (f_1, \dots, f_m)$. Since $V(J) = V(J')$ and $I/J \cong I'/J'$, the assertion follows.

3.7. *Remark.* The assumptions on $b(I/I^2)$ and $X(I/I^2)$ in Theorem 3.6 are for $m \geqslant 2$ equivalent to

(i) $\mathrm{dimm}\,(V(I/I^2) \cap \mathrm{Supp}(I)) \leqslant m - 1$,

(ii) $\mathrm{dimm}\, X_k(I) \leqslant m - k \quad$ for $k = 2, \dots, m-1$,

(iii) $X_m(I) = \emptyset$.

Therefore Theorem 3.6 applies in particular to locally complete intersections. By a *locally complete intersection ideal* we mean an ideal I in a ring R such that

$$\mu(I_x) \leqslant \mathrm{height}(I_x) \quad \text{for all } x \in V(I).$$

Note that, by this definition, I_x need not be generated by a regular sequence in R_x (which would be automatically the case if R were supposed to be Cohen-Macaulay).

Further we do not require $V(I)$ to be of pure codimension. For a finitely generated locally complete intersection ideal I in an n-dimensional ring we have

$$\dim X_k(I) \leqslant n-k \quad \text{for } k \geqslant 2.$$

Therefore Theorem 3.6 implies

3.8. Corollary. *Let R be an n-dimensional noetherian ring and $I \subset R$ a locally complete intersection ideal such that $V(I)$ has no zero-dimensional components. Then I can be generated up to radical by n elements.*

In the case of Cohen-Macaulay rings this result was obtained by Lyubeznik [L] for height$(I) \geqslant 2$, and by Mandal [M] also for height one.

In general, Corollary 3.8 is not correct if $V(I)$ has zero-dimensional components. For example let R be the coordinate ring of a smooth n-dimensional affine algebraic variety X over an algebraically closed field, and I the ideal of a single point $x \in X$. If I is generated up to radical by n elements, then the class of $\{x\}$ in the Chow group $A^n(X)$ of codimension n cycles is a torsion element. This is not always the case (see e.g. [MM] or [R]).

References

[B] Boratyński, M.: Every curve on a nonsingular surface can be defined by two equations. Proc. AMS 96 (1986) 391 – 393.

[EE] Eisenbud, D. and E.-G. Evans jr.: Every algebraic set in n-space is the intersection of n hypersurfaces. Invent. math. 19 (1973) 107 – 112.

[F] Forster, O.: Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ring. Math. Zeitschr. 84 (1964) 80 – 87.

[FR] Forster, O. und K.J. Ramspott: Analytische Modulgarben und Endromisbündel. Invent. Math. 2 (1966) 145 – 170.

[H1] Hamm, H.A.: Zum Homotopietyp Steinscher Räume. Journ. f.d.r.u.a. Math. 338 (1983) 121 – 135.

[H2] Hamm, H.A.: Zum Homotopietyp q-vollständiger Räume. Journ. f.d.r.u.a. Math. 364 (1986) 1 - 9.

[L] Lyubeznik, G.: Some theorems on set theoretic intersections. Preprint 1985.

[M] Mandal, S.: On set theoretic intersections in affine spaces. Preprint 1985.

[MM] Mohan Kumar, N. and M.P. Murthy: Algebraic cycles and vector bundles over affine three-folds. Annals of Math. 116 (1982) 579 – 591.

[MS] Murthy, M.P. and R.G. Swan: Vector bundles over affine surfaces. Invent. math. 36 (1976) 125 – 165.

[R] Rojtman, A.A.: The torsion of the group of 0-cycles modulo rational equivalence. Annals of Math. 111 (1980) 553 – 569.

[S] Serre, J.-P.: Modules projectifs et espaces fibrés à fibre vectorielle. Sém. Dubreil-Pisot 1957/58, exp. 23 (1958).

[St] Storch, U.: Bemerkung zu einem Satz von M. Kneser. Arch. Math. 23 (1972) 403 – 404.

[Sw] Swan, R.G.: The number of generators of a module. Math. Zeitschr. 102 (1967) 318 – 322.

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