The Theorem of Gauß-Bonnet in Complex Analysis[1]

Otto Forster

Abstract. The theorem of Gauß-Bonnet is interpreted within the framework of Complex Analysis of one and several variables.

Geodesic triangles

In 1828, Gauß proved in his *Disquisitiones generales circa superficies curvas* [3] the following theorem: Let ABC be a geodesic triangle on a smooth oriented surface X in Euclidean 3-space with angles α, β, γ. (Geodesic means that the three sides of the triangle are geodesic lines.) Then the *spherical excess* $\alpha+\beta+\gamma-\pi$ equals the integral of the curvature K over the triangle:

$$\alpha+\beta+\gamma-\pi = \int_{ABC} K dS.$$

Here dS is the surface element (2-dimensional volume element) on X. To define the curvature K, Gauß introduced a map (known today as *Gauß map*), which can be constructed as follows: Let $N(x)$, $x \in X$, be a unit normal field on the surface X. Then N defines a map Γ from X to the unit sphere S^2. Let dS be an infinitesimal surface element at $x \in X$. Then the curvature $K(x)$ is defined as the ratio of the areas of $\Gamma(dS)$ and dS. This definition apparently depends on the embedding of X in 3-space. However Gauß proved in the same Disquisitiones his famous *Theorema egregium* that K depends only on the inner geometry (i.e. the metric tensor) of the surface X. If we write in classical notation the metric on X in local coordinates as $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$, then K is a function of E, F, G and its derivatives up to second order.

O. Bonnet (1819 - 1892) considered also the case when the sides of the triangle are not necessarily geodesic and calculated the correction terms that have to be added. However we will not need this in the sequel.

Euler characteristic

Let T be a triangulation of a closed oriented connected surface X. We denote by $n_0(T)$ the number of vertices, by $n_1(T)$ the number of edges and by $n_2(T)$ the number of triangles of this triangulation. Then

$$\chi(T) := n_0(T) - n_1(T) + n_2(T)$$

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is the Euler characteristic of T. It is well known (and was essentially proved by Euler for the case of surfaces homeomorphic to S^2) that $\chi(T)$ does not depend on the triangulation but only on the surface and may therefore be denoted by $\chi(X)$. The Euler characteristic can be expressed by other topological invariants of the surface as

$$\chi(X) = b_0(X) - b_1(X) + b_2(X),$$

where $b_i(X)$ are the *Betti numbers* of X. Since for a closed connected oriented surface $b_0(X) = b_2(X) = 1$ and $b_1(X)$ is even, $b_1(X) = 2g(X)$, where $g(X)$ is called the *genus* of X, we have also

$$\chi(X) = 2 - 2g(X).$$

We are now in a position to state the theorem of Gauß-Bonnet.

Theorem (Gauß-Bonnet). *Let X be a smooth closed oriented surface in $\mathbf{R}^3$. Then*

$$\int_X K dS = 2\pi\chi(X).$$

Proof. We use a geodesic triangulation T of the surface with triangles Δ_ν, $\nu = 1, \ldots, n_2(T)$. Let $\alpha_\nu, \beta_\nu, \gamma_\nu$ be the angles of Δ_ν. Then

$$\int_{\Delta_\nu} K dS = \alpha_\nu + \beta_\nu + \gamma_\nu - \pi.$$

Summation over ν yields

$$\int_X K dS = \sum(\alpha_\nu + \beta_\nu + \gamma_\nu) - \pi n_2(T).$$

Now the sum of angles at every vertice of the triangulation is 2π, hence

$$\sum(\alpha_\nu + \beta_\nu + \gamma_\nu) = 2\pi n_0(T).$$

On the other hand, every edge of the triangulation belongs to two triangles, which implies

$$2n_1(T) = 3n_2(T).$$

Putting everything together, we get

$$\begin{aligned} \int_X K dS &= \pi(2n_0(T) - n_2(T)) \\ &= \pi(2n_0(T) - 2n_1(T) + 2n_2(T)) = 2\pi\chi(T). \end{aligned}$$

This proves the theorem.

Note that for the proof of the theorem one does not need the invariance of the Euler characteristic. On the contrary, since the left hand side of the Gauß-Bonnet formula does not depend on the triangulation, the theorem implies that the Euler characteristic depends only on the surface and its metric. With a little extra work one can see that it is not necessary to suppose the triangulation as geodesic. (If a non geodesic triangulation is given, in a sufficiently fine subdivision one can replace all edges by geodesics, which leaves the Euler characteristic unchanged.) Thus the right hand side does not depend on the metric, so also $\int_X K dS$ does not depend on the metric.

Isothermal coordinates

If we look at the underlying conformal structure of a surface in Euclidean 3-space we get a Riemann surface. Indeed this point of view was already taken by Gauß, who introduced isothermal coordinates. These are local coordinates such that the metric takes the form

$$ds^2 = \lambda(x, y)(dx^2 + dy^2).$$

Therefore (x, y) defines a conformal map of a coordinate neighborhood of the surface to the Euclidean plane. The existence of isothermal coordinates is equivalent to the solution of the so called Beltrami equation (see Ahlfors [1]). Already Gauß proved the existence of isothermal coordinates for the real analytic case. Let us briefly describe his idea. If the metric is given locally by the positive definite quadratic form $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$, then in every point there exist two complex conjugate *isotropic* directions which annihilate this form. So if we embed the surface into a 2-dimensional complex analytic manifold we can find there local coordinates (ξ, η) such that ds^2 transforms to $ds^2 = \lambda d\xi d\eta$. Then the coordinates (x, y) with $\xi = x + iy$, $\eta = x - iy$ are isothermal.
Using isothermal coordinates (x, y), the formula for the Gauß curvature of the metric $ds^2 = \lambda(dx^2 + dy^2)$ simplifies to

$$K = -\frac{1}{2\lambda}\Delta \log \lambda,$$

where Δ is the Laplace operator. Note that the 2-dimensional volume element with respect to these coordinates is $dS = \lambda dx \wedge dy$, hence

$$KdS = -\tfrac{1}{2}\left(\Delta \log \lambda\right) dx \wedge dy.$$

If we introduce the complex coordinate $z = x + iy$ and use Wirtinger calculus, the metric becomes $ds^2 = \lambda |dz|^2$ and

$$KdS = -i \left(\frac{\partial^2}{\partial\bar{z}\partial z} \log \lambda\right) dz \wedge d\bar{z} = i\,\bar{\partial}\,\partial\,\log \lambda.$$

Here the last expression is to be understood in the sense of calculus of exterior differential forms:

$$\partial f = \frac{\partial f}{\partial z}\,dz, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}}\,d\bar{z},$$

$$\partial(f \wedge dg) = \frac{\partial f}{\partial z}\,dz \wedge dg, \quad \bar{\partial}(f \wedge dg) = \frac{\partial f}{\partial \bar{z}}\,d\bar{z} \wedge dg.$$

We have $d = \partial + \bar{\partial}$ and $\partial\,\partial = \bar{\partial}\,\bar{\partial} = 0$, hence one can also write

$$KdS = i\,d\,(\partial\,\log \lambda).$$

Abelian differentials

On our surface X, which we regard as a compact Riemann surface, we consider now an abelian differential σ, i.e. a meromorphic differential form. The degree of σ, defined as the difference of the number of zeroes and the number of poles (counted with multiplicities), equals $2g-2$, where g is the genus of X. This fact is equivalent to the theorem of Gauß-Bonnet, as we shall see now. Let (U_ν, z_ν) be a covering of X by complex charts. We may suppose that every U_ν contains at most one zero or one pole of σ, and that in this case the zero or pole occurs for $z_\nu = 0$. With respect to the coordinates z_ν the metric is given by $ds^2 = \lambda_\nu |dz_\nu|^2$ and the differential form may be written as $\sigma = f_\nu dz_\nu$ with a meromorphic function f_ν. It follows that on the intersections $U_\nu \cap U_\mu$ we have

$$\lambda_\nu/\lambda_\mu = |f_\nu/f_\mu|^2.$$

Therefore there exists a globally defined function φ with

$$\varphi = \frac{\lambda_\nu}{|f_\nu|^2} \quad \text{on } U_\nu \text{ for all } \nu,$$

which is smooth except for singularities at the zeroes and poles of σ. Since $\log|f_\nu|$ is harmonic, we have

$$KdS = i\, d\,(\partial \log \varphi)$$

outside the poles and zeroes of σ. Let $X_\varepsilon = X - \bigcup D_{k,\varepsilon}$, where the $D_{k,\varepsilon}$ are small disks around the singularities of σ. Then

$$\int_X KdS = i \lim_{\varepsilon\to 0} \int_{X_\varepsilon} d\,(\partial \log \varphi) = -i \sum \lim_{\varepsilon \to 0} \int_{\partial D_{k,\varepsilon}} \partial \log \varphi$$

by Stokes' theorem (the two-dimensional case of the Gauß integral formula). To evaluate the integrals over the circles we note that at a zero or pole of σ the function φ is of the form $\varphi = \psi/|z|^{2m}$ with a smooth function ψ without zeroes and m the order of σ at the zero or pole ($m < 0$ in the latter case). Therefore

$$\lim_{\varepsilon\to 0} \int\limits_{\partial D_{k,\varepsilon}} \partial \log \varphi = \lim_{\varepsilon \to 0} \int\limits_{|z|=\varepsilon} \partial\,(\log|z|^{-2m}) = -m \lim_{\varepsilon\to 0} \int\limits_{|z|=\varepsilon} \frac{dz}{z} = -2\pi i m.$$

Summing up, we get

$$\int_X KdS = -2\pi \deg(\sigma).$$

The Gauß-Bonnet theorem now implies $\deg(\sigma) = 2g-2$. Conversely, if one proves the formula $\deg(\sigma) = 2g-2$ by other means (for example by representing the Riemann surface X as a branched covering of the Riemann sphere and studying the ramification points) one gets another proof of the theorem of Gauß-Bonnet.

Chern classes

The above developments have been greatly generalized by Chern [2] (good textbooks are [4],[6]) to higher dimensional manifolds. Chern defined curvature forms for vector bundles on an n-dimensional complex manifold X which represent, via the de Rham isomorphism, cohomology classes that are topological invariants of the vector bundle.

Let E be a holomorphic vector bundle of rank r on X and let h be a hermitian metric on E. With respect to a local trivialization of E, the metric is given by a positive definite hermitian $r \times r$-matrix $h = (h_{\mu\nu})$. Now one can define the curvature form of the metric as

$$\Theta = \bar{\partial} h^{-1} \partial h.$$

This is only defined locally, but can be viewed as a global 2-form with coefficients in the endomorphism bundle $End(E)$. (For $n = 1$ and the tangent bundle the form Θ is equal, up to a factor i, to the form KdS from above.) With an indeterminate t, we write

$$\det\Big(1 + \frac{i}{2\pi}\,\Theta\, t\Big) = 1 + c_1 t + \ldots + c_r t^r.$$

Then c_k is a closed differential form of degree $2k$ which represents the k-th Chern class of E. As a special case, let X be an n-dimensional compact complex manifold with a hermitian metric $g = (g_{\mu\nu})$, (i.e. a metric on the tangent bundle). Then the n-th Chern class is given by $c_n = (\frac{i}{2\pi})^n \det(\bar{\partial} g^{-1} \partial g)$ and Chern's generalization of the Gauß-Bonnet theorem reads

$$\Big(\frac{i}{2\pi}\Big)^n \int_X \det(\bar{\partial} g^{-1} \partial g) = \chi(X) = \sum_{k=0}^{n} (-1)^n b_k(X).$$

Todd classes

On a compact Riemann surface X the genus g can also be analytically defined as the dimension of the first cohomology group $H^1(X, \mathcal{O})$ of the sheaf $\mathcal{O}$ of holomorphic functions. Therefore the Euler-Poincaré characteristic of $\mathcal{O}$ has the value

$$\chi(X, \mathcal{O}) := \dim H^0(X, \mathcal{O}) - \dim H^1(X, \mathcal{O}) = 1 - g,$$

and the Gauß-Bonnet theorem can be written as

$$\int_X KdS = \pi \chi(X, \mathcal{O}).$$

The generalization of this form of the Gauß-Bonnet theorem to higher dimensions is a special case of Hirzebruch's Riemann-Roch theorem [5] and involves Todd classes. To define the Todd classes of a complex vector bundle E of rank n, consider the power series in n indeterminates $x_1, \ldots, x_n$

$$\prod_{\nu=1}^{n} \frac{x_\nu}{1 - e^{-x_\nu}} = \sum_{k=0}^{\infty} F_k(x_1, \ldots, x_n),$$

where F_k is a homogeneous polynomial of degree k in $x_1, \ldots, x_n$. Since F_k is symmetric in $x_1, \ldots, x_n$, it can be expressed as a polynomial in the elementary symmetric functions $s_j(x_1, \ldots, x_n)$,

$$F_k(x_1, \ldots, x_n) = \widetilde{F}_k(s_1, \ldots, s_n).$$

If we substitute s_j by the j-th Chern class $c_j(E)$, we get the k-th Todd class of E

$$\mathrm{td}_k(E) = \widetilde{F}_k(c_1(E), \ldots, c_n(E)).$$

For example, $\mathrm{td}_1 = \frac{1}{2}c_1$ and $\mathrm{td}_2 = (c_1^2 + c_2)/12$. The Todd classes of a compact complex n-dimensional manifold X are defined as the Todd classes of the tangent bundle of X. With these definitions we can now state the Hirzebruch-Riemann-Roch formula for the sheaf $\mathcal{O}$.

$$\int_X \mathrm{td}_n(X) = \chi(X, \mathcal{O}) = \sum_{k=0}^{n} (-1)^k \dim H^k(X, \mathcal{O}).$$

Of course this formula, which is only a special case of the Riemann-Roch theorem and the Atiyah-Singer index theorem, is much more difficult to prove than the classical Gauß-Bonnet theorem. But I hope that its connection to this classical theorem gives a good motivation to take up the study.

References

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Otto Forster
Math. Institut der LMU,
Theresienstr. 39
D-80333 München, Germany
Email: forster@mathematik.uni-muenchen.de