Chapter 8

Heath–Jarrow–Morton (HJM) Methodology

→ original article by Heath, Jarrow and Morton [10], MR[20](Chapter 13.1), Z[28](Chapter 4.4), etc.

As we have seen in the previous section, short rate models are not always flexible enough to calibrating them to the observed initial yield curve. Heath, Jarrow and Morton (HJM, 1992) [10] proposed a new framework for modelling the entire forward curve directly.

The setup is as in Chapter 6. We fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $\mathbb{P}$ is considered as objective probability measure, and a $d$-dimensional Brownian motion $W$.

8.1 Forward Curve Movements

We assume that we are given an $\mathbb{R}$-valued and $\mathbb{R}^d$-valued stochastic process $\alpha(t, T)$ and $\sigma(t, T) = (\sigma_1(t, T), \ldots, \sigma_d(t, T))$, respectively, with two indices, $t, T$, such that

HJM.1 for every $\omega$, the maps $(t, T) \mapsto \alpha(t, T, \omega)$ and $(t, T) \mapsto \sigma(t, T, \omega)$ are continuous for $0 \leq t \leq T$,

HJM.2 for every $T$, the processes $\alpha(t, T)$ and $\sigma(t, T)$, $0 \leq t \leq T$, are adapted,

HJM.3 $\int_0^T \int_0^T \mathbb{E} |\alpha(s, t)| \, ds \, dt < \infty$ for all $T$,
HJM.4 $\int_0^T \int_0^T \mathbb{E} \|\sigma(s,t)\|^2 \, ds \, dt < \infty$ for all $T$.

For a given continuous initial forward curve $T \mapsto f(0,T)$ it is then assumed that, for every $T$, the forward rate process $f(\cdot, T)$ follows the Itô dynamics

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T) \, ds + \int_0^t \sigma(s, T) \, dW(s), \quad t \leq T. \tag{8.1}$$

This is a very general setup. The only substantive economic restrictions are the continuous sample paths assumption for the forward rate process, and the finite number, $d$, of random drivers $W_1, \ldots, W_d$.

Notice that the integrals in (8.1) are well defined by HJM.1–HJM.4. Moreover, it follows from Corollary 8.4.2 that the short rate process

$$r(t) = f(t, t) = f(0, t) + \int_0^t \alpha(s, t) \, ds + \int_0^t \sigma(s, t) \, dW(s)$$

has a predictable version (again denoted by $r(t)$) and, by Schwarz’ inequality and the Itô isometry,

$$\int_0^T \mathbb{E} |r(t)| \, dt \leq \int_0^T |f(0, t)| \, dt + \int_0^T \int_0^t \mathbb{E} |\alpha(s, t)| \, ds \, dt$$

$$+ T^{\frac{1}{2}} \left( \int_0^T \mathbb{E} \left( \int_0^t \sigma(s, t) \, dW(s) \right)^2 \, dt \right)^{\frac{1}{2}}$$

$$\leq \int_0^T |f(0, t)| \, dt + \int_0^T \int_0^T \mathbb{E} |\alpha(s, t)| \, ds \, dt$$

$$+ T^{\frac{1}{2}} \left( \int_0^T \int_0^T \mathbb{E} \|\sigma(s, t)\|^2 \, ds \, dt \right)^{\frac{1}{2}} < \infty.$$ 

Hence the savings account $B(t) = e^{\int_0^t r(s) \, ds}$ is well defined. More can be said about the zero-coupon bond prices.

**Lemma 8.1.1.** For every maturity $T$, the zero-coupon bond price process $P(t, T) = \exp \left( - \int_t^T f(t, u) \, du \right)$, $0 \leq t \leq T$, is an Itô process of the form

$$P(t, T) = P(0, T) + \int_0^t P(s, T) (r(s) + b(s, T)) \, ds + \int_0^t P(s, T) a(s, T) \, dW(s) \tag{8.2}$$
where
\[ a(s, T) := -\int_s^T \sigma(s, u) \, du, \]
\[ b(s, T) := -\int_s^T \alpha(s, u) \, du + \frac{1}{2} \|a(s, T)\|^2. \]

Proof. Using the classical Fubini Theorem and Theorem 8.4.1 for stochastic integrals twice, we calculate
\[
\log P(t, T) = -\int_t^T f(t, u) \, du
\]
\[
= -\int_t^T f(0, u) \, du - \int_t^T \int_0^t \alpha(s, u) \, ds \, du - \int_t^T \int_0^T \sigma(s, u) \, dW(s) \, du
\]
\[
= -\int_t^T f(0, u) \, du - \int_0^t \int_s^T \alpha(s, u) \, du \, ds - \int_t^T \int_0^s \sigma(s, u) \, du \, dW(s)
\]
\[
+ \int_0^t f(0, u) \, du + \int_0^t \int_s^T \alpha(s, u) \, du \, ds + \int_0^t \int_s^T \sigma(s, u) \, du \, dW(s)
\]
\[
= -\int_0^t f(0, u) \, du + \int_0^t \left( b(s, T) - \frac{1}{2} \|a(s, T)\|^2 \right) ds + \int_0^t a(s, T) \, dW(s)
\]
\[
+ \int_0^t \left( f(0, u) \, du + \int_0^u \alpha(s, u) \, ds + \int_0^u \sigma(s, u) \, dW(s) \right) du
\]
\[
= \log P(0, T) + \int_0^t \left( r(s) + b(s, T) - \frac{1}{2} \|a(s, T)\|^2 \right) ds
\]
\[
+ \int_0^t a(s, T) \, dW(s),
\]

and we have used the fact that \( \int_0^t \sigma(s, u)1_{s \leq u} \, dW(s) = \int_0^u \sigma(s, u) \, dW(s) \).

Itô’s formula now implies (8.2) (\( \rightarrow \) exercise).

We write \( Z(t, T) = \frac{P(t, T)}{B(t)} \) for the discounted bond price processes.
**Corollary 8.1.2.** We have, for \(0 \leq t \leq T\),
\[
Z(t, T) = P(0, T) + \int_0^t Z(s, T)b(s, T) \, ds + \int_0^t Z(s, T)a(s, T) \, dW(s).
\]

**Proof.** Itô’s formula (→ exercise).

### 8.2 Absence of Arbitrage

In this section we investigate the restrictions on the dynamics (8.1) under the assumption of no arbitrage. In what follows we let \(\gamma \in \mathcal{L}\) be such that \(\mathcal{E}(\gamma \cdot W)\) is a uniformly integrable martingale with \(\mathcal{E}_\infty(\gamma \cdot W) > 0\). Girsanov’s Change of Measure Theorem 6.2.2 then implies that
\[
\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_\infty(\gamma \cdot W)
\]
defines an equivalent probability measure \(\mathbb{Q} \sim \mathbb{P}\), and
\[
\tilde{W}(t) := W(t) - \int_0^t \gamma(s) \, ds
\]
is a \(\mathbb{Q}\)-Brownian motion. We call \(\mathbb{Q}\) an ELMM for the bond market if the discounted bond price processes, \(Z(t, T) = \frac{P(t, T)}{B(t)}\), \(0 \leq t \leq T\), are \(\mathbb{Q}\)-local martingales, for all \(T\).

**Theorem 8.2.1 (HJM Drift Condition).** \(\mathbb{Q}\) is an ELMM if and only if
\[
b(t, T) = -a(t, T) \cdot \gamma(t) \quad \forall T \quad dt \otimes d\mathbb{P} \quad a.s.
\]
(8.3)

In this case, the \(\mathbb{Q}\)-dynamics of the forward rates \(f(t, T)\), \(0 \leq t \leq T\), are of the form
\[
f(t, T) = f(0, T) + \int_0^t \left( \sigma(s, T) \cdot \int_s^T \sigma(s, u) \, du \right) ds + \int_0^t \sigma(s, T) \, d\tilde{W}(s).
\]
(8.4)

**Proof.** In view of Corollary 8.1.2 we find that
\[
dZ(t, T) = Z(t, T) (b(t, T) + a(t, T) \cdot \gamma(t) ) \, dt + Z(t, T)a(t, T) \, d\tilde{W}(t).
\]
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Hence \( Z(t,T), 0 \leq t \leq T, \) is a \( Q \)-local martingale if and only if \( b(t,T) = -a(t,T) \cdot \gamma(t) \, dt \otimes d\mathbb{P} \)-a.s. Since \( a(t,T) \) and \( b(t,T) \) are continuous in \( T \), we deduce that \( Q \) is an ELMM if and only if (8.3) holds.

Differentiating both sides of (8.3) in \( T \) yields

\[
-\alpha(t,T) + \sigma(t,T) \cdot \int_t^T \sigma(t,u) \, du = \sigma(t,T) \cdot \gamma(t) \quad \forall T \quad dt \otimes d\mathbb{P} \text{ a.s.}
\]

Inserting this in (8.1) gives (8.4).

Remark 8.2.2. It follows from (8.2) and (8.3) that

\[
dP(t,T) = P(t,T) \left( r(t) + a(t,T) \cdot (-\gamma(t)) \right) dt + P(t,T)a(t,T) dW(t).
\]

Whence the interpretation of \(-\gamma\) as the market price of risk for the bond market.

The striking feature of the HJM framework is that the distribution of \( f(t,T) \) and \( P(t,T) \) under \( Q \) only depends on the volatility process \( \sigma(t,T) \) (and not on the \( \mathbb{P} \)-drift \( \alpha(t,T) \)). Hence option pricing only depends on \( \sigma \). This situation is similar to the Black–Scholes stock price model.

We can give sufficient conditions for \( Z(t,T) \) to be a true \( Q \)-martingale.

Corollary 8.2.3. Suppose that (8.3) holds. Then \( Q \) is an EMM if either

1. the Novikov condition

\[
\mathbb{E}_Q \left[ \exp \left( \frac{1}{2} \int_0^T \| \sigma(t,T) \|^2 \, dt \right) \right] < \infty \quad \forall T \quad (8.5)
\]

holds; OR

2. the forward rates are positive: \( f(t,T) \geq 0 \ \forall t \leq T. \)

Proof. We have \( Z(t,T) = P(0,T)\mathcal{E}_t(\sigma(\cdot,T) \cdot \tilde{W}) \). Hence the Novikov condition (8.5) is sufficient for \( Z(t,T) \) to be a \( Q \)-martingale (see [24, Proposition (1.26), Chapter IV]).

If \( f(t,T) \geq 0, \) then \( 0 \leq P(t,T) \leq 1 \) and \( B(t) \geq 1 \). Hence \( 0 \leq Z(t,T) \leq 1 \).

But a uniformly bounded local martingale is a true martingale. \( \square \)
8.3 Short Rate Dynamics

What is the interplay between the short rate models of the last chapter and the present HJM framework? Let us consider the simplest HJM model: a constant \( \sigma(t,T) \equiv \sigma > 0 \). Suppose that \( Q \) is an ELMM. Then (8.4) implies

\[
f(t,T) = f(0,T) + \sigma^2 t \left( T - \frac{t}{2} \right) + \sigma \tilde{W}(t).
\]

Hence for the short rates

\[
r(t) = f(t,t) = f(0,t) + \sigma^2 \frac{t^2}{2} + \sigma \tilde{W}(t).
\]

This is just the Ho–Lee model of Section 7.5.4.

In general, we have the following

**Proposition 8.3.1.** Suppose that \( f(0,T), \alpha(t,T) \) and \( \sigma(t,T) \) are differentiable in \( T \) with \( \int_0^T |\partial_t f(0,u)| \, du < \infty \) and such that HJM.1–HJM.4 are satisfied when \( \alpha(t,T) \) and \( \sigma(t,T) \) are replaced by \( \partial_T \alpha(t,T) \) and \( \partial_T \sigma(t,T) \), respectively.

Then the short rate process is an Itô process of the form

\[
r(t) = r(0) + \int_0^t \zeta(u) \, du + \int_0^t \sigma(u,u) \, dW(u) \quad (8.6)
\]

where

\[
\zeta(u) := \alpha(u,u) + \partial_u f(0,u) + \int_u^t \partial_{uu} \alpha(s,u) \, ds + \int_u^t \partial_u \sigma(s,u) \, dW(s).
\]

**Proof.** Recall first that

\[
r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t) \, ds + \int_0^t \sigma(s,t) \, dW(s).
\]

Applying the Fubini Theorem 8.4.1 to the stochastic integral gives

\[
\int_0^t \sigma(s,t) \, dW(s) = \int_0^t \sigma(s,s) \, dW(s) + \int_0^t (\sigma(s,t) - \sigma(s,s)) \, dW(s)
\]

\[
= \int_0^t \sigma(s,s) \, dW(s) + \int_0^t \int_s^t \partial_u \sigma(s,u) \, du \, dW(s)
\]

\[
= \int_0^t \sigma(s,s) \, dW(s) + \int_0^t \int_0^u \partial_u \sigma(s,u) \, dW(s) \, du.
\]
Moreover, from the classical Fubini Theorem we deduce, similarly,

\[
\int_0^t \alpha(s,t) \, ds = \int_0^t \alpha(s,s) \, ds + \int_0^t \int_0^u \partial_u \alpha(s,u) \, ds \, du,
\]
and finally

\[
f(0,t) = r(0) + \int_0^t \partial_u f(0,u) \, du.
\]

Combining these formulas, we obtain (8.6).

\[\square\]

## 8.4 Fubini’s Theorem

In this section we prove Fubini’s Theorem for stochastic integrals. For the classical version of Fubini’s Theorem, we refer to the standard textbooks in integration theory.

In what follows we let \( A \) denote a closed convex subset in \([0,T]^2\) and \( A^c = [0,T]^2 \setminus A \) its complement, e.g. \( A = \{(t,s) \in [0,T]^2 \mid t \leq s\} \).

**Theorem 8.4.1 (Fubini’s Theorem for stochastic integrals).** Consider the \( \mathbb{R}^d \)-valued stochastic process \( \phi = \phi(t,s) \) with two indices, \( 0 \leq t,s \leq T \), satisfying the following properties:

1. for every \( \omega \), the map \((t,s) \mapsto \phi(t,s,\omega)\) is continuous on the interior of \( A \) and \( A^c \)

2. for every \( s \in [0,T] \), the process \( \phi(t,s) \), \( 0 \leq t \leq T \), is adapted,

3. \( \int_0^T \int_0^T \mathbb{E} \|\phi(t,s)\|^2 \, dt \, ds < \infty. \)

Then the stochastic process \( \psi(s) := \int_0^t \phi(t,s) \, dW(t) \) has a \( \mathcal{B}[0,T] \otimes \mathcal{F}_T \)-measurable version (denoted again by \( \psi \)), and \( \lambda(t) := \int_0^T \phi(t,s) \, ds \) is piecewise continuous.

Moreover, \( \int_0^T \psi(s) \, ds = \int_0^T \lambda(t) \, dW(t) \), that is,

\[
\int_0^T \left( \int_0^t \phi(t,s) \, dW(t) \right) \, ds = \int_0^T \left( \int_0^T \phi(t,s) \, ds \right) \, dW(t). \tag{8.7}
\]
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Proof. We may assume that $||\phi|| \leq N$, otherwise we replace $\phi$ by $\phi 1_{\{||\phi|| \leq N\}}$ and let $N \to \infty$. Notice that $\tau(s, \omega) := \inf\{t \mid ||\phi(t, s, \omega)|| > N\} \land T$ is $\mathcal{B}[0, T] \otimes \mathcal{F}_T$-measurable and a stopping time for every fixed $s$ (→ exercise).

That $\lambda$ is piecewise continuous and adapted follows from the classical Fubini Theorem (→ exercise).

Now let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[0, T]$, and define

$$\phi_n(t, s) := \sum_{i=0}^{n-1} \phi(t_i, s) 1_{(t_i, t_{i+1})}(t).$$

It is then clear that

$$\psi_n(s) := \int_0^T \phi_n(t, s) dW(t) = \sum_{i=0}^{n-1} \phi(t_i, s) (W(t_{i+1}) - W(t_i)) \quad (8.8)$$

is $\mathcal{B}[0, T] \otimes \mathcal{F}_T$-measurable. Moreover,

$$\int_0^T \psi_n(s) ds = \int_0^T \left( \sum_{i=0}^{n-1} \phi(t_i, s) (W(t_{i+1}) - W(t_i)) \right) ds
= \sum_{i=0}^{n-1} \left( \int_0^T \phi(t_i, s) ds \right) (W(t_{i+1}) - W(t_i))
= \int_0^T \left( \int_0^T \phi_n(t, s) ds \right) dW(t). \quad (8.9)$$

From the Itô isometry and dominated convergence we have

$$\lim_{n \to \infty} \mathbb{E} \left[ (\psi_n(s) - \psi(s))^2 \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T (\phi_n(t, s) - \phi(t, s))^2 dt \right] = 0 \quad \forall s. \quad (8.10)$$

Let $A := \{(s, \omega) \mid \lim_n \psi_n(s, \omega) \text{ exists}\}$. Then $A$ is $\mathcal{B}[0, T] \otimes \mathcal{F}_T$-measurable and so is the process

$$\widehat{\psi}(s, \omega) := \begin{cases} 
\lim_n \psi_n(s, \omega), & \text{if } (s, \omega) \in A \\
0, & \text{otherwise.} 
\end{cases} \quad (8.11)$$

But in view of (8.10) we have $\psi(s) = \widehat{\psi}(s)$ a.s. Hence $\psi$ has a $\mathcal{B}[0, T] \otimes \mathcal{F}_T$-measurable version, which we denote again by $\psi$, so that the integral $\int_0^T \psi(s) ds$ is well defined and $\mathcal{F}_T$-measurable.
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From the Itô isometry and dominated convergence again we then have

\[
\mathbb{E} \left[ \left( \int_0^T \psi_n(s) \, ds - \int_0^T \psi(s) \, ds \right)^2 \right] \leq T \int_0^T \mathbb{E} \left[ (\psi_n(s) - \psi(s))^2 \right] \, ds
\]

\[
= T \int_0^T \mathbb{E} \left[ \int_0^T (\phi_n(t, s) - \phi(t, s))^2 \, dt \right] \, ds \to 0 \quad \text{for} \quad n \to \infty. \quad (8.12)
\]

On the other hand,

\[
\mathbb{E} \left[ \left( \int_0^T \left( \int_0^T \phi_n(t, s) \, ds \right) \, dW(t) - \int_0^T \lambda(t) \, dW(t) \right)^2 \right]
\]

\[
= \mathbb{E} \left[ \int_0^T \left( \int_0^T \phi_n(t, s) \, ds - \int_0^T \phi(t, s) \, ds \right)^2 \, dt \right]
\]

\[
\leq T \mathbb{E} \left[ \int_0^T \int_0^T (\phi_n(t, s) - \phi(t, s))^2 \, ds \, dt \right] \to 0 \quad \text{for} \quad n \to \infty. \quad (8.13)
\]

Combining (8.12) and (8.13) with (8.9) proves the theorem. \(\square\)

**Corollary 8.4.2.** Let \( \phi \) be as in Theorem 8.4.1. Then the process

\[\psi(s) := \int_0^s \phi(t, s) \, dW(t), \quad s \in [0, T],\]

has a predictable version.

**Proof.** This follows by similar arguments as in the proof of Theorem 8.4.1. Replace (8.8) by the predictable process

\[\psi_n(s) := \int_0^s \phi_n(t, s) \, dW(t) = \sum_{i=0}^{n-1} \phi(t_i \wedge s, s) \left( W(t_{i+1} \wedge s) - W(t_i \wedge s) \right),\]

and modify (8.10) and (8.11) accordingly (\(\to\) exercise). \(\square\)
Chapter 9

Forward Measures

We consider the HJM setup (Chapter 8) and assume there exists an EMM $Q \sim P$ of the form $dQ/dP = \mathcal{E}(\gamma \cdot W)$, as in Section 8.2, under which all discounted bond price processes

$$\frac{P(t,T)}{B(t)}, \quad t \in [0,T],$$

are strictly positive martingales.

9.1 $T$-Bond as Numeraire

Fix $T > 0$. Since

$$\frac{1}{P(0,T)B(T)} > 0 \quad \text{and} \quad \mathbb{E}_Q\left[ \frac{1}{P(0,T)B(T)} \right] = 1$$

we can define an equivalent probability measure $Q^T \sim Q$ on $\mathcal{F}_T$ by

$$\frac{dQ^T}{dQ} = \frac{1}{P(0,T)B(T)}.$$

For $t \leq T$ we have

$$\frac{dQ^T}{dQ} |_{\mathcal{F}_t} = \mathbb{E}_Q\left[ \frac{dQ^T}{dQ} | \mathcal{F}_t \right] = \frac{P(t,T)}{P(0,T)B(t)}.$$

This probability measure has already been introduced in Section 7.6. It is called the $T$-forward measure.
Lemma 9.1.1. For any $S > 0$,

$$\frac{P(t, S)}{P(t, T)}, \quad t \in [0, S \wedge T],$$

is a $Q^T$-martingale.

Proof. Let $s \leq t \leq S \wedge T$. Bayes’ rule gives

$$\mathbb{E}_{Q^T} \left[ \frac{P(t, S)}{P(t, T)} \mid \mathcal{F}_s \right] = \frac{\mathbb{E}_Q \left[ \frac{P(t, T)}{P(0, T) B(t)} \frac{P(t, S)}{P(0, T) B(s)} \right]}{P(s, T) B(s)} = \frac{P(s, S)}{P(s, T) B(s)} = \frac{P(s, S)}{P(s, T)}.$$

We thus have an entire collection of EMMs now! Each $Q^T$ corresponds to a different numeraire, namely the $T$-bond. Since $Q$ is related to the risk-free asset, one usually calls $Q$ the risk neutral measure.

$T$-forward measures give simpler pricing formulas. Indeed, let $X$ be a $T$-claim such that

$$\frac{X}{B(T)} \in L^1(Q, \mathcal{F}_T).$$

Its arbitrage price at time $t \leq T$ is then given by

$$\pi(t) = \mathbb{E}_Q \left[ e^{-\int_t^T r(s) \, ds} X \mid \mathcal{F}_t \right].$$

To compute $\pi(t)$ we have to know the joint distribution of $\exp \left[ -\int_t^T r(s) \, ds \right]$ and $X$, and integrate with respect to that distribution. Thus we have to compute a double integral, which in most cases turns out to be rather hard work. If $B(T)/B(t)$ and $X$ were independent under $Q$ (which is not realistic! it holds, for instance, if $r$ is deterministic) we would have

$$\pi(t) = P(t, T) \mathbb{E}_Q [X \mid \mathcal{F}_t],$$

a much nicer formula, since

- we only have to compute the single integral $\mathbb{E}_Q[X \mid \mathcal{F}_t]$;
• the bond price $P(t, T)$ can be observed at time $t$ and does not have to be computed.

The good news is that the above formula holds — not under $Q$ though, but under $Q^T$:

** Proposition 9.1.2.** Let $X$ be a $T$-claim such that (9.1) holds. Then

$$E_{Q^T} [|X|] < \infty$$

(9.2) and

$$\pi(t) = P(t, T)E_{Q^T} [X | \mathcal{F}_t].$$

(9.3)

**Proof.** Bayes’s rule yields

$$E_{Q^T} [|X|] = E_Q \left[ \frac{|X|}{P(0, T)B(T)} \right] < \infty \quad \text{(by (9.1))},$$

whence (9.2). And

$$\pi(t) = P(0, T)B(t)E_Q \left[ \frac{X}{P(0, T)B(T)} | \mathcal{F}_t \right]$$

$$= P(0, T)B(t) \frac{P(t, T)}{P(0, T)B(t)} E_{Q^T} [X | \mathcal{F}_t]$$

$$= P(t, T)E_{Q^T} [X | \mathcal{F}_t],$$

which proves (9.3). \qed

## 9.2 An Expectation Hypothesis

Under the forward measure the expectation hypothesis holds. That is, the expression of the forward rates $f(t, T)$ as conditional expectation of the future short rate $r(T)$.

To see that, we write $W$ for the driving $Q$-Brownian motion. The forward rates then follow the dynamics

$$f(t, T) = f(0, T) + \int_0^t \left( \sigma(s, T) \cdot \int_s^T \sigma(s, u) du \right) ds + \int_0^t \sigma(s, T) dW(s).$$

(9.4)
The \(Q\)-dynamics of the discounted bond price process is

\[
\frac{P(t, T)}{B(t)} = P(0, T) + \int_0^t \frac{P(s, T)}{B(s)} \left(-\int_t^T \sigma(s, u) \, du\right) \, dW(s). \tag{9.5}
\]

This equation has a unique solution

\[
\frac{P(t, T)}{B(t)} = P(0, T) \mathcal{E}_t \left(\left(-\int_t^T \sigma(\cdot, u) \, du\right) \cdot W\right).
\]

We thus have

\[
\left.\frac{dQ^T}{dQ}\right|_F = \mathcal{E}_t \left(\left(-\int_t^T \sigma(\cdot, u) \, du\right) \cdot W\right). \tag{9.6}
\]

Girsanov’s theorem applies and

\[
W^T(t) = W(t) + \int_0^t \left(\int_s^T \sigma(s, u) \, du\right) \, ds, \quad t \in [0, T],
\]

is a \(Q^T\)-Brownian motion. Equation (9.4) now reads

\[
f(t, T) = f(0, T) + \int_0^t \sigma(s, T) \, dW^T(s).
\]

Hence, if

\[
\mathbb{E}_Q^T \left[\int_0^T ||\sigma(s, T)||^2 \, ds\right] < \infty
\]

then

\[
(f(t, T))_{t \in [0, T]} \text{ is a } Q^T\text{-martingale.}
\]

Summarizing we have thus proved

**Lemma 9.2.1.** Under the above assumptions, the expectation hypothesis holds under the forward measures

\[
f(t, T) = \mathbb{E}_Q^T [r(T) \mid \mathcal{F}_t].
\]
9.3 Option Pricing in Gaussian HJM Models

We consider a European call option on an $S$-bond with expiry date $T < S$ and strike price $K$. Its price at time $t = 0$ (for simplicity only) is

$$\pi = \mathbb{E}_Q \left[ e^{-\int_0^T r(s)\, ds} \left( P(T, S) - K \right)^+ \right].$$

We proceed as in Section 7.6 and decompose

$$\pi = \mathbb{E}_Q \left[ B(T) - 1 P(T, S) 1( P(T, S) \geq K) \right] - K \mathbb{E}_Q \left[ B(T)^{-1} 1(P(T, S) \geq K) \right] = P(0, S) Q^S [P(T, S) \geq K] - K P(0, T) Q^T [P(T, S) \geq K].$$

This option pricing formula holds in general.

We already know that

$$\frac{dP(t, T)}{P(t, T)} = r(t) \, dt + v(t, T) \, dW(t)$$

and hence

$$P(t, T) = P(0, T) \exp \left[ \int_0^t v(s, T) \, dW(s) + \int_0^t \left( r(s) - \frac{1}{2} \|v(s, T)\|_2 \right) \, ds \right]$$

where

$$v(t, T) := -\int_t^T \sigma(t, u) \, du. \quad (9.7)$$

We also know that $(P(t, T)/P(t, S))_{t \in [0, T]}$ is a $Q^S$-martingale and $(P(t, S)/P(t, T))_{t \in [0, T]}$ is a $Q^T$-martingale. In fact ($\rightarrow$ exercise)

$$\frac{P(t, T)}{P(t, S)} = \frac{P(0, T)}{P(0, S)} \times \exp \left[ \int_0^t \sigma_{T, S}(s) \, dW(s) - \frac{1}{2} \int_0^t \left( \|v(s, T)\|_2^2 - \|v(s, S)\|_2^2 \right) \, ds \right]$$

$$= \frac{P(0, T)}{P(0, S)} \exp \left[ \int_0^t \sigma_{T, S}(s) \, dW_S(s) - \frac{1}{2} \int_0^t \|\sigma_{T, S}(s)\|_2^2 \, ds \right]$$

where

$$\sigma_{T, S}(s) := v(s, T) - v(s, S) = \int_T^S \sigma(s, u) \, du, \quad (9.8)$$
and
\[
\frac{P(t, S)}{P(t, T)} = \frac{P(0, S)}{P(0, T)} \times \exp \left[ -\int_0^t \sigma_{T,S}(s) dW(s) - \frac{1}{2} \int_0^t (\|v(s, S)\|^2 - \|v(s, T)\|^2) \, ds \right]
\]
\[
= \frac{P(0, S)}{P(0, T)} \exp \left[ -\int_0^t \sigma_{T,S}(s) dW^T(s) - \frac{1}{2} \int_0^t \|\sigma_{T,S}(s)\|^2 \, ds \right].
\]

Now observe that
\[
\mathbb{Q}^S[P(T, S) \geq K] = \mathbb{Q}^S \left[ \frac{P(T, T)}{P(T, S)} \leq \frac{1}{K} \right],
\]
\[
\mathbb{Q}^T[P(T, S) \geq K] = \mathbb{Q}^T \left[ \frac{P(T, S)}{P(T, T)} \geq K \right].
\]

This suggests to look at those models for which \(\sigma_{T,S}\) is deterministic, and hence \(P(T, T)\) and \(P(T, S)\) are log-normally distributed under the respective forward measures.

We thus assume that \(\sigma(t, T) = (\sigma_1(t, T), \ldots, \sigma_d(t, T))\) are deterministic functions of \(t\) and \(T\), and hence forward rates \(f(t, T)\) are Gaussian distributed.

We obtain the following closed form option price formula.

**Proposition 9.3.1.** Under the above Gaussian assumption, the option price is
\[
\pi = P(0, S)\Phi[d_1] - KP(0, T)\Phi[d_2],
\]
where
\[
d_{1,2} = \frac{\log \left[ \frac{P(0, S)}{KP(0, T)} \right] + \frac{1}{2} \int_0^T \|\sigma_{T,S}(s)\|^2 \, ds}{\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2 \, ds}},
\]
\(\sigma_{T,S}(s)\) is given in (9.8) and \(\Phi\) is the standard Gaussian CDF.

**Proof.** It is enough to observe that
\[
\frac{\log \frac{P(T,T)}{P(T,S)} - \log \frac{P(0,T)}{P(0,S)} + \frac{1}{2} \int_0^T \|\sigma_{T,S}(s)\|^2 \, ds}{\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2 \, ds}}
\]
and
\[
\frac{\log \frac{P(T,S)}{P(T,T)}}{\log \frac{P(0,S)}{P(0,T)}} + \frac{1}{2} \int_0^T \|\sigma_{T,S}(s)\|^2 \, ds
\]
\[
\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2 \, ds}
\]
are standard Gaussian distributed under \(Q^S\) and \(Q^T\), respectively. \(\square\)

Of course, the Vasicek option price formula from Section 7.6.1 can now be obtained as a corollary of Proposition 9.3.1 (→ exercise).
Chapter 10

Forwards and Futures

→ B[3](Chapter 20), or Hull (2002) [11]

We discuss two common types of term contracts: forwards, which are mainly traded OTC, and futures, which are actively traded on many exchanges.

The underlying is in both cases a $T$-claim $\mathcal{Y}$, for some fixed future date $T$. This can be an exchange rate, an interest rate, a commodity such as copper, any traded or non-traded asset, an index, etc.

10.1 Forward Contracts

A forward contract on $\mathcal{Y}$, contracted at $t$, with time of delivery $T > t$, and with the forward price $f(t; T, \mathcal{Y})$ is defined by the following payment scheme:

• at $T$, the holder of the contract (long position) pays $f(t; T, \mathcal{Y})$ and receives $\mathcal{Y}$ from the underwriter (short position);

• at $t$, the forward price is chosen such that the present value of the forward contract is zero, thus

$$\mathbb{E}_Q \left[ e^{-\int_t^T r(s) ds} (\mathcal{Y} - f(t; T, \mathcal{Y})) \left| \mathcal{F}_t \right. \right] = 0.$$ 

This is equivalent to

$$f(t; T, \mathcal{Y}) = \frac{1}{P(t, T)} \mathbb{E}_Q \left[ e^{-\int_t^T r(s) ds} \mathcal{Y} \left| \mathcal{F}_t \right. \right] = \mathbb{E}_Q^{\mathcal{F}_T} \left[ \mathcal{Y} \left| \mathcal{F}_t \right. \right].$$
Examples  The forward price at $t$ of

1. a dollar delivered at $T$ is 1;

2. an $S$-bond delivered at $T \leq S$ is $\frac{P(t,S)}{P(t,T)}$;

3. any traded asset $S$ delivered at $T$ is $\frac{S(t)}{P(t,T)}$.

The forward price $f(s;T,Y)$ has to be distinguished from the (spot) price at time $s$ of the forward contract entered at time $t \leq s$, which is

$$
\mathbb{E}_Q \left[ e^{-\int_s^T r(u) \, du} (Y - f(t;T,Y)) \mid \mathcal{F}_s \right] = \mathbb{E}_Q \left[ e^{-\int_s^T r(u) \, du} Y \mid \mathcal{F}_s \right] - P(t,T) f(t;T,Y).
$$

10.2 Futures Contracts

A futures contract on $Y$ with time of delivery $T$ is defined as follows:

- at every $t \leq T$, there is a market quoted futures price $F(t;T,Y)$, which makes the futures contract on $Y$, if entered at $t$, equal to zero;

- at $T$, the holder of the contract (long position) pays $F(T;T,Y)$ and receives $Y$ from the underwriter (short position);

- during any time interval $(s,t]$ the holder of the contract receives (or pays, if negative) the amount $F(t;T,Y) - F(s;T,Y)$ (this is called marking to market).

So there is a continuous cash-flow between the two parties of a futures contract. They are required to keep a certain amount of money as a safety margin.

The volumes in which futures are traded are huge. One of the reasons for this is that in many markets it is difficult to trade (hedge) directly in the underlying object. This might be an index which includes many different (illiquid) instruments, or a commodity such as copper, gas or electricity, etc. Holding a (short position in a) futures does not force you to physically deliver the underlying object (if you exit the contract before delivery date), and selling short makes it possible to hedge against the underlying.
10.2. FUTURES CONTRACTS

Suppose $\mathcal{Y} \in L^1(\mathbb{Q})$. Then the futures price process is given by the $\mathbb{Q}$-martingale

$$F(t; T, \mathcal{Y}) = \mathbb{E}_\mathbb{Q}[\mathcal{Y} | \mathcal{F}_t].$$

(10.1)

Often, this is just how futures prices are defined. We now give a heuristic argument for (10.1) based on the above characterization of a futures contract.

First, our model economy is driven by Brownian motion and changes in a continuous way. Hence there is no reason to believe that futures prices evolve discontinuously, and we may assume that

$$F(t) = F(t; T, \mathcal{Y})$$

is a continuous semimartingale (or Itô process).

Now suppose we enter the futures contract at time $t < T$. We face a continuum of cashflows in the interval $(t, T]$. Indeed, let $t = t_0 < \cdots < t_N = t$ be a partition of $[t, T]$. The present value of the corresponding cashflows $F(t_i) - F(t_{i-1})$ at $t_i$, $i = 1, \ldots, N$, is given by $\mathbb{E}_\mathbb{Q}[\Sigma | \mathcal{F}_t]$ where

$$\Sigma := \sum_{i=1}^N \frac{1}{B(t_i)} (F(t_i) - F(t_{i-1})).$$

But the futures contract has present value zero, hence

$$\mathbb{E}_\mathbb{Q}[\Sigma | \mathcal{F}_t] = 0.$$

This has to hold for any partition $(t_i)$. We can rewrite $\Sigma$ as

$$\sum_{i=1}^N \frac{1}{B(t_{i-1})} (F(t_i) - F(t_{i-1})) + \sum_{i=1}^N \left( \frac{1}{B(t_i)} - \frac{1}{B(t_{i-1})} \right) (F(t_i) - F(t_{i-1})).$$

If we let the partition become finer and finer this expression converges in probability towards

$$\int_t^T \frac{1}{B(s)} dF(s) + \int_t^T d \left\langle \frac{1}{B}, F \right\rangle_s = \int_t^T \frac{1}{B(s)} dF(s),$$

since the quadratic variation of $1/B$ (finite variation) and $F$ (continuous) is zero. Under the appropriate integrability assumptions (uniform integrability) we conclude that

$$\mathbb{E}_\mathbb{Q} \left[ \int_t^T \frac{1}{B(s)} dF(s) | \mathcal{F}_t \right] = 0,$$
and that
\[ M(t) = \int_0^t \frac{1}{B(s)} dF(s) = \mathbb{E}_Q \left[ \int_0^T \frac{1}{B(s)} dF(s) \mid \mathcal{F}_t \right], \quad t \in [0, T], \]
is a \( \mathbb{Q} \)-martingale. If, moreover
\[ \mathbb{E}_Q \left[ \int_0^T B(s)^2 d\langle M, M \rangle_s \right] = \mathbb{E}_Q [\langle F, F \rangle_T] < \infty \]
then
\[ F(t) = \int_0^t B(s) dM(s), \quad t \in [0, T], \]
is a \( \mathbb{Q} \)-martingale, which implies (10.1).

## 10.3 Interest Rate Futures

→ Z[28](Section 5.4)

Interest rate futures contracts may be divided into futures on short term instruments and futures on coupon bonds. We only consider an example from the first group.

Eurodollars are deposits of US dollars in institutions outside of the US. LIBOR is the interbank rate of interest for Eurodollar loans. The Eurodollar futures contract is tied to the LIBOR. It was introduced by the International Money Market (IMM) of the Chicago Mercantile Exchange (CME) in 1981, and is designed to protect its owner from fluctuations in the 3-months (=1/4 years) LIBOR. The maturity (delivery) months are March, June, September and December.

Fix a maturity date \( T \) and let \( L(T) \) denote the 3-months LIBOR for the period \( [T, T + 1/4] \), prevailing at \( T \). The market quote of the Eurodollar futures contract on \( L(T) \) at time \( t \leq T \) is
\[ 1 - L_F(t, T) \quad [100 \text{ per cent}] \]
where \( L_F(t, T) \) is the corresponding futures rate (compare with the example in Section 4.2.2). As \( t \) tends to \( T \), \( L_F(t, T) \) tends to \( L(T) \). The futures price, used for the marking to market, is defined by
\[ F(t; T, L(T)) = 1 - \frac{1}{4} L_F(t, T) \quad [\text{Mio. dollars}]. \]
Consequently, a change of 1 basis point (0.01%) in the futures rate $L_F(t, T)$ leads to a cashflow of 

$$10^6 \times 10^{-4} \times \frac{1}{4} = 25 \text{ \{dollars\}}.$$ 

We also see that the final price $F(T; T, L(T)) = 1 - \frac{1}{4}L(T) = \mathcal{Y}$ is not $P(T, T + 1/4) = 1 - \frac{1}{4}L(T)P(T, T + 1/4)$ as one might suppose. In fact, the underlying $\mathcal{Y}$ is a synthetic value. At maturity there is no physical delivery. Instead, settlement is made in cash.

On the other hand, since 

$$1 - \frac{1}{4}L_F(t, T) = F(t; T, L(T))$$

$$= \mathbb{E}_Q [F(T; T, L(T)) \mid \mathcal{F}_t] = 1 - \frac{1}{4} \mathbb{E}_Q [L(T) \mid \mathcal{F}_t],$$

we obtain an explicit formula for the futures rate 

$$L_F(t, T) = \mathbb{E}_Q [L(T) \mid \mathcal{F}_t].$$

### 10.4 Forward vs. Futures in a Gaussian Setup

Let $S$ be the price process of a traded asset. Hence the $\mathbb{Q}$-dynamics of $S$ is of the form 

$$\frac{dS(t)}{S(t)} = r(t) dt + \rho(t) dW(t),$$

for some volatility process $\rho$. Fix a delivery date $T$. The forward and futures prices of $S$ for delivery at $T$ are 

$$f(t; T, S(T)) = \frac{S(t)}{P(t, T)}, \quad F(t; T, S(T)) = \mathbb{E}_Q[S(T) \mid \mathcal{F}_t].$$

Under Gaussian assumption we can establish the relationship between the two prices.

**Proposition 10.4.1.** Suppose $\rho(t)$ and $v(t, T)$ are deterministic functions in $t$, where 

$$v(t, T) = -\int_t^T \sigma(t, u) du$$
is the volatility of the T-bond (see (9.7)). Then

\[ F(t; T, S(T)) = f(t; T, S(T)) \exp \left( \int_t^T (v(s, T) - \rho(s)) \cdot v(s, T) \, ds \right) \]

for \( t \leq T \).

Hence, if the instantaneous correlation of \( dS(t) \) and \( dP(t, T) \) is negative

\[ \frac{d(S, P(\cdot, T))_t}{dt} = S(t)P(t, T)\rho(t) \cdot v(t, T) \leq 0 \]

then the futures price dominates the forward price.

Proof. Write \( \mu(s) := v(s, T) - \rho(s) \). It is clear that

\[
\begin{align*}
    f(t; T, S(T)) &= \frac{S(0)}{P(0, T)} \exp \left( - \int_0^t \mu(s) \, dW(s) - \frac{1}{2} \int_0^t \|\mu(s)\|^2 \, ds \right) \\
    &\quad \times \exp \left( \int_0^t \mu(s) \cdot v(s, T) \, ds \right),
\end{align*}
\]

and hence

\[
\begin{align*}
    f(T; T, S(T)) &= f(t; T, S(T)) \exp \left( - \int_t^T \mu(s) \, dW(s) - \frac{1}{2} \int_t^T \|\mu(s)\|^2 \, ds \right) \\
    &\quad \times \exp \left( \int_t^T \mu(s) \cdot v(s, T) \, ds \right).
\end{align*}
\]

By assumption \( \mu(s) \) is deterministic. Consequently,

\[
\mathbb{E}_Q \left[ \exp \left( - \int_t^T \mu(s) \, dW(s) - \frac{1}{2} \int_t^T \|\mu(s)\|^2 \, ds \right) \mid \mathcal{F}_t \right] = 1
\]

and

\[
\begin{align*}
    F(t; T, S(T)) &= \mathbb{E}_Q [f(T; T, S(T)) \mid \mathcal{F}_t] \\
    &= f(t; T, S(T)) \exp \left( \int_t^T \mu(s) \cdot v(s, T) \, ds \right),
\end{align*}
\]

as desired. \qed
Similarly, one can show ($\rightarrow$ exercise)

**Lemma 10.4.2.** In a Gaussian HJM framework ($\sigma(t,T)$ deterministic) we have the following relations (convexity adjustments) between instantaneous and simple futures and forward rates

$$f(t,T) = \mathbb{E}[r(T) \mid \mathcal{F}_t] - \int_t^T \left( \sigma(s,T) \cdot \int_s^T \sigma(s,u) du \right) ds,$$

$$F(t;T,S) = \mathbb{E}[F(T,S) \mid \mathcal{F}_t]$$

$$- \frac{P(t,T)}{(S-T)P(t,S)} \left( e^{\int_t^T (\int_0^s \sigma(s,v) dv - \int_0^u \sigma(s,u) du) ds} - 1 \right)$$

for $t \leq T < S$.

Hence, if

$$\sigma(s,v) \cdot \sigma(s,u) \geq 0 \quad \text{for all } s \leq \min(u,v)$$

then futures rates are always greater than the corresponding forward rates.