Interest Rate Models

Damir Filipović
University of Munich

Contents

1 Introduction **7**

2 Interest Rates and Related Contracts **9**
- 2.1 Zero-Coupon Bonds . . . 9
- 2.2 Interest Rates . . . 11
  - 2.2.1 Market Example: LIBOR . . . 12
  - 2.2.2 Simple vs. Continuous Compounding . . . 12
  - 2.2.3 Forward vs. Future Rates . . . 13
- 2.3 Bank Account and Short Rates . . . 14
- 2.4 Coupon Bonds, Swaps and Yields . . . 15
  - 2.4.1 Fixed Coupon Bonds . . . 16
  - 2.4.2 Floating Rate Notes . . . 16
  - 2.4.3 Interest Rate Swaps . . . 17
  - 2.4.4 Yield and Duration . . . 20
- 2.5 Market Conventions . . . 22
  - 2.5.1 Day-count Conventions . . . 22
  - 2.5.2 Coupon Bonds . . . 23
  - 2.5.3 Accrued Interest, Clean Price and Dirty Price . . . 24
  - 2.5.4 Yield-to-Maturity . . . 25
- 2.6 Caps and Floors . . . 25
- 2.7 Swaptions . . . 29

3 Statistics of the Yield Curve **33**
- 3.1 Principal Component Analysis (PCA) . . . 33
- 3.2 PCA of the Yield Curve . . . 35
- 3.3 Correlation . . . 36

4 Estimating the Yield Curve **39**
4.1 A Bootstrapping Example 39
4.2 General Case 44
4.2.1 Bond Markets 45
4.2.2 Money Markets 46
4.2.3 Problems 48
4.2.4 Parametrized Curve Families 49

5 Why Yield Curve Models? **65**

6 Arbitrage Pricing **67**
6.1 Self-Financing Portfolios 67
6.2 Arbitrage and Martingale Measures 69
6.3 Hedging and Pricing 73

7 Short Rate Models **77**
7.1 Generalities 77
7.2 Diffusion Short Rate Models 79
7.2.1 Examples 82
7.3 Inverting the Yield Curve 83
7.4 Affine Term Structures 83
7.5 Some Standard Models 85
7.5.1 Vasicek Model 85
7.5.2 Cox–Ingersoll–Ross Model 86
7.5.3 Dothan Model 87
7.5.4 Ho–Lee Model 88
7.5.5 Hull–White Model 89
7.6 Option Pricing in Affine Models 90
7.6.1 Example: Vasicek Model (a, b, β const, $\alpha = 0$). 92

8 HJM Methodology **95**

9 Forward Measures **97**
9.1 T-Bond as Numeraire 97
9.2 An Expectation Hypothesis 99
9.3 Option Pricing in Gaussian HJM Models 101

10 Forwards and Futures **105**
10.1 Forward Contracts . . . 105
10.2 Futures Contracts . . . 106
10.3 Interest Rate Futures . . . 108
10.4 Forward vs. Futures in a Gaussian Setup . . . 109

11 Multi-Factor Models **113**
11.1 No-Arbitrage Condition . . . 115
11.2 Affine Term Structures . . . 117
11.3 Polynomial Term Structures . . . 118
11.4 Exponential-Polynomial Families . . . 122
11.4.1 Nelson–Siegel Family . . . 122
11.4.2 Svensson Family . . . 123

12 Market Models **127**
12.1 Models of Forward LIBOR Rates . . . 129
12.1.1 Discrete-tenor Case . . . 130
12.1.2 Continuous-tenor Case . . . 140

13 Default Risk **145**
13.1 Transition and Default Probabilities . . . 145
13.1.1 Historical Method . . . 146
13.1.2 Structural Approach . . . 148
13.2 Intensity Based Method . . . 150
13.2.1 Construction of Intensity Based Models . . . 156
13.2.2 Computation of Default Probabilities . . . 157
13.2.3 Pricing Default Risk . . . 157
13.2.4 Measure Change . . . 160

Chapter 1

Introduction

These notes have been written for a graduate course on fixed income models that I held in the fall term 2002–2003 at the Department of Operations Research and Financial Engineering at Princeton University.

The number of books on fixed income models is growing, yet it is difficult to find a convenient textbook for a one-semester course like this. There are several reasons for this:

- Until recently, many textbooks on mathematical finance have treated stochastic interest rates as an appendix to the elementary arbitrage pricing theory, which usually requires constant (zero) interest rates.
- Interest rate theory is not standardized yet: there is no well-accepted "standard" general model such as the Black–Scholes model for equities.
- The very nature of fixed income instruments causes difficulties, other than for stock derivatives, in implementing and calibrating models. These issues should therefore not been left out.

I will frequently refer to the following books:

B[3]: Björk (98) [3]. A pedagogically well written introduction to mathematical finance. Chapters 15–20 are on interest rates.

BM[6]: Brigo–Mercurio (01) [6]. This is a book on interest rate modelling written by two quantitative analysts in financial institutions. Much emphasis is on the practical implementation and calibration of selected models.

JW[12]: James–Webber (00) [12]. An encyclopedic treatment of interest rates and their related financial derivatives.

J[14]: Jarrow (96) [14]. Introduction to fixed-income securities and interest rate options. Discrete time only.

MR[20]: Musiela–Rutkowski (97) [20]. A comprehensive book on financial mathematics with a large part (Part II) on interest rate modelling. Much emphasis is on market pricing practice.

R[23]: Rebonato (98) [23]. Written by a practitionar. Much emphasis on market practice for pricing and handling interest rate derivatives.

Z[28]: Zagst (02) [28]. A comprehensive textbook on mathematical finance, interest rate modelling and risk management.

Since this text had been written, new good books on interest rates have been published. I want to mention in particular the excellent introductory textbook by Cairns (04) [7].

I did not intend to write an entire text but rather collect fragments of the material that can be found in the above books and further references.

Munich, October 2005

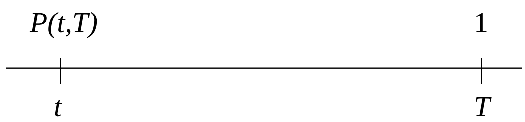
Chapter 2

Interest Rates and Related Contracts

Literature: B[3](Chapter 15), BM[6](Chapter 1), and many more

2.1 Zero-Coupon Bonds

A dollar today is worth more than a dollar tomorrow. The time t value of a dollar at time $T \geq t$ is expressed by the *zero-coupon bond* with *maturity* T, $P(t,T)$, for briefty also T-*bond*. This is a contract which guarantees the holder one dollar to be paid at the maturity date T.



→ future cashflows can be discounted, such as coupon-bearing bonds

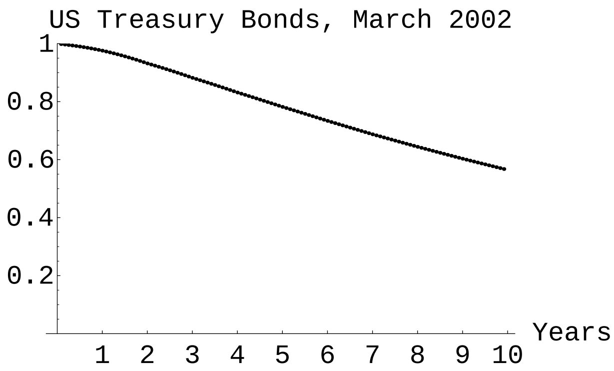
$$C_1 P(t,t_1) + \cdots + C_{n-1} P(t,t_{n-1}) + (1 + C_n) P(t,T).$$

In theory we will assume that

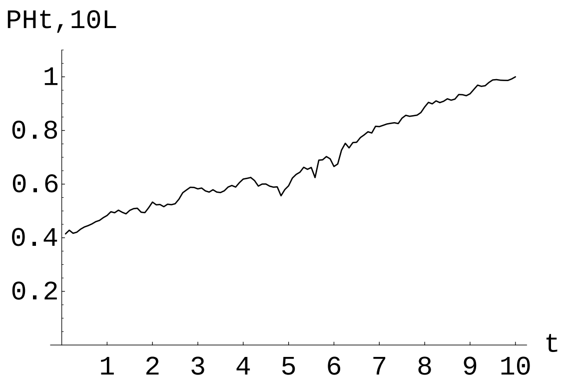
- there exists a frictionless market for T-bonds for every $T > 0$.
- $P(T,T) = 1$ for all T.
- $P(t,T)$ is continuously differentiable in T.

In reality this assumptions are not always satisfied: zero-coupon bonds are not traded for all maturities, and $P(T,T)$ might be less than one if the issuer of the T-bond defaults. Yet, this is a good starting point for doing the mathematics. More realistic models will be introduced and discussed in the sequel.

The third condition is purely technical and implies that the *term structure* of zero-coupon bond prices $T \mapsto P(t,T)$ is a smooth curve.



Note that $t \mapsto P(t,T)$ is a stochastic process since bond prices $P(t,T)$ are not known with certainty before t.



A reasonable assumption would also be that $T \mapsto P(t,T) \leq 1$ is a decreasing curve (which is equivalent to positivity of interest rates). However, already classical interest rate models imply zero-coupon bond prices greater than 1. Therefore we leave away this requirement.

2.2 Interest Rates

The term structure of zero-coupon bond prices does not contain much visual information (strictly speaking it does). A better measure is given by the implied interest rates. There is a variety of them.

A prototypical *forward rate agreement (FRA)* is a contract involving three time instants $t < T < S$: the current time t, the expiry time $T > t$, and the maturity time $S > T$.

- At t: sell one T-bond and buy $\frac{P(t,T)}{P(t,S)}$ S-bonds = zero net investment.
- At T: pay one dollar.
- At S: obtain $\frac{P(t,T)}{P(t,S)}$ dollars.

The net effect is a forward investment of one dollar at time T yielding $\frac{P(t,T)}{P(t,S)}$ dollars at S with certainty.

We are led to the following definitions.

- The *simple (simply-compounded) forward rate* for $[T, S]$ prevailing at t is given by

$$1+(S-T)F(t;T,S) := \frac{P(t,T)}{P(t,S)} \Leftrightarrow F(t;T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

- The *simple spot rate* for $[t, T]$ is

$$F(t,T) := F(t;t,T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right).$$

- The *continuously compounded forward rate* for $[T, S]$ prevailing at t is given by

$$e^{R(t;T,S)(S-T)} := \frac{P(t,T)}{P(t,S)} \Leftrightarrow R(t;T,S) = -\frac{\log P(t,S) - \log P(t,T)}{S-T}.$$

- The *continuously compounded spot rate* for $[T, S]$ is

$$R(t,T) := R(t;t,T) = -\frac{\log P(t,T)}{T-t}.$$

- The *instantaneous forward rate* with maturity T prevailing at time t is defined by

$$f(t,T) := \lim_{S \downarrow T} R(t;T,S) = -\frac{\partial \log P(t,T)}{\partial T}. \tag{2.1}$$

  The function $T \mapsto f(t,T)$ is called the *forward curve* at time t.

- The *instantaneous short rate* at time t is defined by

$$r(t) := f(t,t) = \lim_{T \downarrow t} R(t,T).$$

Notice that (2.1) together with the requirement $P(T,T) = 1$ is equivalent to

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

2.2.1 Market Example: LIBOR

"Interbank rates" are rates at which deposits between banks are exchanged, and at which swap transactions (see below) between banks occur. The most important interbank rate usually considered as a reference for fixed income contracts is the *LIBOR (London InterBank Offered Rate)*[1] for a series of possible maturities, ranging from *overnight* to 12 months. These rates are quoted on a simple compounding basis. For example, the three-months forward LIBOR for the period $[T, T+1/4]$ at time t is given by

$$L(t,T) = F(t;T,T+1/4).$$

2.2.2 Simple vs. Continuous Compounding

One dollar invested for one year at an interest rate of R per annum growths to $1+R$. If the rate is compounded twice per year the terminal value is $(1+R/2)^2$, etc. It is a mathematical fact that

$$\left(1+\frac{R}{m}\right)^m \to e^R \quad \text{as } m \to \infty.$$

[1] To be more precise: this is the rate at which high-credit financial institutions can *borrow* in the interbank market.

Moreover,

$$e^R = 1 + R + o(R) \quad \text{for } R \text{ small.}$$

Example: $e^{0.04} = 1.04081$.

Since the exponential function has nicer analytic properties than power functions, we often consider continuously compounded interest rates. This makes the theory more tractable.

2.2.3 Forward vs. Future Rates

Can forward rates predict the future spot rates?

Consider a deterministic world. If markets are efficient (i.e. no arbitrage = no riskless, systematic profit) we have necessarily

$$P(t, S) = P(t, T)P(T, S), \quad \forall t \leq T \leq S. \tag{2.2}$$

Proof. Suppose that $P(t, S) > P(t, T)P(T, S)$ for some $t \leq T \leq S$. Then we follow the strategy:

- At t: sell one S-bond, and buy $P(T, S)$ T-bonds.

  Net cost: $-P(t, S) + P(t, T)P(T, S) < 0$.

- At T: receive $P(T, S)$ dollars and buy one S-bond.

- At S: pay one dollar, receive one dollar.

(Where do we use the assumption of a deterministic world?)

The net is a riskless gain of $-P(t, S) + P(t, T)P(T, S)$ $(\times 1/P(t, S))$. This is a pure arbitrage opportunity, which contradicts the assumption.

If $P(t, S) < P(t, T)P(T, S)$ the same profit can be realized by changing sign in the strategy. □

Taking logarithm in (2.2) yields

$$\int_T^S f(t, u)\, du = \int_T^S f(T, u)\, du, \quad \forall t \leq T \leq S.$$

This is equivalent to

$$f(t, S) = f(T, S) = r(S), \quad \forall t \leq T \leq S$$

(as time goes by we walk along the forward curve: the forward curve is shifted). In this case, the forward rate with maturity S prevailing at time $t \leq S$ is exactly the future short rate at S.

The real world is not deterministic though. We will see that in general the forward rate $f(t, T)$ is the conditional expectation of the short rate $r(T)$ under a particular probability measure (forward measure), depending on T.

Hence the forward rate is a biased estimator for the future short rate. Forecasts of future short rates by forward rates have little or no predictive power.

2.3 Bank Account and Short Rates

The return of a one dollar investment today ($t = 0$) over the period $[0, \Delta t]$ is given by

$$\frac{1}{P(0, \Delta t)} = \exp\left(\int_0^{\Delta t} f(0, u)\, du\right) = 1 + r(0)\Delta t + o(\Delta t).$$

Instantaneous reinvestment in $2\Delta t$-bonds yields

$$\frac{1}{P(0, \Delta t)} \frac{1}{P(\Delta t, 2\Delta t)} = (1 + r(0)\Delta t)(1 + r(\Delta t)\Delta t) + o(\Delta t)$$

at time $2\Delta t$, etc. This strategy of "rolling over"[2] just maturing bonds leads in the limit to the *bank account (money-market account)* $B(t)$. Hence $B(t)$ is the asset which growths at time t instantaneously at short rate $r(t)$

$$B(t + \Delta t) = B(t)(1 + r(t)\Delta t) + o(\Delta t).$$

For $\Delta t \to 0$ this converges to

$$dB(t) = r(t)B(t)dt$$

and with $B(0) = 1$ we obtain

$$B(t) = \exp\left(\int_0^t r(s)\, ds\right).$$

[2]This limiting process is made rigorous in [4].

B is a risk-free asset insofar as its future value at time $t + \Delta t$ is known (up to order Δt) at time t. In stochastic terms we speak of a predictable process. For the same reason we speak of $r(t)$ as the *risk-free rate of return* over the infinitesimal period $[t, t + dt]$.

B is important for relating amounts of currencies available at different times: in order to have one dollar in the bank account at time T we need to have

$$\frac{B(t)}{B(T)} = \exp\left(-\int_t^T r(s)\,ds\right)$$

dollars in the bank account at time $t \leq T$. This *discount factor* is stochastic: it is not known with certainty at time t. There is a close connection to the deterministic (=known at time t) discount factor given by $P(t, T)$. Indeed, we will see that the latter is the conditional expectation of the former under the risk neutral probability measure.

Proxies for the Short Rate

→ JW[12](Chapter 3.5)

The short rate $r(t)$ is a key interest rate in all models and fundamental to no-arbitrage pricing. But it cannot be directly observed.

The overnight interest rate is not usually considered to be a good proxy for the short rate, because the motives and needs driving overnight borrowers are very different from those of borrowers who want money for a month or more.

The overnight fed funds rate is nevertheless comparatively stable and perhaps a fair proxy, but empirical studies suggest that it has low correlation with other spot rates.

The best available proxy is given by one- or three-month spot rates since they are very liquid.

2.4 Coupon Bonds, Swaps and Yields

In most bond markets, there is only a relatively small number of zero-coupon bonds traded. Most bonds include coupons.

2.4.1 Fixed Coupon Bonds

A *fixed coupon bond* is a contract specified by

- a number of future dates $T_1 < \cdots < T_n$ (the *coupon dates*) (T_n is the *maturity* of the bond),
- a sequence of (deterministic) coupons $c_1, \ldots, c_n$,
- a nominal value N,

such that the owner receives c_i at time T_i, for $i = 1, \ldots, n$, and N at terminal time T_n. The price $p(t)$ at time $t \leq T_1$ of this coupon bond is given by the sum of discounted cashflows

$$p(t) = \sum_{i=1}^{n} P(t, T_i) c_i + P(t, T_n) N.$$

Typically, it holds that $T_{i+1} - T_i \equiv \delta$, and the coupons are given as a fixed percentage of the nominal value: $c_i \equiv K\delta N$, for some fixed interest rate K. The above formula reduces to

$$p(t) = \left(K\delta \sum_{i=1}^{n} P(t, T_i) + P(t, T_n) \right) N.$$

2.4.2 Floating Rate Notes

There are versions of coupon bonds for which the value of the coupon is not fixed at the time the bond is issued, but rather reset for every coupon period. Most often the resetting is determined by some market interest rate (e.g. LIBOR).

A *floating rate note* is specified by

- a number of future dates $T_0 < T_1 < \cdots < T_n$,
- a nominal value N.

The deterministic coupon payments for the fixed coupon bond are now replaced by

$$c_i = (T_i - T_{i-1}) F(T_{i-1}, T_i) N,$$

where $F(T_{i-1}, T_i)$ is the prevailing simple market interest rate, and we note that $F(T_{i-1}, T_i)$ is determined already at time T_{i-1} (this is why here we have T_0 in addition to the coupon dates $T_1, \ldots, T_n$), but that the cash-flow c_i is at time T_i.

The value $p(t)$ of this note at time $t \le T_0$ is obtained as follows. Without loss of generality we set $N = 1$. By definition of $F(T_{i-1}, T_i)$ we then have

$$c_i = \frac{1}{P(T_{i-1}, T_i)} - 1.$$

The time t value of -1 paid out at T_i is $-P(t, T_i)$. The time t value of $\frac{1}{P(T_{i-1},T_i)}$ paid out at T_i is $P(t, T_{i-1})$:

- At t: buy a T_{i-1}-bond. Cost: $P(t, T_{i-1})$.
- At T_{i-1}: receive one dollar and buy $1/P(T_{i-1}, T_i)$ T_i-bonds. Zero net investment.
- At T_i: receive $1/P(T_{i-1}, T_i)$ dollars.

The time t value of c_i therefore is

$$P(t, T_{i-1}) - P(t, T_i).$$

Summing up we obtain the (surprisingly easy) formula

$$p(t) = P(t, T_n) + \sum_{i=1}^{n} (P(t, T_{i-1}) - P(t, T_i)) = P(t, T_0).$$

In particular, for $t = T_0$: $p(T_0) = 1$.

2.4.3 Interest Rate Swaps

An interest rate swap is a scheme where you exchange a payment stream at a *fixed* rate of interest for a payment stream at a *floating* rate (typically LIBOR).

There are many versions of interest rate swaps. A *payer interest rate swap* settled in arrears is specified by

- a number of future dates $T_0 < T_1 < \cdots < T_n$ with $T_i - T_{i-1} \equiv \delta$

  (T_n is the *maturity* of the swap),

- a fixed rate K,
- a nominal value N.

Of course, the equidistance hypothesis is only for convenience of notation and can easily be relaxed. Cashflows take place only at the coupon dates $T_1, \dots, T_n$. At T_i, the holder of the contract

- pays fixed $K\delta N$,
- and receives floating $F(T_{i-1}, T_i)\delta N$.

The net cashflow at T_i is thus

$$(F(T_{i-1}, T_i) - K)\delta N,$$

and using the previous results we can compute the value at $t \leq T_0$ of this cashflow as

$$N(P(t, T_{i-1}) - P(t, T_i) - K\delta P(t, T_i)). \tag{2.3}$$

The total value $\Pi_p(t)$ of the swap at time $t \leq T_0$ is thus

$$\Pi_p(t) = N\left(P(t, T_0) - P(t, T_n) - K\delta \sum_{i=1}^{n} P(t, T_i)\right).$$

A *receiver interest rate swap* settled in arrears is obtained by changing the sign of the cashflows at times $T_1, \dots, T_n$. Its value at time $t \leq T_0$ is thus

$$\Pi_r(t) = -\Pi_p(t).$$

The remaining question is how the "fair" fixed rate K is determined. The *forward swap rate* $R_{swap}(t)$ at time $t \leq T_0$ is the fixed rate K above which gives $\Pi_p(t) = \Pi_r(t) = 0$. Hence

$$R_{swap}(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}.$$

The following alternative representation of $R_{swap}(t)$ is sometimes useful. Since $P(t, T_{i-1}) - P(t, T_i) = F(t; T_{i-1}, T_i)\delta P(t, T_i)$, we can rewrite (2.3) as

$$N\delta P(t, T_i)\left(F(t; T_{i-1}, T_i) - K\right).$$

Summing up yields

$$\Pi_p(t) = N\delta \sum_{i=1}^{n} P(t,T_i)\left(F(t;T_{i-1},T_i) - K\right),$$

and thus we can write the swap rate as weighted average of simple forward rates

$$R_{swap}(t) = \sum_{i=1}^{n} w_i(t) F(t;T_{i-1},T_i),$$

with weights

$$w_i(t) = \frac{P(t,T_i)}{\sum_{j=1}^{n} P(t,T_j)}.$$

These weights are random, but there seems to be empirical evidence that the variability of $w_i(t)$ is small compared to that of $F(t;T_{i-1},T_i)$. This is used for approximations of swaption (see below) price formulas in LIBOR market models: the swap rate volatility is written as linear combination of the forward LIBOR volatilities ("Rebonato's formula" → BM[6], p.248).

Swaps were developed because different companies could borrow at different rates in different markets.

Example

→ JW[12](p.11)

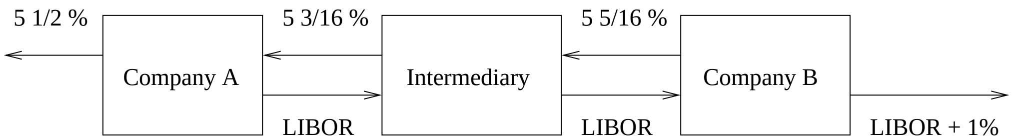
- Company A: is borrowing fixed for five years at 5 1/2%, but could borrow floating at LIBOR plus 1/2%.
- Company B: is borrowing floating at LIBOR plus 1%, but could borrow fixed for five years at 6 1/2%.

By agreeing to swap streams of cashflows both companies could be better off, and a mediating institution would also make money.

- Company A pays LIBOR to the intermediary in exchange for fixed at 5 3/16% (receiver swap).
- Company B pays the intermediary fixed at 5 5/16% in exchange for LIBOR (payer swap).

Net:

- Company A is now paying LIBOR plus 5/16% instead of LIBOR plus 1/2%.
- Company B is paying fixed at 6 5/16% instead of 6 1/2%.
- The intermediary receives fixed at 1/8%.



Everyone seems to be better off. But there is implicit credit risk; this is why Company B had higher borrowing rates in the first place. This risk has been partly taken up by the intermediary, in return for the money it makes on the spread.

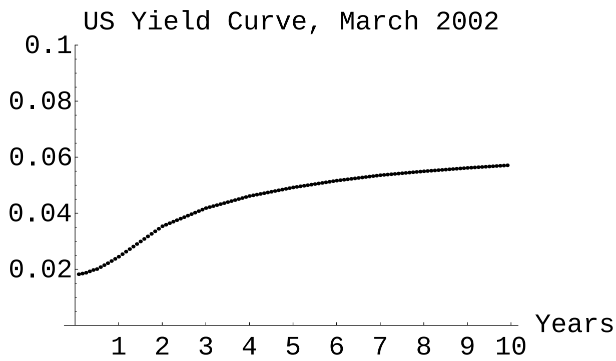
2.4.4 Yield and Duration

For a zero-coupon bond $P(t,T)$ the *zero-coupon yield* is simply the continuously compounded spot rate $R(t,T)$. That is,

$$P(t,T) = e^{-R(t,T)(T-t)}.$$

Accordingly, the function $T \mapsto R(t,T)$ is referred to as *(zero-coupon) yield curve*.

The term "yield curve" is ambiguous. There is a variety of other terminologies, such as zero-rate curve (Z[28]), zero-coupon curve (BM[6]). In JW[12] the yield curve is is given by simple spot rates, and in BM[6] it is a combination of simple spot rates (for maturities up to 1 year) and annually compounded spot rates (for maturities greater than 1 year), etc.



Now let $p(t)$ be the time t market value of a fixed coupon bond with coupon dates $T_1 < \cdots < T_n$, coupon payments $c_1, \ldots, c_n$ and nominal value N (see Section 2.4.1). For simplicity we suppose that c_n already contains N, that is,

$$p(t) = \sum_{i=1}^{n} P(t, T_i) c_i, \quad t \le T_1.$$

Again we ask for the bond's "internal rate of interest"; that is, the constant (over the period $[t, T_n]$) continuously compounded rate which generates the market value of the coupon bond: the *(continuously compounded) yield-to-maturity* $y(t)$ of this bond at time $t \le T_1$ is defined as the unique solution to

$$p(t) = \sum_{i=1}^{n} c_i e^{-y(t)(T_i - t)}.$$

Remark 2.4.1. → *R[23](p.21). It is argued by Schaefer (1977) that the yield-to-maturity is an inadequate statistics for the bond market:*

- *coupon payments occurring at the* same point in time *are discounted by* different discount factors*, but*
- *coupon payments at* different points in time *from the same bond are* discounted by the same rate*.*

To simplify the notation we assume now that $t = 0$, and write $p = p(0)$, $y = y(0)$, etc. The *Macaulay duration* of the coupon bond is defined as

$$D_{Mac} := \frac{\sum_{i=1}^{n} T_i c_i e^{-y T_i}}{p}.$$

The duration is thus a weighted average of the coupon dates $T_1, \ldots, T_n$, and it provides us in a certain sense with the "mean time to coupon payment". As such it is an important concept for interest rate risk management: it acts as a measure of the first order sensitivity of the bond price w.r.t. changes in the yield-to-maturity (see Z[28](Chapter 6.1.3) for a thourough treatment). This is shown by the obvious formula

$$\frac{dp}{dy} = \frac{d}{dy}\left(\sum_{i=1}^{n} c_i e^{-yT_i}\right) = -D_{Mac}p.$$

A first order sensitivity measure of the bond price w.r.t. *parallel shifts* of the entire zero-coupon yield curve $T \mapsto R(0,T)$ is given by the *duration* of the bond

$$D := \frac{\sum_{i=1}^{n} T_i c_i e^{-y_i T_i}}{p} = \sum_{i=1}^{n} \frac{c_i P(0,T_i)}{p} T_i,$$

with $y_i := R(0,T_i)$. In fact, we have

$$\frac{d}{ds}\left(\sum_{i=1}^{n} c_i e^{-(y_i+s)T_i}\right)|_{s=0} = -Dp.$$

Hence duration is essentially for bonds (w.r.t. parallel shift of the yield curve) what delta is for stock options. The bond equivalent of the gamma is *convexity*:

$$C := \frac{d^2}{ds^2}\left(\sum_{i=1}^{n} c_i e^{-(y_i+s)T_i}\right)|_{s=0} = \sum_{i=1}^{n} c_i e^{-y_i T_i}(T_i)^2.$$

2.5 Market Conventions

2.5.1 Day-count Conventions

Time is measured in years.

If t and T denote two dates expressed as day/month/year, it is not clear what $T - t$ should be. The market evaluates the year fraction between t and T in different ways.

The *day-count convention* decides upon the time measurement between two dates t and T.

Here are three examples of day-count conventions:

- Actual/365: a year has 365 days, and the day-count convention for $T - t$ is given by

$$\frac{\text{actual number of days between } t \text{ and } T}{365}.$$

- Actual/360: as above but the year counts 360 days.

- 30/360: months count 30 and years 360 days. Let $t = (d_1, m_1, y_1)$ and $T = (d_2, m_2, y_2)$. The day-count convention for $T - t$ is given by

$$\frac{\min(d_2, 30) + (30 - d_1)^+}{360} + \frac{(m_2 - m_1 - 1)^+}{12} + y_2 - y_1.$$

  Example: The time between t=January 4, 2000 and T=July 4, 2002 is given by

$$\frac{4 + (30 - 4)}{360} + \frac{7 - 1 - 1}{12} + 2002 - 2000 = 2.5.$$

When extracting information on interest rates from data, it is important to realize for which day-count convention a specific interest rate is quoted.

→ BM[6](p.4), Z[28](Sect. 5.1)

2.5.2 Coupon Bonds

→ MR[20](Sect. 11.2), Z[28](Sect. 5.2), J[14](Chapter 2)

Coupon bonds issued in the American (European) markets typically have semi-annual (annual) coupon payments.

Debt securities issued by the U.S. Treasury are divided into three classes:

- *Bills:* zero-coupon bonds with time to maturity less than one year.

- *Notes:* coupon bonds (semi-annual) with time to maturity between 2 and 10 years.

- *Bonds:* coupon bonds (semi-annual) with time to maturity between 10 and 30 years[3].

[3]Recently, the issuance of 30 year treasury bonds has been stopped.

In addition to bills, notes and bonds, Treasury securities called *STRIPS* (separate trading of registered interest and principal of securities) have traded since August 1985. These are the coupons or principal (=nominal) amounts of Treasury bonds trading separately through the Federal Reserve's book-entry system. They are *synthetically* created zero-coupon bonds of longer maturities than a year. They were created in response to investor demands.

2.5.3 Accrued Interest, Clean Price and Dirty Price

Remember that we had for the price of a coupon bond with coupon dates $T_1, \ldots, T_n$ and payments $c_1, \ldots, c_n$ the price formula

$$p(t) = \sum_{i=1}^{n} c_i P(t, T_i), \quad t \leq T_1.$$

For $t \in (T_1, T_2]$ we have

$$p(t) = \sum_{i=2}^{n} c_i P(t, T_i),$$

etc. Hence there are systematic discontinuities of the price trajectory at $t = T_1, \ldots, T_n$ which is due to the coupon payments. This is why prices are differently quoted at the exchange.

The *accrued interest* at time $t \in (T_{i-1}, T_i]$ is defined by

$$AI(i;t) := c_i \frac{t - T_{i-1}}{T_i - T_{i-1}}$$

(where now time differences are taken according to the day-count convention). The quoted price, or *clean price*, of the coupon bond at time t is

$$p_{clean}(t) := p(t) - AI(i;t), \quad t \in (T_{i-1}, T_i].$$

That is, whenever we buy a coupon bond quoted at a clean price of $p_{clean}(t)$ at time $t \in (T_{i-1}, T_i]$, the cash price, or *dirty price*, we have to pay is

$$p(t) = p_{clean}(t) + AI(i;t).$$

2.5.4 Yield-to-Maturity

The quoted *(annual) yield-to-maturity* $\hat{y}(t)$ on a Treasury bond at time $t = T_i$ is defined by the relationship

$$p_{clean}(T_i) = \sum_{j=i+1}^{n} \frac{r_c N/2}{(1+\hat{y}(T_i)/2)^{j-i}} + \frac{N}{(1+\hat{y}(T_i)/2)^{n-i}},$$

and at $t \in [T_i, T_{i+1})$

$$p_{clean}(t) = \sum_{j=i+1}^{n} \frac{r_c N/2}{(1+\hat{y}(t)/2)^{j-i-1+\tau}} + \frac{N}{(1+\hat{y}(t)/2)^{n-i-1+\tau}},$$

where r_c is the (annualized) coupon rate, N the nominal amount and

$$\tau = \frac{T_{i+1} - t}{T_{i+1} - T_i}$$

is again given by the day-count convention, and we assume here that

$$T_{i+1} - T_i \equiv 1/2 \quad \text{(semi-annual coupons)}.$$

2.6 Caps and Floors

→ BM[6](Sect. 1.6), Z[28](Sect. 5.6.2)

Caps

A *caplet* with reset date T and settlement date $T + \delta$ pays the holder the difference between a simple market rate $F(T, T+\delta)$ (e.g. LIBOR) and the strike rate κ. Its cashflow at time $T + \delta$ is

$$\delta(F(T, T+\delta) - \kappa)^+.$$

A *cap* is a strip of caplets. It thus consists of

- a number of future dates $T_0 < T_1 < \cdots < T_n$ with $T_i - T_{i-1} \equiv \delta$ (T_n is the *maturity* of the cap),
- a *cap rate* κ.

Cashflows take place at the dates $T_1, \ldots, T_n$. At T_i the holder of the cap receives

$$\delta(F(T_{i-1}, T_i) - \kappa)^+. \tag{2.4}$$

Let $t \leq T_0$. We write

$$Cpl(i;t), \quad i = 1, \ldots, n,$$

for the time t price of the ith caplet with reset date T_{i-1} and settlement date T_i, and

$$Cp(t) = \sum_{i=1}^{n} Cpl(i;t)$$

for the time t price of the cap.

A cap gives the holder a protection against rising interest rates. It guarantees that the interest to be paid on a floating rate loan never exceeds the predetermined cap rate κ.

It can be shown ($\rightarrow$ exercise) that the cashflow (2.4) at time T_i is the equivalent to $(1 + \delta\kappa)$ times the cashflow at date T_{i-1} of a put option on a T_i-bond with strike price $1/(1 + \delta\kappa)$ and maturity T_{i-1}, that is,

$$(1 + \delta\kappa) \left(\frac{1}{1 + \delta\kappa} - P(T_{i-1}, T_i) \right)^+ .$$

This is an important fact because many interest rate models have explicit formulae for bond option values, which means that caps can be priced very easily in those models.

Floors

A *floor* is the converse to a cap. It protects against low rates. A floor is a strip of *floorlets*, the cashflow of which is – with the same notation as above – at time T_i

$$\delta(\kappa - F(T_{i-1}, T_i))^+.$$

Write $Fll(i;t)$ for the price of the ith floorlet and

$$Fl(t) = \sum_{i=1}^{n} Fll(i;t)$$

for the price of the floor.

Caps, Floors and Swaps

Caps and floors are strongly related to swaps. Indeed, one can show the parity relation (→ exercise)

$$Cp(t) - Fl(t) = \Pi_p(t),$$

where $\Pi_p(t)$ is the value at t of a payer swap with rate κ, nominal one and the same tenor structure as the cap and floor.

Let $t = 0$. The cap/floor is said to be *at-the-money (ATM)* if

$$\kappa = R_{swap}(0) = \frac{P(0,T_0) - P(0,T_n)}{\delta \sum_{i=1}^n P(0,T_i)},$$

the forward swap rate. The cap (floor) is *in-the-money (ITM)* if $\kappa < R_{swap}(0)$ ($\kappa > R_{swap}(0)$), and *out-of-the-money (OTM)* if $\kappa > R_{swap}(0)$ ($\kappa < R_{swap}(0)$).

Black's Formula

It is market practice to price a cap/floor according to *Black's formula*. Let $t \le T_0$. Black's formula for the value of the ith caplet is

$$Cpl(i;t) = \delta P(t,T_i) \left(F(t;T_{i-1},T_i)\Phi(d_1(i;t)) - \kappa\Phi(d_2(i;t)) \right),$$

where

$$d_{1,2}(i;t) := \frac{\log\left(\frac{F(t;T_{i-1},T_i)}{\kappa}\right) \pm \frac{1}{2}\sigma(t)^2(T_{i-1}-t)}{\sigma(t)\sqrt{T_{i-1}-t}}$$

(Φ stands for the standard Gaussian cumulative distribution function), and $\sigma(t)$ is the *cap volatility* (it is the same for all caplets).

Correspondingly, Black's formula for the value of the ith floorlet is

$$Fll(i;t) = \delta P(t,T_i) \left(\kappa\Phi(-d_2(i;t)) - F(t;T_{i-1},T_i)\Phi(-d_1(i;t)) \right).$$

Cap/floor prices are quoted in the market in term of their implied volatilities. Typically, we have $t = 0$, and T_0 and $\delta = T_i - T_{i-1}$ being equal to three months.

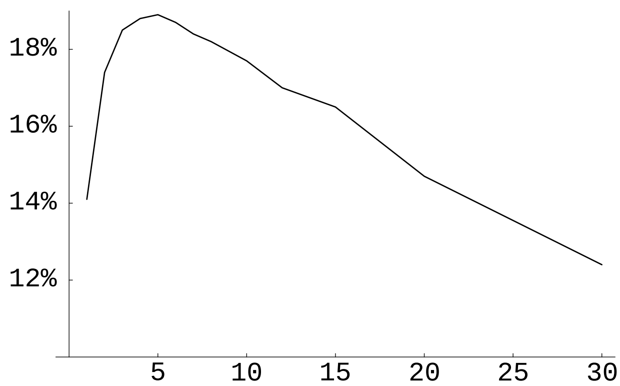
An example of a US dollar ATM market cap volatility curve is shown in Table 2.1 and Figure 2.1 (→ JW[12](p.49)).

It is a challenge for any market realistic interest rate model to match the given volatility curve.

Table 2.1: US dollar ATM cap volatilities, 23 July 1999

Maturity (in years)	ATM vols (in %)
1	14.1
2	17.4
3	18.5
4	18.8
5	18.9
6	18.7
7	18.4
8	18.2
10	17.7
12	17.0
15	16.5
20	14.7
30	12.4

Figure 2.1: US dollar ATM cap volatilities, 23 July 1999



2.7 Swaptions

A European *payer (receiver) swaption* with *strike rate* K is an option giving the right to enter a payer (receiver) swap with fixed rate K at a given future date, the *swaption maturity*. Usually, the swaption maturity coincides with the first reset date of the underlying swap. The underlying swap lenght $T_n - T_0$ is called the *tenor* of the swaption.

Recall that the value of a payer swap with fixed rate K at its first reset date, T_0, is

$$\Pi_p(T_0, K) = N \sum_{i=1}^{n} P(T_0, T_i)\delta(F(T_0; T_{i-1}, T_i) - K).$$

Hence the payoff of the swaption with strike rate K at maturity T_0 is

$$N \left(\sum_{i=1}^{n} P(T_0, T_i)\delta(F(T_0; T_{i-1}, T_i) - K) \right)^+ . \tag{2.5}$$

Notice that, contrary to the cap case, this payoff cannot be decomposed into more elementary payoffs. This is a fundamental difference between caps/floors and swaptions. Here the correlation between different forward rates will enter the valuation procedure.

Since $\Pi_p(T_0, R_{swap}(T_0)) = 0$, one can show ($\to$ exercise) that the payoff (2.5) of the payer swaption at time T_0 can also be written as

$$N\delta(R_{swap}(T_0) - K)^+ \sum_{i=1}^{n} P(T_0, T_i),$$

and for the receiver swaption

$$N\delta(K - R_{swap}(T_0))^+ \sum_{i=1}^{n} P(T_0, T_i).$$

Accordingly, at time $t \leq T_0$, the payer (receiver) swaption with strike rate K is said to be *ATM*, *ITM*, *OTM*, if

$$K = R_{swap}(t), \quad K < (>) R_{swap}(t), \quad K > (<) R_{swap}(t),$$

respectively.

Black's Formula

Black's formula for the price at time $t \leq T_0$ of the payer ($Swpt_p(t)$) and receiver ($Swpt_r(t)$) swaption is

$$Swpt_p(t) = N\delta\left(R_{swap}(t)\Phi(d_1(t)) - K\Phi(d_2(t))\right)\sum_{i=1}^{n} P(t,T_i),$$

$$Swpt_r(t) = N\delta\left(K\Phi(-d_2(t)) - R_{swap}(t)\Phi(-d_1(t))\right)\sum_{i=1}^{n} P(t,T_i),$$

with

$$d_{1,2}(t) := \frac{\log\left(\frac{R_{swap}(t)}{K}\right) \pm \frac{1}{2}\sigma(t)^2(T_0 - t)}{\sigma(t)\sqrt{T_0 - t}},$$

and $\sigma(t)$ is the prevailing Black's swaption volatility.

Swaption prices are quoted in terms of implied volatilities in matrix form. An $x \times y$-swaption is the swaption with maturity in x years and whose underlying swap is y years long.

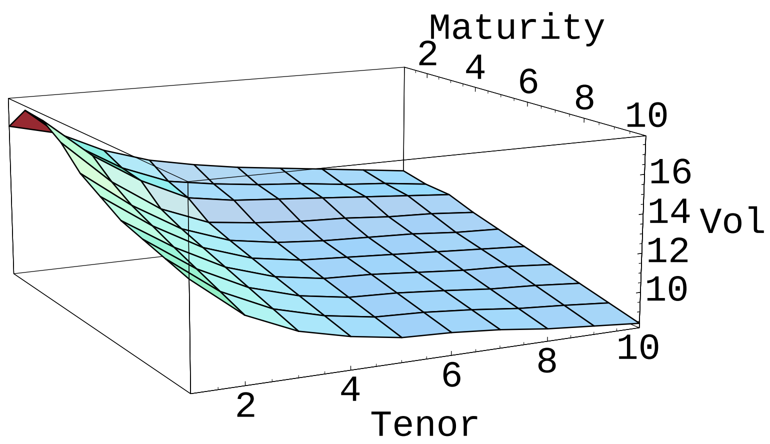
A typical example of implied swaption volatilities is shown in Table 2.2 and Figure 2.2 (→ BM[6](p.253)).

An interest model for swaptions valuation must fit the given today's volatility surface.

Table 2.2: Black's implied volatilities (in %) of ATM swaptions on May 16, 2000. Maturities are 1,2,3,4,5,7,10 years, swaps lengths from 1 to 10 years.

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	16.4	15.8	14.6	13.8	13.3	12.9	12.6	12.3	12.0	11.7
2y	17.7	15.6	14.1	13.1	12.7	12.4	12.2	11.9	11.7	11.4
3y	17.6	15.5	13.9	12.7	12.3	12.1	11.9	11.7	11.5	11.3
4y	16.9	14.6	12.9	11.9	11.6	11.4	11.3	11.1	11.0	10.8
5y	15.8	13.9	12.4	11.5	11.1	10.9	10.8	10.7	10.5	10.4
7y	14.5	12.9	11.6	10.8	10.4	10.3	10.1	9.9	9.8	9.6
10y	13.5	11.5	10.4	9.8	9.4	9.3	9.1	8.8	8.6	8.4

Figure 2.2: Black's implied volatilities (in %) of ATM swaptions on May 16, 2000.



Chapter 3

Some Statistics of the Yield Curve

3.1 Principal Component Analysis (PCA)

$\rightarrow$ JW[12](Chapter 16.2), [22]

- Let $x(1), \ldots, x(N)$ be a sample of a random $n \times 1$ vector x.
- Form the empirical $n \times n$ covariance matrix $\hat{\Sigma}$,

$$\hat{\Sigma}_{ij} = \frac{\sum_{k=1}^{N}(x_i(k) - \mu[x_i])(x_j(k) - \mu[x_j])}{N-1}$$
$$= \frac{\sum_{k=1}^{N} x_i(k)x_j(k) - N\mu[x_i]\mu[x_j]}{N-1},$$

  where

$$\mu[x_i] := \frac{1}{N}\sum_{k=1}^{N} x_i(k) \quad \text{(mean of } x_i\text{)}.$$

  We assume that $\hat{\Sigma}$ is non-degenerate (otherwise we can express an x_i as linear combination of the other x_js).

- There exists a unique orthogonal matrix $A = (p_1, \ldots, p_n)$ (that is, $A^{-1} = A^T$ and $A_{ij} = p_{j;i}$) consisting of orthonormal $n \times 1$ Eigenvectors p_i of $\hat{\Sigma}$ such that

$$\hat{\Sigma} = ALA^T,$$

where $L = diag(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$ (the Eigenvalues of $\hat{\Sigma}$).

- Define $z := A^T x$. Then

$$Cov[z_i, z_j] = \sum_{k,l=1}^{n} A_{ik}^T Cov[x_k, x_l] A_{jl}^T = \left(A^T \hat{\Sigma} A\right)_{ij} = \lambda_i \delta_{ij}.$$

  Hence the z_is are uncorrelated.

- The *principal components (PCs)* are the $n \times 1$ vectors $p_1, \dots, p_n$:

$$x = Az = z_1 p_1 + \cdots z_n p_n.$$

  The importance of component p_i is determined by the size of the corresponding Eigenvalue, λ_i, which indicates the amount of variance explained by p_i. The key statistics is the proportion

$$\frac{\lambda_i}{\sum_{j=1}^n \lambda_j},$$

  the *explained variance* by p_i.

- Normalization: let $\tilde{w} := (L^{1/2})^{-1} z$, where $L^{1/2} := diag(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$, and $w = \tilde{w} - \mu[\tilde{w}]$ ($\mu[\tilde{w}]$=mean of $\tilde{w}$). Then

$$\mu[w] = 0, \qquad Cov[w_i, w_j] = Cov[\tilde{w}_i, \tilde{w}_j] = \delta_{ij},$$

  and

$$x = \mu[x] + AL^{1/2} w = \mu[x] + \sum_{j=1}^{n} p_j \sqrt{\lambda_j} w_j.$$

  In components

$$x_i = \mu[x_i] + \sum_{j=1}^{n} A_{ij} \sqrt{\lambda_j} w_j.$$

- Sometimes the following view is useful (→ R[23](Chapter 3)): set

$$\sigma_i := Var[x_i]^{1/2} = \left(\hat{\Sigma}_{ii}\right)^{1/2} = \left(\sum_{j=1}^{n} A_{ij}^2 \lambda_j\right)^{1/2}$$

$$v_i := \frac{x_i - \mu[x_i]}{\sigma_i} = \frac{\sum_{j=1}^n A_{ij} \sqrt{\lambda_j} w_j}{\sigma_i}, \quad i = 1, \dots, n.$$

Then we have $\mu[v_i] = 0$, $\mu[v_i^2] = 1$ and

$$x_i = \mu[x_i] + \sigma_i v_i.$$

It can be appropriate to assume a parametric functional form (→ reduction of parameters) of the correlation structure of x,

$$Corr[x_i, x_j] = Cov[v_i, v_j] = \frac{\hat{\Sigma}_{ij}}{\sigma_i \sigma_j} = \frac{\sum_{k=1}^{n} A_{ik} A_{jk} \lambda_k}{\sigma_i \sigma_j} = \rho(\pi; i, j),$$

where π is some low-dimensional parameter (this is adapted to the calibration of market models → BM[6](Chapter 6.9)).

3.2 PCA of the Yield Curve

Now let $x = (x_1, \ldots, x_n)^T$ be the increments of the forward curve, say

$$x_i = R(t + \Delta t; t + \Delta t + \tau_{i-1}, t + \Delta t + \tau_i) - R(t; t + \tau_{i-1}, t + \tau_i),$$

for some maturity spectrum $0 = \tau_0 < \cdots < \tau_n$.

PCA typically leads to the following picture (→ R[23]p.61): UK market in the years 1989-1992 (the original maturity spectrum has been divided into eight distinct buckets, i.e. $n = 8$).

The first three principal components are

$$p_1 = \begin{pmatrix} 0.329 \\ 0.354 \\ 0.365 \\ 0.367 \\ 0.364 \\ 0.361 \\ 0.358 \\ 0.352 \end{pmatrix}, \quad p_2 = \begin{pmatrix} -0.722 \\ -0.368 \\ -0.121 \\ 0.044 \\ 0.161 \\ 0.291 \\ 0.316 \\ 0.343 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0.490 \\ -0.204 \\ -0.455 \\ -0.461 \\ -0.176 \\ 0.176 \\ 0.268 \\ 0.404 \end{pmatrix}.$$

- The first PC is roughly flat (parallel shift → average rate),
- the second PC is upward sloping (tilt → slope),
- the third PC hump-shaped (flex → curvature).

Figure 3.1: First Three PCs.

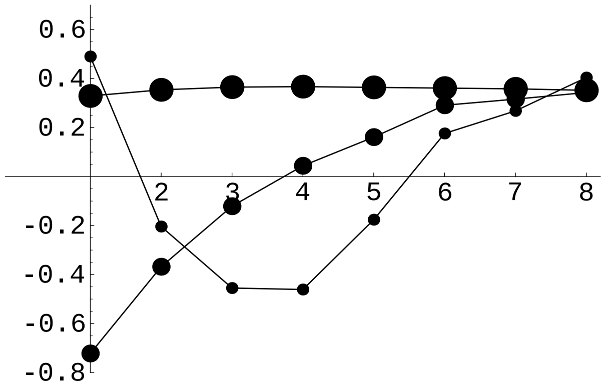


Table 3.1: Explained Variance of the Principal Components (PCs).

PC	Explained Variance (%)
1	92.17
2	6.93
3	0.61
4	0.24
5	0.03
6–8	0.01

The first three PCs explain more than 99 % of the variance of x ($\rightarrow$ Table 3.1).

PCA of the yield curve goes back to the seminal paper by Litterman and Scheinkman (91) [18] (Prof. J. Scheinkman is at the Department of Economics, Princeton University).

3.3 Correlation

$\rightarrow$ R[23](p.58)

A typical example of correlation among forward rates is provided by

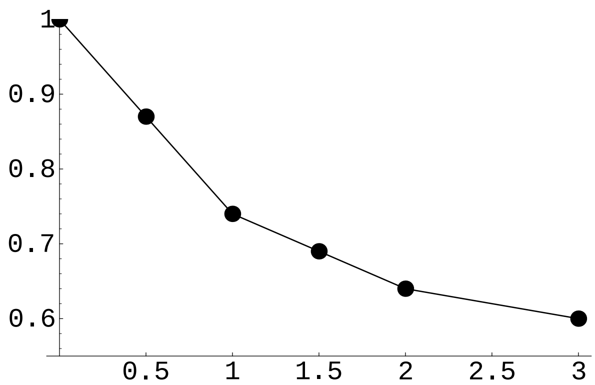
Brown and Schaefer (1994). The data is from the US Treasury yield curve 1987–1994. The following matrix (→ Figure 3.2)

$$\begin{pmatrix} 1 & 0.87 & 0.74 & 0.69 & 0.64 & 0.6 \\ & 1 & 0.96 & 0.93 & 0.9 & 0.85 \\ & & 1 & 0.99 & 0.95 & 0.92 \\ & & & 1 & 0.97 & 0.93 \\ & & & & 1 & 0.95 \\ & & & & & 1 \end{pmatrix}$$

shows the correlation for changes of forward rates of maturities

$$0, \quad 0.5, \quad 1, \quad 1.5, \quad 2, \quad 3 \quad \text{years.}$$

Figure 3.2: Correlation between the short rate and instantaneous forward rates for the US Treasury curve 1987–1994



→ Decorrelation occurs quickly.

→ Exponentially decaying correlation structure is plausible.

Chapter 4

Estimating the Yield Curve

4.1 A Bootstrapping Example

→ JW[12](p.129–136)

This is a naive bootstrapping method of fitting to a money market yield curve. The idea is to build up the yield curve

from shorter maturities to longer maturities.

We take Yen data from 9 January, 1996 (→ JW[12](Section 5.4)). The spot date t_0 is 11 January, 1996. The day-count convention is Actual/360,

$$\delta(T, S) = \frac{\text{actual number of days between } T \text{ and } S}{360}.$$

Table 4.1: Yen data, 9 January 1996.

LIBOR (%)		Futures		Swaps (%)	
o/n	0.49	20 Mar 96	99.34	2y	1.14
1w	0.50	19 Jun 96	99.25	3y	1.60
1m	0.53	18 Sep 96	99.10	4y	2.04
2m	0.55	18 Dec 96	98.90	5y	2.43
3m	0.56			7y	3.01
				10y	3.36

- The first column contains the LIBOR (=simple spot rates) $F(t_0, S_i)$ for maturities

$$\{S_1, \dots, S_5\} = \{12/1/96,\ 18/1/96,\ 13/2/96,\ 11/3/96,\ 11/4/96\}$$

hence for 1, 7, 33, 60 and 91 days to maturity, respectively. The zero-coupon bonds are

$$P(t_0, S_i) = \frac{1}{1 + F(t_0, S_i)\,\delta(t_0, S_i)}.$$

- The futures are quoted as

$$\text{futures price for settlement day } T_i = 100(1 - F_F(t_0; T_i, T_{i+1})),$$

where $F_F(t_0; T_i, T_{i+1})$ is the futures rate for period $[T_i, T_{i+1}]$ prevailing at t_0, and

$$\{T_1, \dots, T_5\} = \{20/3/96,\ 19/6/96,\ 18/9/96,\ 18/12/96,\ 19/3/97\},$$

hence $\delta(T_i, T_{i+1}) \equiv 91/360$.

We treat futures rates as if they were simple forward rates, that is, we set

$$F(t_0; T_i, T_{i+1}) = F_F(t_0; T_i, T_{i+1}).$$

To calculate zero-coupon bond from futures prices we need $P(t_0, T_1)$. We use geometric interpoliation

$$P(t_0, T_1) = P(t_0, S_4)^q\, P(t_0, S_5)^{1-q},$$

which is equivalent to using linear interpolation of continuously compounded spot rates

$$R(t_0, T_1) = q\, R(t_0, S_4) + (1 - q)\, R(t_0, S_5),$$

where

$$q = \frac{\delta(T_1, S_5)}{\delta(S_4, S_5)} = \frac{22}{31} = 0.709677.$$

Then we use the relation

$$P(t_0, T_{i+1}) = \frac{P(t_0, T_i)}{1 + \delta(T_i, T_{i+1})\, F(t_0; T_i, T_{i+1})}$$

to derive $P(t_0, T_2), \dots, P(t_0, T_5)$.

- Yen swaps have semi-annual cashflows at dates

$$\{U_1, \dots, U_{20}\} = \left\{ \begin{array}{cc} 11/7/96, & 13/1/97, \\ 11/7/97, & 12/1/98, \\ 13/7/98, & 11/1/99, \\ 12/7/99, & 11/1/00, \\ 11/7/00, & 11/1/01, \\ 11/7/01, & 11/1/02, \\ 11/7/02, & 13/1/03, \\ 11/7/03, & 12/1/04, \\ 12/7/04, & 11/1, 05, \\ 11/7/05, & 11/1/06 \end{array} \right\}.$$

For a swap with maturity U_n the swap rate at t_0 is given by

$$R_{swap}(t_0, U_n) = \frac{1 - P(t_0, U_n)}{\sum_{i=1}^{n} \delta(U_{i-1}, U_i)\, P(t_0, U_i)}, \quad (U_0 := t_0).$$

From the data we have $R_{swap}(t_0, U_i)$ for $i = 4, 6, 8, 10, 14, 20$.

We obtain $P(t_0, U_1)$, $P(t_0, U_2)$ (and hence $R_{swap}(t_0, U_1), R_{swap}(t_0, U_2)$) by linear interpolation of the continuously compounded spot rates

$$R(t_0, U_1) = \frac{69}{91} R(t_0, T_2) + \frac{22}{91} R(t_0, T_3)$$
$$R(t_0, U_2) = \frac{65}{91} R(t_0, T_4) + \frac{26}{91} R(t_0, T_5).$$

All remaining swap rates are obtained by linear interpolation. For maturity U_3 this is

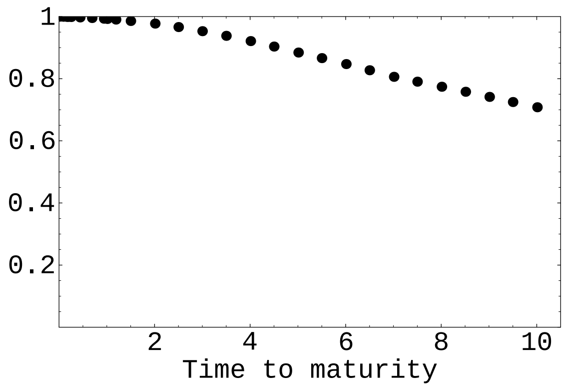
$$R_{swap}(t_0, U_3) = \frac{1}{2}(R_{swap}(t_0, U_2) + R_{swap}(t_0, U_4)).$$

We have ($\rightarrow$ exercise)

$$P(t_0, U_n) = \frac{1 - R_{swap}(t_0, U_n) \sum_{i=1}^{n-1} \delta(U_{i-1}, U_i)\, P(t_0, U_i)}{1 + R_{swap}(t_0, U_n) \delta(U_{n-1}, U_n)}.$$

This gives $P(t_0, U_n)$ for $n = 3, \dots, 20$.

Figure 4.1: Zero-coupon bond curve



In Figure 4.1 is the implied zero-coupon bond price curve

$$P(t_0, t_i), \quad i = 0, \ldots, 29$$

(we have 29 points and set $P(t_0, t_0) = 1$).

The spot and forward rate curves are in Figure 4.2. Spot and forward rates are continuously compounded

$$R(t_0, t_i) = -\frac{\log P(t_0, t_i)}{\delta(t_0, t_i)}$$

$$R(t_0, t_i, t_{i+1}) = -\frac{\log P(t_0, t_{i+1}) - \log P(t_0, t_i)}{\delta(t_i, t_{i+1})}, \quad i = 1, \ldots, 29.$$

The forward curve, reflecting the derivative of $T \mapsto -\log P(t_0, T)$, is very unsmooth and sensitive to slight variations (errors) in prices.

Figure 4.3 shows the spot rate curves from LIBOR, futures and swaps. It is evident that the three curves are not coincident to a common underlying curve. Our naive method made no attempt to meld the three curves together.

→ The entire yield curve is constructed from relatively few instruments. The method exactly reconstructs market prices (this is desirable for interest rate option traders). But it produces an unstable, non-smooth forward curve.

Figure 4.2: Spot rates (lower curve), forward rates (upper curve)

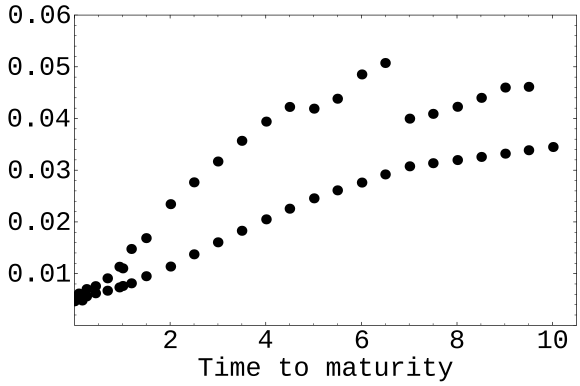
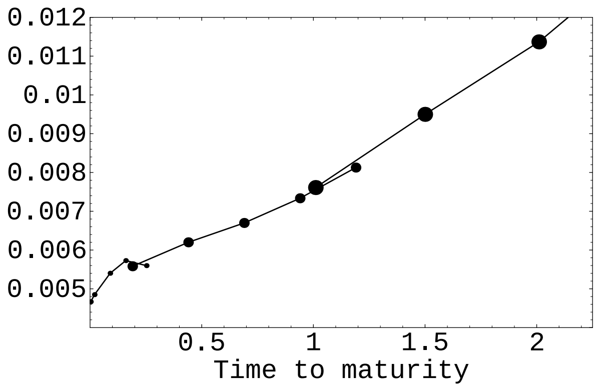


Figure 4.3: Comparison of money market curves



→ Another method would be to estimate a smooth yield curve parametrically from the market rates (for fund managers, long term strategies).

The main difficulties with our method are:

- Futures rates are treated as forward rates. In reality futures rates are greater than forward rates. The amount by which the futures rate is above the forward rate is called the convexity adjustment, which is model dependend. An example is

$$\text{forward rate} = \text{futures rate} - \frac{1}{2}\sigma^2\tau^2,$$

where τ is the time to maturity of the futures contract, and σ is the volatility parameter.

- LIBOR rates beyond the "stup date" $T_1 = 20/3/96$ (that is, at $S_5 = 11/4/96$) are ignored once $P(t_0, T_1)$ is found. In general, the segments of LIBOR, futures and swap markets overlap.
- Swap rates are inappropriately interpolated. The linear interpolation produces a "sawtooth" in the forward rate curve. However, in some markets intermediate swaps are indeed priced as if their prices were found by linear interpolation.

4.2 General Case

The general problem of finding today's (t_0) term structure of zero-coupon bond prices (or the *discount function*)

$$x \mapsto D(x) := P(t_0, t_0 + x)$$

can be formulated as

$$p = C \cdot d + \epsilon,$$

where p is a vector of n market prices, C the related cashflow matrix, and $d = (D(x_1), \ldots, D(x_N))$ with cashflow dates $t_0 < T_1 < \cdots < T_N$,

$$T_i - t_0 = x_i,$$

and ϵ a vector of pricing errors. Reasons for including errors are

- prices are never exactly simultaneous,
- round-off errors in the quotes (bid-ask spreads, etc),
- liquidity effects,
- tax effects (high coupons, low coupons),
- allows for smoothing.

4.2.1 Bond Markets

Data:

- vector of quoted/market bond prices $p = (p_1, \ldots, p_n)$,
- dates of all cashflows $t_0 < T_1 < \cdots < T_N$,
- bond i with cashflows (coupon and principal payments) $c_{i,j}$ at time T_j (may be zero), forming the $n \times N$ cashflow matrix

$$C = (c_{i,j})_{\substack{1\le i\le n \\ 1\le j\le N}} .$$

Example (→ JW[12], p.426): UK government bond (gilt) market, September 4, 1996, selection of nine gilts. The coupon payments are semiannual. The spot date is 4/9/96, and the day-count convention is actual/365.

Table 4.2: Market prices for UK gilts, 4/9/96.

	coupon (%)	next coupon	maturity date	dirty price (p_i)
bond 1	10	15/11/96	15/11/96	103.82
bond 2	9.75	19/01/97	19/01/98	106.04
bond 3	12.25	26/09/96	26/03/99	118.44
bond 4	9	03/03/97	03/03/00	106.28
bond 5	7	06/11/96	06/11/01	101.15
bond 6	9.75	27/02/97	27/08/02	111.06
bond 7	8.5	07/12/96	07/12/05	106.24
bond 8	7.75	08/03/97	08/09/06	98.49
bond 9	9	13/10/96	13/10/08	110.87

Hence $n = 9$ and $N = 1 + 3 + 6 + 7 + 11 + 12 + 19 + 20 + 25 = 104$,

$$T_1 = 26/09/96, \quad T_2 = 13/10/96, \quad T_3 = 06/11/97, \ldots .$$

No bonds have cashflows at the same date. The 9×104 cashflow matrix is

$$C = \begin{pmatrix} 0 & 0 & 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 4.875 & 0 & 0 & 0 & 0 & \ldots \\ 6.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.125 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.5 & 0 & 0 & \ldots \\ 0 & 0 & 3.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.875 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 4.25 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.875 & 0 & \ldots \\ 0 & 4.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \end{pmatrix}$$

4.2.2 Money Markets

Money market data can be put into the same price–cashflow form as above.

LIBOR (rate L, maturity T): $p = 1$ and $c = 1 + (T - t_0)L$ at T.

FRA (forward rate F for $[T, S]$): $p = 0$, $c_1 = -1$ at $T_1 = T$, $c_2 = 1+(S-T)F$ at $T_2 = S$.

Swap (receiver, swap rate K, tenor $t_0 \le T_0 < \cdots < T_n$, $T_i - T_{i-1} \equiv \delta$): since

$$0 = -D(T_0 - t_0) + \delta K \sum_{j=1}^{n} D(T_j - t_0) + (1 + \delta K) D(T_n - t_0),$$

- if $T_0 = t_0$: $p = 1$, $c_1 = \cdots = c_{n-1} = \delta K$, $c_n = 1 + \delta K$,
- if $T_0 > t_0$: $p = 0$, $c_0 = -1$, $c_1 = \cdots = c_{n-1} = \delta K$, $c_n = 1 + \delta K$.

$\rightarrow$ at t_0: LIBOR and swaps have notional price 1, FRAs and forward swaps have notional price 0.

Example ($\rightarrow$ JW[12], p.428): US money market on October 6, 1997. The day-count convention is Actual/360. The spot date t_0 is 8/10/97.

LIBOR is for o/n (1/365), 1m (33/360), and 3m (92/360).

Futures are three month rates ($\delta = 91/360$). We take them as forward rates. That is, the quote of the futures contract with maturity date (settlement day) T is

$$100(1 - F(t_0; T, T + \delta)).$$

Swaps are annual ($\delta = 1$). The first payment date is 8/10/98.

Table 4.3: US money market, October 6, 1997.

	Period	Rate	Maturity Date
LIBOR	o/n	5.59375	9/10/97
	1m	5.625	10/11/97
	3m	5.71875	8/1/98
Futures	Oct-97	94.27	15/10/97
	Nov-97	94.26	19/11/97
	Dec-97	94.24	17/12/97
	Mar-98	94.23	18/3/98
	Jun-98	94.18	17/6/98
	Sep-98	94.12	16/9/98
	Dec-98	94	16/12/98
Swaps	2	6.01253	
	3	6.10823	
	4	6.16	
	5	6.22	
	7	6.32	
	10	6.42	
	15	6.56	
	20	6.56	
	30	6.56	

Here $n = 3 + 7 + 9 = 19$, $N = 3 + 14 + 30 = 47$, $T_1 = 9/10/97$, $T_2 = 15/10/97$ (first future), $T_3 = 10/11/97$, ... The first 14 columns of the 19×47 cashflow matrix C are

$$
\begin{array}{cccccccccccccc}
c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & 0 & 0 & c_{47} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & c_{58} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & c_{69} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{7,10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{8,11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & c_{9,13} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{10,14} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{12,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{13,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{14,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{15,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{16,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{17,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{18,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{19,12} & 0 & 0
\end{array}
$$

with

$$
\begin{array}{c}
c_{11} = 1.00016, \quad c_{23} = 1.00516, \quad c_{36} = 1.01461, \\
\hline
c_{47} = 1.01448, \quad c_{58} = 1.01451, \quad c_{69} = 1.01456, \quad c_{7,10} = 1.01459, \\
c_{8,11} = 1.01471, \quad c_{9,13} = 1.01486, \quad c_{10,14} = 1.01517 \\
\hline
c_{11,12} = 0.060125, \quad c_{12,12} = 0.061082, \quad c_{13,12} = 0.0616, \\
c_{14,12} = 0.0622, \quad c_{15,12} = 0.0632, \quad c_{16,12} = 0.0642, \\
c_{17,12} = c_{18,12} = c_{19,12} = 0.0656.
\end{array}
$$

4.2.3 Problems

Typically, we have $n \ll N$. Moreover, many entries of C are zero (different cashflow dates). This makes ordinary least square (OLS) regression

$$\min_{d \in \mathbb{R}^N} \{ \|\epsilon\|^2 \mid \epsilon = p - C \cdot d \} \quad (\Rightarrow C^T p = C^T C d^*)$$

unfeasible.

One could chose the data set such that cashflows are at same points in time (say four dates each year) and the cashflow matrix C is not entirely full of zeros (Carleton–Cooper (1976)). Still regression only yields values $D(x_i)$ at the payment dates $t_0 + x_i$

$\rightarrow$ interpolation technics necessary.

But there is nothing to regularize the discount factors (discount factors of similar maturity can be very different). As a result this leads to a ragged spot rate (yield) curve, and even worse for forward rates.

4.2.4 Parametrized Curve Families

Reduction of parameters and smooth yield curves can be achieved by using parametrized families of smooth curves

$$D(x) = D(x; z) = \exp\left(-\int_0^x \phi(u; z)\, du\right), \quad z \in \mathcal{Z},$$

with state space $\mathcal{Z} \subset \mathbb{R}^m$.

For regularity reasons (see below) it is best to estimate the forward curve

$$\mathbb{R}_+ \ni x \mapsto f(t_0, t_0 + x) = \phi(x) = \phi(x; z).$$

This leads to a nonlinear optimization problem

$$\min_{z \in \mathcal{Z}} \| p - C \cdot d(z) \|,$$

with

$$d_i(z) = \exp\left(-\int_0^{x_i} \phi(u; z)\, du\right)$$

for some payment tenor $0 < x_1 < \cdots < x_N$.

Linear Families

Fix a set of basis functions $\psi_1, \ldots, \psi_m$ (preferably *with compact support*), and let

$$\phi(x; z) = z_1 \psi_1(x) + \cdots + z_m \psi_m(x).$$

Cubic B-splines A cubic spline is a piecewise cubic polynomial that is everywhere twice differentiable. It interpolates values at $m+1$ *knot points* $\xi_0 < \cdots < \xi_m$. Its general form is

$$\sigma(x) = \sum_{i=0}^{3} a_i x^i + \sum_{j=1}^{m-1} b_j (x - \xi_j)_+^3,$$

hence it has $m+3$ parameters $\{a_0, \ldots, a_4, b_1, \ldots, b_{m-1}\}$ (a kth degree spline has $m+k$ parameters). The spline is uniquely characterized by specification of σ' or σ'' at ξ_0 and ξ_m.

Introduce six extra knot points

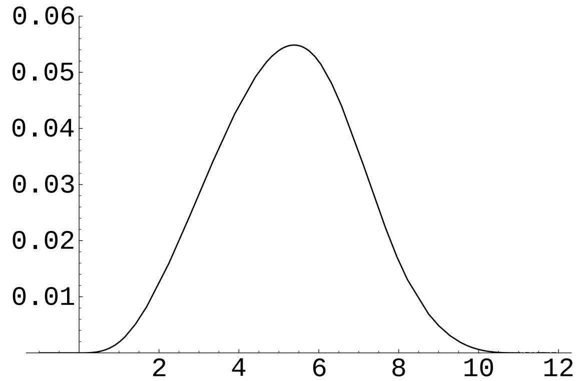
$$\xi_{-3} < \xi_{-2} < \xi_{-1} < \xi_0 < \cdots < \xi_m < \xi_{m+1} < \xi_{m+2} < \xi_{m+3}.$$

A basis for the cubic splines on $[\xi_0, \xi_m]$ is given by the $m+3$ *B-splines*

$$\psi_k(x) = \sum_{j=k}^{k+4} \left(\prod_{i=k, i \neq j}^{k+4} \frac{1}{\xi_i - \xi_j} \right) (x - \xi_j)_+^3, \quad k = -3, \ldots, m-1.$$

The B-spline ψ_k is zero outside $[\xi_k, \xi_{k+4}]$.

Figure 4.4: B-spline with knot points $\{0, 1, 6, 8, 11\}$.



Estimating the Discount Function B-splines can also be used to estimate the discount function directly (Steeley (1991)),

$$D(x;z) = z_1\psi_1(x) + \cdots + z_m\psi_m(x).$$

With

$$d(z) = \begin{pmatrix} D(x_1;z) \\ \vdots \\ D(x_N;z) \end{pmatrix} = \begin{pmatrix} \psi_1(x_1) & \cdots & \psi_m(x_1) \\ \vdots & & \vdots \\ \psi_1(x_N) & \cdots & \psi_m(x_N) \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} =: \Psi \cdot z$$

this leads to the linear optimization problem

$$\min_{z\in\mathbb{R}^m} \|p - C\Psi z\|.$$

If the $n \times m$ matrix $A := C\Psi$ has full rank m, the unique unconstrained solution is

$$z^* = (A^T A)^{-1} A^T p.$$

A reasonable constraint would be

$$D(0;z) = \psi_1(0)z_1 + \cdots + \psi_m(0)z_m = 1.$$

Example We take the UK government bond market data from the last section (Table 4.2). The maximum time to maturity, x_{104}, is 12.11 [years]. Notice that the first bond is a zero-coupon bond. Its exact yield is

$$y = -\frac{365}{72}\log\frac{103.822}{105} = -\frac{1}{0.197}\log 0.989 = 0.0572.$$

- As a basis we use the 8 (resp. first 7) B-splines with the 12 knot points

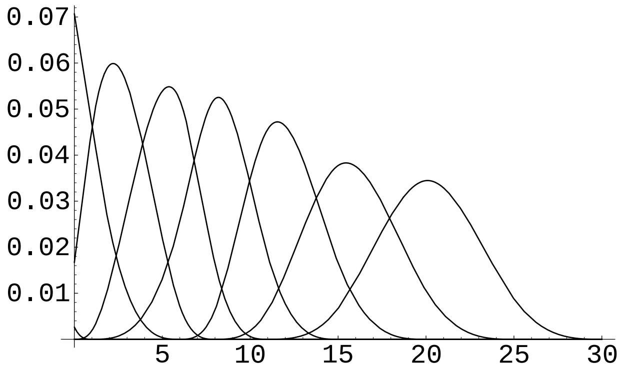
  $$\{-20, -5, -2, 0, 1, 6, 8, 11, 15, 20, 25, 30\}$$

  (see Figure 4.5).

  The estimation with all 8 B-splines leads to

  $$\min_{z\in\mathbb{R}^8} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.23$$

Figure 4.5: B-splines with knots $\{-20, -5, -2, 0, 1, 6, 8, 11, 15, 20, 25, 30\}$.



with

$$z^* = \begin{pmatrix} 13.8641 \\ 11.4665 \\ 8.49629 \\ 7.69741 \\ 6.98066 \\ 6.23383 \\ -4.9717 \\ 855.074 \end{pmatrix},$$

and the discount function, yield curve (cont. comp. spot rates), and forward curve (cont. comp. 3-monthly forward rates) shown in Figure 4.7.

The estimation with only the first 7 B-splines leads to

$$\min_{z \in \mathbb{R}^7} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.32$$

with

$$z^* = \begin{pmatrix} 17.8019 \\ 11.3603 \\ 8.57992 \\ 7.56562 \\ 7.28853 \\ 5.38766 \\ 4.9919 \end{pmatrix},$$

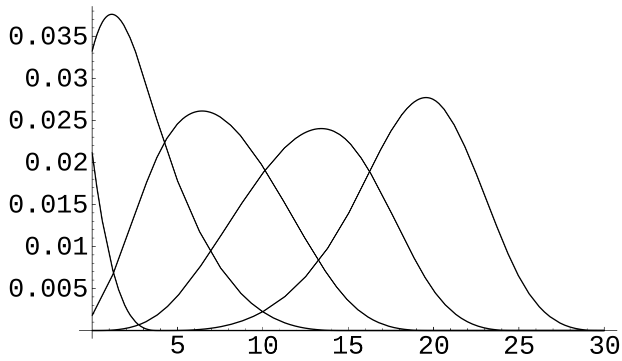
and the discount function, yield curve (cont. comp. spot rates), and forward curve (cont. comp. 3-month forward rates) shown in Figure 4.8.

- Next we use only 5 B-splines with the 9 knot points

$$\{-10, -5, -2, 0, 4, 15, 20, 25, 30\}$$

(see Figure 4.6).

Figure 4.6: Five B-splines with knot points $\{-10, -5, -2, 0, 4, 15, 20, 25, 30\}$.



The estimation with this 5 B-splines leads to

$$\min_{z \in \mathbb{R}^5} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.39$$

with

$$z^* = \begin{pmatrix} 15.652 \\ 19.4385 \\ 12.9886 \\ 7.40296 \\ 6.23152 \end{pmatrix},$$

and the discount function, yield curve (cont. comp. spot rates), and forward curve (cont. comp. 3-monthly forward rates) shown in Figure 4.9.

Figure 4.7: Discount function, yield and forward curves for estimation with 8 B-splines. The dot is the exact yield of the first bond.

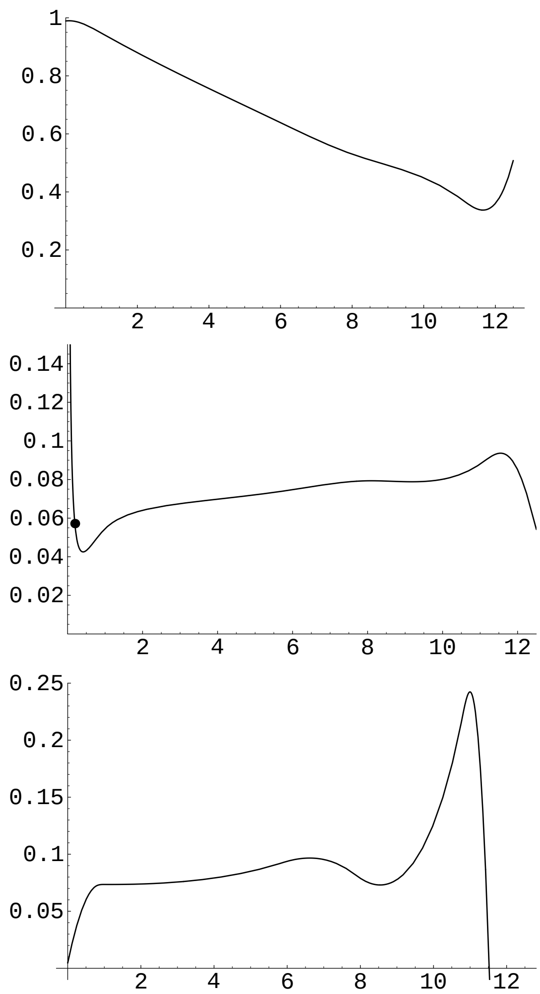


Figure 4.8: Discount function, yield and forward curves for estimation with 7 B-splines. The dot is the exact yield of the first bond.

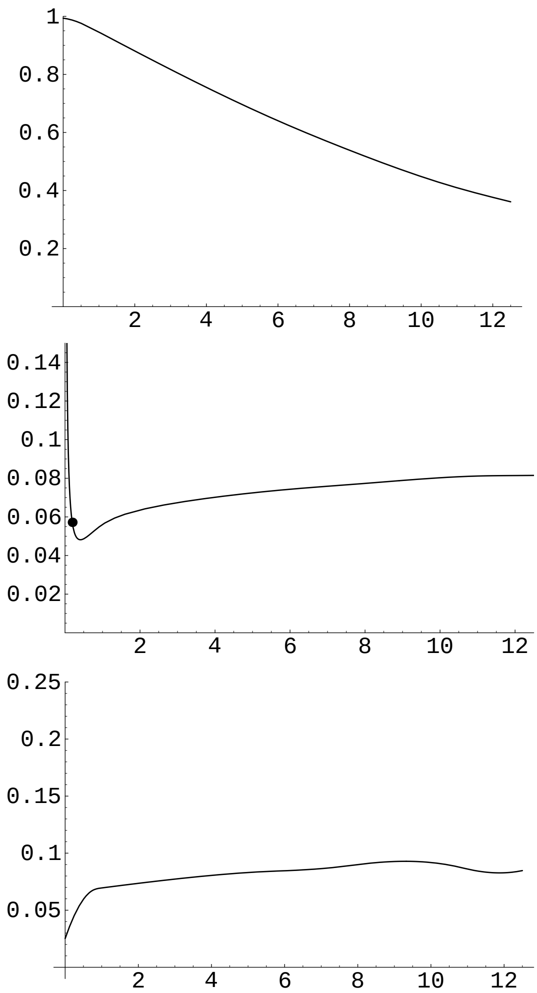
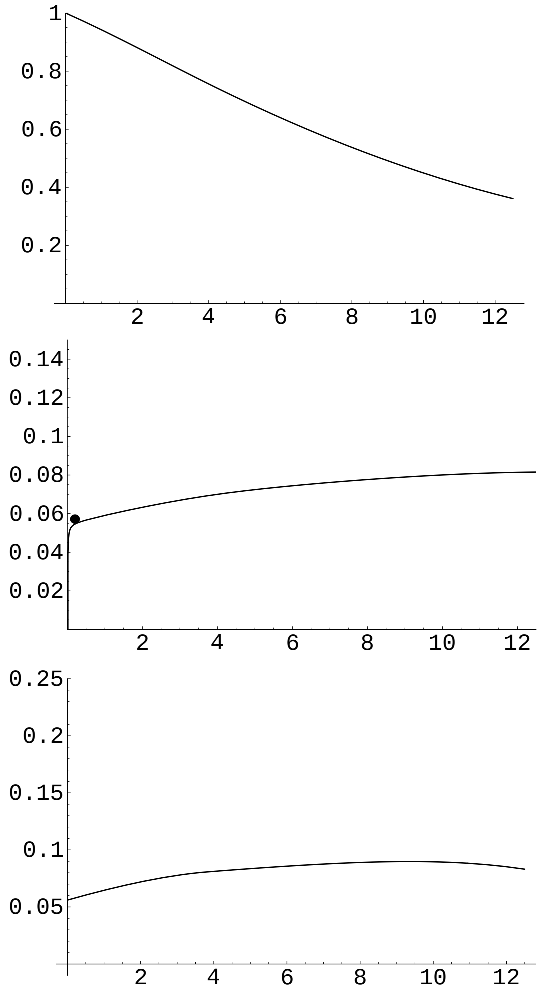


Figure 4.9: Discount function, yield and forward curves for estimation with 5 B-splines. The dot is the exact yield of the first bond.



Discussion

- In general, splines can produce bad fits.
- Estimating the discount function leads to unstable and non-smooth yield and forward curves. Problems mostly at short and long term maturities.
- Splines are not useful for extrapolating to long term maturities.
- There is a trade-off between the quality (or regularity) and the correctness of the fit. The curves in Figures 4.8 and 4.9 are more regular than those in Figure 4.7, but their correctness criteria (0.32 and 0.39) are worse than for the fit with 8 B-splines (0.23).
- The B-spline fits are extremely sensitive to the number and location of the knot points.

→ Need criterions asserting smooth yield and forward curves that do not fluctuate too much and flatten towards the long end.

→ Direct estimation of the yield or forward curve.

→ Optimal selection of number and location of knot points for splines.

→ Smoothing splines.

Smoothing Splines The least squares criterion

$$\min_z \|p - C \cdot d(z)\|^2$$

has to be replaced/extended by criterions for the smoothness of the yield or forward curve.

Example: Lorimier (95). In her PhD thesis 1995, Sabine Lorimier suggests a spline method where the number and location of the knots are determined by the observed data itself.

For ease of notation we set $t_0 = 0$ (today). The data is given by N observed zero-coupon bonds $P(0,T_1),\ldots,P(0,T_N)$ at $0 < T_1 < \cdots < T_N \equiv T$, and consequently the N yields

$$Y_1,\ldots,Y_N, \quad P(0,T_i) = \exp(-T_i Y_i).$$

Let $f(u)$ denote the forward curve. The fitting requirement now is for the forward curve

$$\int_0^{T_i} f(u)\,du + \epsilon_i/\sqrt{\alpha} = T_i Y_i, \tag{4.1}$$

with an arbitrary constant $\alpha > 0$. The aim is to minimize $\|\epsilon\|^2$ as well as the smoothness criterion

$$\int_0^T (f'(u))^2\,du. \tag{4.2}$$

Introduce the Sobolev space

$$H = \{g \mid g' \in L^2[0,T]\}$$

with scalar product

$$\langle g, h\rangle_H = g(0)h(0) + \int_0^T g'(u)h'(u)\,du,$$

and the nonlinear functional on H

$$F(f) := \left[\int_0^T (f'(u))^2\,du + \alpha \sum_{i=1}^N \left(Y_i T_i - \int_0^{T_i} f(u)\,du\right)^2\right].$$

The optimization problem then is

$$\min_{f\in H} F(f). \tag{*}$$

The parameter α tunes the trade-off between smoothness and correctness of the fit.

Theorem 4.2.1. *Problem (*) has a unique solution f, which is a second order spline characterized by*

$$f(u) = f(0) + \sum_{k=1}^N a_k h_k(u) \tag{4.3}$$

where $h_k \in C^1[0,T]$ is a second order polynomial on $[0,T_k]$ with

$$h_k'(u) = (T_k - u)^+, \quad h_k(0) = T_k, \quad k = 1,\dots,N, \tag{4.4}$$

and $f(0)$ and a_k solve the linear system of equations

$$\sum_{k=1}^{N} a_k T_k = 0, \tag{4.5}$$

$$\alpha\left(Y_k T_k - f(0)T_k - \sum_{l=1}^{N} a_l \langle h_l, h_k\rangle_H\right) = a_k, \quad k = 1, \ldots, N. \tag{4.6}$$

Proof. Integration by parts yields

$$\begin{aligned}\int_0^{T_k} g(u)\,du &= T_k g(T_k) - \int_0^{T_k} u g'(u)\,du \\ &= T_k g(0) + T_k \int_0^{T_k} g'(u)\,du - \int_0^{T_k} u g'(u)\,du \\ &= T_k g(0) + \int_0^{T} (T_k - u)^+ g'(u)\,du = \langle h_k, g\rangle_H,\end{aligned}$$

for all $g \in H$. In particular,

$$\int_0^{T_k} h_l\,du = \langle h_l, h_k\rangle_H.$$

A (local) minimizer f of F satisfies

$$\frac{d}{d\epsilon} F(f + \epsilon g)|_{\epsilon=0} = 0$$

or equivalently

$$\int_0^T f' g'\,du = \alpha \sum_{k=1}^{N} \left(Y_k T_k - \int_0^{T_k} f\,du\right) \int_0^{T_k} g\,du, \quad \forall g \in H. \tag{4.7}$$

In particular, for all $g \in H$ with $\langle g, h_k\rangle_H = 0$ we obtain

$$\langle f - f(0), g\rangle_H = \int_0^T f'(u) g'(u)\,du = 0.$$

Hence

$$f - f(0) \in span\{h_1, \ldots, h_N\}$$

what proves (4.3), (4.4) and (4.5) (set $u = 0$). Hence we have

$$\int_0^T f'(u)g'(u)\,du = \sum_{k=1}^N a_k \left(-T_k g(0) + \int_0^{T_k} g(u)\,du \right) = \sum_{k=1}^N a_k \int_0^{T_k} g(u)\,du,$$

and (4.7) can be rewritten as

$$\sum_{k=1}^N \left(a_k - \alpha \left(Y_k T_k - f(0)T_k - \sum_{l=1}^N a_l \langle h_l, h_k \rangle_H \right) \right) \int_0^{T_k} g(u)\,du = 0$$

for all $g \in H$. This is true if and only if (4.6) holds.

Thus we have shown that (4.7) is equivalent to (4.3)–(4.6).

Next we show that (4.7) is a sufficient condition for f to be a global minimizer of F. Let $g \in H$, then

$$\begin{aligned} F(g) &= \int_0^T \left((g' - f') + f'\right)^2 du + \alpha \sum_{k=1}^N \left(Y_k T_k - \int_0^{T_k} g\,du \right)^2 \\ &\overset{(4.7)}{=} F(f) + \int_0^T (g' - f')^2\,du + \alpha \sum_{k=1}^N \left(\int_0^{T_k} f\,du - \int_0^{T_k} g\,du \right)^2 \\ &\geq F(f), \end{aligned}$$

where we used (4.7) with $g - f \in H$.

It remains to show that f exists and is unique; that is, that the linear system (4.5)–(4.6) has a unique solution $(f(0), a_1, \ldots, a_N)$. The corresponding $(N+1) \times (N+1)$ matrix is

$$A = \begin{pmatrix} 0 & T_1 & T_2 & \cdots & T_N \\ \alpha T_1 & \alpha\langle h_1, h_1 \rangle_H + 1 & \alpha\langle h_1, h_2 \rangle_H & \cdots & \alpha\langle h_1, h_N \rangle_H \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \alpha T_N & \alpha\langle h_N, h_1 \rangle_H & \alpha\langle h_N, h_2 \rangle_H & \cdots & \alpha\langle h_N, h_N \rangle_H + 1 \end{pmatrix}. \tag{4.8}$$

Let $\lambda = (\lambda_0, \ldots, \lambda_N)^T \in \mathbb{R}^{N+1}$ such that $A\lambda = 0$, that is,

$$\sum_{k=1}^N T_k \lambda_k = 0$$

$$\alpha T_k \lambda_0 + \alpha \sum_{l=1}^N \langle h_k, h_l \rangle_H \lambda_l + \lambda_k = 0, \quad k = 1, \ldots, N.$$

Multiplying the latter equation with λ_k and summing up yields

$$\alpha \left\| \sum_{k=1}^{N} \lambda_k h_k \right\|_H^2 + \sum_{k=1}^{N} \lambda_k^2 = 0.$$

Hence $\lambda = 0$, whence A is non-singular. □

The role of α is as follows:

- If $\alpha \to 0$ then by (4.3) and (4.6) we have $f(u) \equiv f(0)$, a constant function. That is, maximal regularity

$$\int_0^T (f'(u))^2 \, du = 0$$

  but no fitting of data, see (4.1).

- If $\alpha \to \infty$ then (4.7) implies that

$$\int_0^{T_k} f(u) \, du = Y_k T_k, \quad k = 1, \ldots, N, \tag{4.9}$$

  a perfect fit. That is, f minimizes (4.2) subject to the constraints (4.9).

To estimate the forward curve from N zero-coupon bonds—that is, yields $Y = (Y_1, \ldots, Y_N)^T$—one has to solve the linear system

$$A \cdot \begin{pmatrix} f(0) \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ Y \end{pmatrix} \quad \text{(see (4.8))}.$$

Of course, if coupon bond prices are given, then the above method has to be modified and becomes nonlinear. With $p \in \mathbb{R}^n$ denoting the market price vector and c_{kl} the cashflows at dates T_l, $k = 1, \ldots, n$, $l = 1, \ldots, N$, this reads

$$\min_{f \in H} \left\{ \int_0^T (f')^2 \, du + \alpha \sum_{k=1}^{n} \left(\log p_k - \log \left[\sum_{l=1}^{N} c_{kl} \exp \left[- \int_0^{T_l} f \, du \right] \right] \right)^2 \right\}.$$

If the coupon payments are small compared to the nominal (=1), then this problem has a unique solution. This and much more is carried out in Lorimier's thesis.

Exponential-Polynomial Families

Exponential-polynomial functions

$$p_1(x)e^{-\alpha_1 x} + \cdots + p_n(x)e^{-\alpha_n x} \quad (p_i\text{=polynomial of degree } n_i)$$

form non-linear families of functions. Popular examples are:

Nelson–Siegel (87) [21] There are 4 parameters $z_1, \ldots, z_4$ and

$$\phi_{NS}(x; z) = z_1 + (z_2 + z_3 x)e^{-z_4 x}.$$

Svensson (94) [27] (Prof. L. E. O. Svensson is at the Economics Department, Princeton University) This is an extension of Nelson–Siegel, including 6 parameters $z_1, \ldots, z_6$,

$$\phi_S(x; z) = z_1 + (z_2 + z_3 x)e^{-z_4 x} + z_5 e^{-z_6 x}.$$

Figure 4.10: Nelson–Siegel curves for $z_1 = 7.69$, $z_2 = -4.13$, $z_4 = 0.5$ and 7 different values for $z_3 = 1.76$, 0.77, -0.22, -1.21, -2.2, -3.19, -4.18.

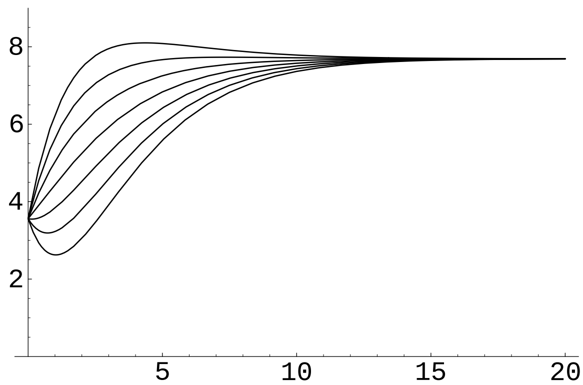


Table 4.4 is taken from a document of the Bank for International Settlements (BIS) 1999 [2].

Table 4.4: Overview of estimation procedures by several central banks. BIS 1999 [2]. NS is for Nelson–Siegel, S for Svensson, wp for weighted prices.

Central Bank	Method	Minimized Error
Belgium	S or NS	wp
Canada	S	sp
Finland	NS	wp
France	S or NS	wp
Germany	S	yields
Italy	NS	wp
Japan	smoothing splines	prices
Norway	S	yields
Spain	S	wp
Sweden	S	yields
UK	S	yields
USA	smoothing splines	bills: wp bonds: prices

Criteria for Curve Families

- Flexibility (do the curves fit a wide range of term structures?)
- Number of factors not too large (curse of dimensionality).
- Regularity (smooth yield or forward curves that flatten out towards the long end).
- Consistency: do the curve families go well with interest rate models? $\rightarrow$ this point will be exploited in the sequel.

Chapter 5

Why Yield Curve Models?

→ R[23](Chapter 5)

Why modelling the entire term structure of interest rates? There is no need when pricing a single European call option on a bond.

But: the payoffs even of “plain-vanilla” fixed income products such as caps, floors, swaptions consist of a sequence of cashflows at $T_1, \ldots, T_n$, where n may be 20 (e.g. a 10y swap with semi-annual payments) or more.

→ The valuation of such products requires the modelling of the entire covariance structure. Historical estimation of such large covariance matrices is statistically not tractable anymore.

→ Need strong structure to be imposed on the co-movements of financial quantities of interest.

→ Specify the dynamics of a small number of variables (e.g. PCA).

→ Correlation structure among observable quantities can now be obtained analytically or numerically.

→ Simultaneous pricing of different options and hedging instruments in a consistent framework.

This is exactly what interest rate (curve) models offer:

- reduction of fitting degrees of freedom → makes problem manageable.

$\Longrightarrow$ It is practically and intellectually rewarding to consider no-arbitrage conditions in much broader generality.

Bibliography

[1] T. R. Bielecki and M. Rutkowski, *Credit risk: modelling, valuation and hedging*, Springer Finance, Springer-Verlag, Berlin, 2002.

[2] BIS, *Zero-coupon yield curves: Technical documentation*, Bank for International Settlements, Basle, March 1999.

[3] T. Björk, *Arbitrage theory in continuous time*, Oxford University Press, 1998.

[4] T. Björk, Y. Kabanov, and W. Runggaldier, *Bond market structure in the presence of marked point processes*, Math. Finance **7** (1997), no. 2, 211–239.

[5] A. Brace, D. Gątarek, and M. Musiela, *The market model of interest rate dynamics*, Math. Finance **7** (1997), no. 2, 127–155.

[6] D. Brigo and F. Mercurio, *Interest rate models—theory and practice*, Springer-Verlag, Berlin, 2001.

[7] A. J. G. Cairns, *Interest rate models*, Princeton University Press, 2004.

[8] J. Cox, J. Ingersoll, and S. Ross, *A theory of the term structure of interest rates*, Econometrica **53** (1985), 385–408.

[9] D. Filipović, *Consistency problems for Heath–Jarrow–Morton interest rate models*, Springer-Verlag, Berlin, 2001.

[10] D. Heath, R. Jarrow, and A. Morton, *Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation*, Econometrica **60** (1992), 77–105.

[11] J. C. Hull, *Options, futures, and other derivatives*, fourth ed., Prentice-Hall International, Inc., 2000.

[12] J. James and N. Webber, *Interest rate modelling*, Wiley, 2000.

[13] R. Jamshidian, *Libor and swap market models and measures*, Finance and Stochastics **1** (1997), 290–330.

[14] R. Jarrow, *Modelling fixed income securities and interest rate options*, McGraw-Hill, 1996.

[15] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, second ed., Springer-Verlag, New York, 1991.

[16] S. Kusuoka, *A remark on default risk models*, Advances in mathematical economics, Vol. 1, Springer, Tokyo, 1999, pp. 69–82.

[17] D. Lando, *On Cox processes and credit-risky securities*, Rev. Derivatives Res. **2** (1998), 99–120.

[18] R. Litterman and J.A. Scheinkman, *Common factors affecting bond returns*, Journal of Fixed Income **1** (1991), 54–61.

[19] R. C. Merton, *On the pricing for corporate debt: the risk structure of interest rates*, J. Finance **29** (1974), 449–470.

[20] M. Musiela and M. Rutkowski, *Martingale methods in financial modelling*, Applications of Mathematics, vol. 36, Springer-Verlag, Berlin-Heidelberg, 1997.

[21] C. Nelson and A. Siegel, *Parsimonious modeling of yield curves*, J. of Business **60** (1987), 473–489.

[22] C. R. Rao, *Principal component and factor analyses*, Statistical methods in finance, North-Holland, Amsterdam, 1996, pp. 489–505.

[23] R. Rebonato, *Interest-rate option models*, second ed., Wiley, 1998.

[24] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Grundlehren der mathematischen Wissenschaften, vol. 293, Springer-Verlag, Berlin-Heidelberg-New York, 1994.

[25] Bernd Schmid, *Pricing credit linked financial instruments*, Lecture Notes in Economics and Mathematical Systems, vol. 516, Springer-Verlag, Berlin, 2002, Theory and empirical evidence.

[26] J. Michael Steele, *Stochastic calculus and financial applications*, Applications of Mathematics, vol. 45, Springer-Verlag, New York, 2001.

[27] L. E. O. Svensson, *Estimating and interpreting forward interest rates: Sweden 1992-1994*, IMF Working Paper No. 114, September 1994.

[28] R. Zagst, *Interest-rate management*, Springer-Verlag, Berlin, 2002.