Bohmian Mechanics

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1 The Theory

1.1 Equations

Bohmian mechanics is a quantum theory without observers [1,2]. This means that neither the act of observation nor the notion of observer play any role in defining the theory – the theory is not about observers and observation – and it explains all non relativistic quantum phenomena. The theory is about something primitive[1], the basic ontology, and the laws for that are given.

Bohmian mechanics is a deterministic theory of point particles. Like Newtonian mechanics it is invariant under Galilei transformations, but unlike Newtonian mechanics it is a first-order theory – acceleration is not a concept entering the law of motion. Rather this law directly determines the velocities of the particles as follows.

For an N-particle system the positions of the N particles form the configuration space variable $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \in \mathbb{R}^{3N}$, $\boldsymbol{Q}_k \in \mathbb{R}^3$ being the position of k-th particle. The motion of the configuration is determined by a time-dependent wave function on configuration space $\psi(q,t) \in \mathbb{C}$, $q = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N) \in \mathbb{R}^{3N}$, that obeys Schrödingers equation (m_k is the mass of the k-th particle and V is the potential)

$$\mathrm{i}\hbar \frac{\partial \psi(q,t)}{\partial t} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \Delta_k \psi(q,t) + V(q)\psi(q,t). \tag{1}$$

The wave function's role is to induce a velocity-vectorfield on configuration space $v^\psi(q)$, the integral curves of which are the trajectories of the particles

$$\frac{\mathrm{d}\boldsymbol{Q}_k}{\mathrm{d}t} = \boldsymbol{v}_k^\psi(Q) = \frac{\hbar}{m_k} \frac{\mathrm{Im}(\psi^* \nabla_k \psi)}{\psi^* \psi} (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N). \tag{2}$$

To solve (1), (2) one needs an initial configuration (i.e. the initial positions of all the particles) and an initial wave function.

In [3] we show how the law of motion may be seen as emerging in a natural manner from the requirement that it be invariant under Galileian transformations.

[1] In contrast, observers are complicated objects, made of many parts: for example one may get a haircut and still be functioning as observer.

Non Newtonian Character Bohmian mechanics is a first-order theory: Given the velocity field, specifying the positions of the particles at one time fixes the trajectories. Newtonian mechanics needs both position and velocity specified for the motion to be determined. Differentiating (2) with respect to time t does not render the theory second order, no more than does a Newtonian theory become a third order theory by differentiating $m\ddot{\boldsymbol{x}} = \boldsymbol{F}$ with respect to time. Nevertheless David Bohm, who worked this theory out in 1952 [4,5], presented this first-order theory also in the Newtonian looking second-order form. It is clearly useful to take the first-order character seriously. For example configuration space trajectories can never *cross* each other. Bell [1] also emphasized the first-order character. He writes the right hand side of (2) as the quotient of the quantum flux $\boldsymbol{j}^\psi$ and the density $\rho = |\psi|^2$, the significance of which I shall address later.

Nonlocal Character The wave function is a function on configuration space. That space is not 'anschaulich' like physical space, it is abstract. But since the wave function is part of the law of the motion it is important for the understanding of Bohmian mechanics that the abstract configuration space be taken seriously. The role played by the configuration space wave function in defining the motion makes Bohmian mechanics nonlocal: Looking at (2), the velocity of each particle depends on the position of all the other particles no matter how far the other particles are away. The amount of action at a distance depends only on the degree of *entanglement* of the wave function on configuration space, that is, the extent to which the wave function differs from a product of single particle wave functions.

Because of the observed violations of Bell's inequalities [1] (see [6] for various examples), nature is nonlocal. A fundamentally correct physical theory must therefore be nonlocal. Bohmian mechanics shows that this can naturally be achieved – by incorporating the *wave function on configuration space* as physical field, which is naturally a nonlocal object.

Names The theory goes under various names, but it's always Bohmian mechanics: Causal Interpretation [4], de Broglie-Bohm pilot wave theory (Bell [1]), or de Broglie-Bohm theory, or pilot wave theory (Valentini in this volume) or simply Bohm's theory. De Broglie's name appears because de Broglie presented reluctantly the equations of Bohmian mechanics (which even earlier had been written down by Madelung, for a fluid picture of the wave evolution) at the Solvay conference 1927, but in the discussion with Pauli at this conference, he found himself unable to defend his ideas. A stochastic version of Bohmian mechanics, where the trajectories are perturbed by white noise, exists, namely, stochastic mechanics (sometimes called Nelson's stochastic mechanics [7]).

Point Particle Theories Other deterministic point particle theories are for example Newtonian mechanics and the relativistic Wheeler–Feynman Electromagnetism [8] of point charges – a theory without fields. (For a relativistic theory of extended charges and fields see the contribution by Michael Kiessling in this volume.)

While in these theories physicists agree that the particles are ontological, the particles in Bohmian mechanics are often seen with distrust and given funny attributes (Adler in this volume calls them hidden 'particles'). On the one hand, this is understandable because orthodox quantum mechanics forbids us to talk about the trajectories of particles. On the other hand, the point particles are ontological in Bohmian mechanics – they are, in fact, what the theory is about. Ignoring them, the theory becomes a theory about nothing, pretty much like orthodox quantum mechanics, where one resorts to the dubious notion of observer as fundamental. To understand Bohmian mechanics and how quantum phenomena emerge from it, it is necessary that the particles be ontological – i.e., be taken seriously.

Other Quantum Theories Without Observers A pure wave function ontology, with no particles as part of the ontology, lies behind the idea of the collapse models of quantum mechanics, and they are discussed in this volume by Alberto Rimini and Gian Carlo Ghirardi. So I say nothing more.

The program of Decoherent Histories or Consistent Histories arises in my understanding also from the wish to have facts in the quantum world without the action of some observer's mind. Roland Omnès reports about that in this volume.

1.2 Frequently Asked Questions

I answer briefly some questions that may arise immediately when the theory is presented as above. Typically the questions are of the nature: *But isn't there a problem with this and that?* and the answer is: *No, there is no problem.*

What About Spin? Can Bohmian mechanics deal with spin?

Instead of the wave function $\psi(q,t) \in \mathbb{C}$ let $\psi(q,t) \in$ (spinspace $=$) $\mathbb{C}^k, k = 1,2,3,...$, i.e. the wave function is a spinor, which defines then the velocity field in an obvious way, namely by using the inner product in spinspace in (2) [9]. The Schrödinger equation (1) is then replaced by the Pauli equation, involving electromagnetic fields and Pauli matrices. Spin is thus a property of the wave function.

What About Fermions or Bosons? Can Bohmian mechanics describe indistinguishable particles?

In (1) and (2) the particles are labeled, as if they were distinguishable (for example by different masses). To formulate Bohmian mechanics for indistinguishable particles one may demand that the law be invariant under permutation of the (artificially introduced) labels, or one may formulate Bohmian mechanics from the beginning on the *right* configuration space of indistinguishable particles. As one might expect both approaches establish that only symmetric or antisymmetric wave functions can be used in the formulation of the law [12].

What About the Newtonian Limit? Does Newtonian mechanics emerge from Bohmian mechanics?

The Newtonian limit or classical limit of Bohmian mechanics is a limit in which the Bohmian trajectories become Newtonian. There are various physical situations in which this is the case. The question itself presents no conceptual difficulties. Concerning the transition of a first-order theory to a second order theory, one may note that the Bohmian trajectories do have at a given time (say the initial time) a position and a velocity, and that the velocity is determined by the wave function. The extra degree of freedom needed for a second order theory (the free choice of an initial velocity) resides in the wave function. (Note also that a one dimensional Bohmian world is special since Bohmian trajectories can never cross, which in one dimension presents a 'topological' barrier for the particle motion, while classical trajectories do not have that. Here classical motion still emerges from the motion of narrow wave packets.) For a recent discussion of the classical limit of quantum mechanics from a Bohmian perspective see [14].

Concerning the appearance of masses (and the potential V) in (1) and (2) we note that actually only naturalized quantities $\mu_k = \frac{m_k}{\hbar}$ (or $\frac{V}{\hbar}$) appear in the law, which only a posteriori, in the Newtonian limit of Bohmian mechanics, are recognized as connected with the inertial mass (or classical potential) [9].

What About Relativity? Can Bohmian mechanics be turned into a relativistic theory?

There are various ideas behind this question. One idea is that relativistic quantum theory is a quantum theory of fields, so, superficially, quantum field theory shows that particles cannot be ontological. The phenomena of pair creation and annihilation support that argument. On the other hand, pair creation and annihilation is a particle phenomenon, and the aim of quantum field theory is to describe particles (for example in scattering situations). Therefore under further scrutiny it may well be the case that particles are still part of the ontology. To come to grips with these aspects of relativistic quantum phenomena may in fact be easy compared to the following: Nature is nonlocal and any relativistic theory of nature must respect that, i.e. it must account in a Lorentz invariant way for faster than light effects to achieve action at a distance. For toy models in which this is achieved see [13], and see also the discussion in [10].

There is also the more metaphysical idea that eventually physics will look quite different from the way it looks now: I am thinking here of future quantum gravity and physics beyond the Planck length. From a Bohmian perspective, the metaphysical answer to this is the following: The moral of Bohmian mechanics is not that the ontology must be a particle ontology, but rather that physics, and I mean now in particular quantum physics, can still be (I would rather like to say must be) about something very clear, about *some clear* ontology.

Otherwise one ends in endless debates about interpretations of mathematical symbols. Furthermore Bohmian mechanics shows that one should not be afraid of thinking of the simplest ontology, one directly suggested by the phenomena, as the right ontology, despite of the droppings of authorities. All the same, it may also happen that the primitive ontology for quantum theory at deeper levels is more elusive.

2 The Explanation Of Phenomena

2.1 Trajectories

Newtonian mechanics applies with a high degree of accuracy over a very wide range of scales: From planetary motion, to apples falling to earth, to bullets shot from a gun, and further on to the motion of gas molecules in a dilute gas. The difference of description of the gas and of the motion of the bullet (or of other macroscopic bodies) is rather striking. In the case of the bullet one is used to taking aim, that is, one thinks of a very precise initial position and velocity of the bullet for computing its trajectory, or, better, for predicting its trajectory, and one could even consider testing such a prediction in experiment, observing for example where and at which time the bullet arrived. Thus the bullet motion is described in detail. However, for the gas a statistical description is used and accepted. I shall come to that in the next section, because what springs to mind first when a mechanical law is written down, is the study of detailed motions like the bullet motion. I will go along with that here but I warn that the computation of particle trajectories in Bohmian mechanics analogous to what one is used to in Newtonian mechanics is not as illuminating. The reason is that in Bohmian mechanics the positions of the particles are, as I shall explain later, typically randomly distributed, given the wave function, just as one would expect from Born's statistical law: At every moment of time, the configuration of the system is $|\psi|^2$ distributed, and that is that, for most of the cases at least[2].

Motion in Ground States Ground state wave functions can be taken to be real; thus by (2) the velocity field is zero, so $\boldsymbol{Q}(t) = \boldsymbol{Q}(0)$ – the particles do not move. Many find this at first disturbing, in particular when it comes to the ground state of the hydrogen atom: First one learns the Bohr-model, where electrons move on orbits around the nucleus, then one learns that that is false because the existence of trajectories contradict the Heisenberg uncertainty principle, and now Bohmian mechanics says that the electron is at rest. This seems even worse, because the Coulomb force is supposed to act on the charge and standing still is not much of a reaction. So this is a good example to get familiar with the radically non Newtonian character of Bohmian mechanics, in which the action of forces does not play a fundamental role.

[2] Exit time, tunneling times and scattering, as well as the classical regime are situations where trajectories are helpful [11,14].

Double Slit Experiment This is one of the experiments that strongly suggests the particle ontology. Particles are sent one at a time through a double slit. There is always only one particle on the way. One can see it coming at the screen, let's say a photo plate! When it arrives it makes a black spot where it lands. At the end, after many particles have passed, the black spots add up to a pattern, which looks like an *interference* pattern of waves, and Bohmian mechanics explains that. Just look at the trajectory picture of the computer solutions (due to Dewdney et al. see [5]) of the equations (1) and (2), where the initial positions of the trajectories are chosen in a random manner. Now note the following: The trajectories curve wildly. Why? Look how the trajectories spread immediately after passing through the slit, the particles moving on straight lines guided by the spherical wave parts originating at the slits, until the two parts of the wave begin to interfere (the typical interference pattern of spherical waves builds up before the photo plate) and the particles, guided by this wave, move so as to reproduce the $|\psi|^2$-distribution. Furthermore if one considers the symmetry

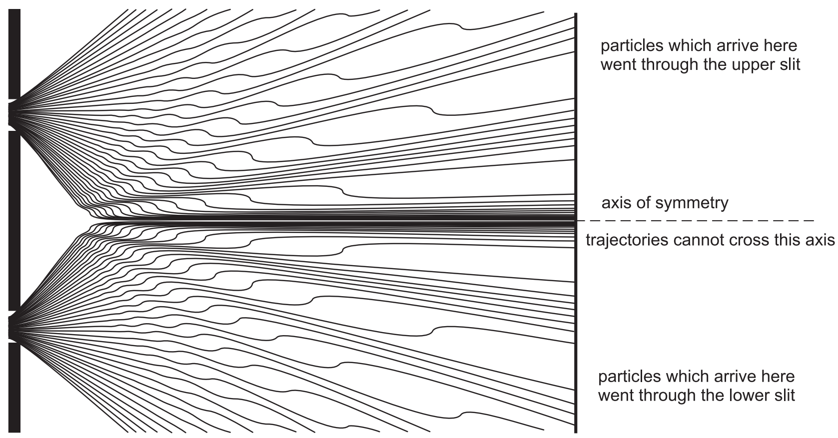

Fig. 1. Possible trajectories in the two slit experiment

line in this experiment and recalls that trajectories can't cross, one sees that trajectories starting in the upper half must go through the upper slit and must hit the upper half of the photoplate. (This is one way of observing through which slit the particle has gone without destroying the interference pattern. Just look where it arrives on the photo plate.)

Tunneling In textbooks tunneling through a barrier is often discussed by stationary wave methods. In one dimension trajectories cannot cross, and one finds immediately that in this stationary picture particle trajectories can only travel towards the barrier. This, one sometimes hears, is bad for Bohmian mechanics, because it does not describe back scattering of particles. But this is wrong. When a particle is sent towards a barrier, it is guided by a wave packet which is not stationary, and that's all. Because trajectories can't cross, if the particle is 'in front' of the packet it goes through the barrier, if it is in the back it turns back. It couldn't be nicer than that.

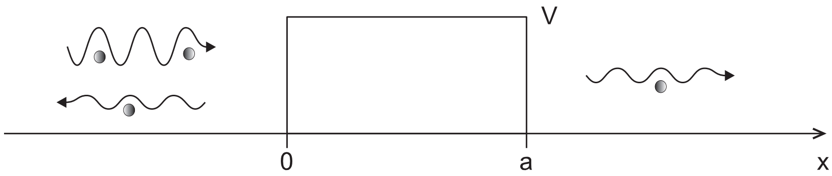

Fig. 2. Tunneling through a barrier: A potential V

Cats and Cloud Chamber Tracks A macroscopic Bohmian trajectory, which looks Newtonian, is that of the cat, when it jumps out of the box in which the atom did not decay. Other Bohmian trajectories are made visible to the eye in cloud chamber tracks.

2.2 Statistical Mechanics

Equilibrium Statistical mechanics and the kinetic theory of gases is by now well established. A famous success is Boltzmann's explanation of the second law of thermodynamics – the meaning of entropy and why it should increase. See Sheldon Goldstein's as well as Herbert Spohn's contributions to this volume. To describe a gas, a statistical description is used, and the regularities derived are not certain but merely overwhelmingly 'probable.' Nevertheless it is clear to most that a statistical description is a matter of convenience, not necessity: it is impractical to specify the initial velocities and positions of all the gas molecules, compute their motion and observe where they are later. We are simply ignorant about these details, about which we don't much care, and that's where the statistical analysis comes to the rescue.

Boltzmann characterized the initial conditions for which the thermodynamic regularities hold as overwhelmingly probable. Nowadays, it is widely recognized that it would be more appropriate to speak in terms of *typicality.* The good initial conditions are typical with respect to a measure of typicality, in the sense that, with respect to this measure, exceptions are rare – they are assigned very small weight by the measure. This measure is determined by the physical law: It must be "equivariant" with the motion of the particles, so that the notion of typicality is independent of time. Typicality is discussed and applied in the contributions by Goldstein, Kiessling and Spohn. I shall return to this in later sections, where I discuss the meaning of typicality statements in more detail.

In the Newtonian case the measure is determined by a strong form of equivariance, namely stationarity, i.e. its density is a stationary solution of the continuity equation for the Hamiltonian flow on phase space, which is called Liouville's equation for Hamiltonian or Newtonian dynamics. Liouville's equation allows for many stationary solutions, and some feel uneasy that the measure of typicality is sometimes not uniquely determined.

Nonequilibrium But we think that *in principle* we could do without statistics in a classical world, in principle we could know all the velocities and positions of all the gas molecules and we could even displace some or all of the molecules according to our taste, i.e., we could change what looks typical into something that looks atypical. More to the point: We have the impression that we could get rid of randomness altogether if we wished to do so. Why? Because our universe is in a state of global nonequilibrium (see [3])! Typicality with respect to an equivariant measure is in contrast an equilibrium notion.

If one applies this idea of complete control of initial conditions at least for small subsystems to Bohmian mechanics, it is often felt disturbing that one cannot control wave function and position of a Bohmian particle the way one would like. Born's statistical law is a law! If the wave function is ψ, the position is $|\psi|^2$-distributed. A common reaction is then this: *If the position of the particle cannot be experimentally controlled and given a sharp value, no matter what the wave function is, then what are the particles good for? Then they play no physical role!* As superficial as this argument is, we may as well apply it to the universe. *We humans* cannot experimentally control the initial conditions of the stars in the milky way, hence the stars play no physical role. Should we deny then that they *are*?

I think what is felt as disturbing is that now it seems that there is a fact to be explained: Why is it that Born's statistical law holds without fail, while the equidistribution of gas molecules in a container may fail to hold (by prior proper manipulation from outside)[3]?

Boltzmann's answer to this could turn out to be surprisingly simple: Maybe the statistical law of Born arises because it is 'overwhelmingly probable,' or, better, typical. Because if that were the case then it would explain why it does not fail (see also Goldstein's article) – no more than the second law of thermodynamics, no more than it is possible to build perpetual motion machines. And it would explain why the name *Born's law* is properly chosen.

2.3 Statistical Bohmian Mechanics

In [3] we showed that the empirical import of Bohmian mechanics emerges from typicality. What is typical is the distribution of particle positions. But before

[3] So as long as one does not talk about particles, i.e. as long as Born's statistical law is not about a distribution of particles but about results of 'observations,' it has seemed acceptable. On what grounds? On no grounds.

I explain more, I recall that in orthodox quantum theory it is often said that randomness is intrinsic. What is meant by that?

Observe that in orthodox quantum theory only the Schrödinger equation (1) appears and that equation has no intrinsic randomness. (The collapse models, as discussed by Rimini in this volume could serve as examples for an intrinsically random theory). The Schrödinger evolution thus cannot account for the random spots on the photo plate in the two slit experiment. (Moreover, it cannot account for any fact, random or nonrandom.) This is related to the measurement problem of orthodox quantum theory, which I describe next.

The Collapse The evolution of a system coupled to a measurement device in which the wave function of the device (Φ) becomes correlated with particular wave functions ($\psi_k, k = 1, 2, ...$) of the system,

$$\psi_k(x)\Phi_0(y) \longrightarrow \psi_k(x)\Phi_k(y), \tag{3}$$

is called a measurement process, where the arrow indicates the evolution of the wave function of the combined system: the measured system and the apparatus. The Φ_k are wave functions corresponding to distinct pointer positions, and Φ_0 is the ready state. Then by the linearity of the Schrödinger equation

$$\sum_k a_k\psi_k(x)\Phi_0(y) \longrightarrow \sum_k a_k\psi_k(x)\Phi_k(y). \tag{4}$$

That is what comes out of the theory – a macroscopic superposition of pointer wave functions. That is not familiar in nature and that is the measurement problem. It is 'solved' by talk: The *observer* collapses the sum in (4) to only one of the wave functions, let's say to $\psi_{k_0}(x)\Phi_{k_0}(y)$. And he does so with the probability $|a_{k_0}|^2$. But no one knows why he should do that. That is the *intrinsic* part. That is why there is nothing to show. It's intrinsic.

The observer is supposed to do two things. He creates facts and he does so in a random fashion, but at the same time obeying Born's statistical law. So he must be very smart! That is why Bell asked whether the observer must have a degree in physics – who else would know about Born's statistical law? But it is clear that the measurement problem cannot be talked away, it is not a philosophical issue. It is a hardware problem!

It is worthwhile to understand the measurement situation in Bohmian mechanics. The Bohmian trajectories of the system and apparatus particles are in configuration space made of the system-coordinates X and the apparatus-coordinates Y. Then look at the following picture:

When the apparatus wave functions have sufficiently disjoint supports, then it is very difficult to have them interfere again. This is because of purely thermodynamical considerations concerning the number of degrees of freedom which have to be controlled to achieve interference in the future. Some have difficulties with such arguments, especially if they go further, to include the recordings of the pointer positions on paper. That is, when the wave function is now one which

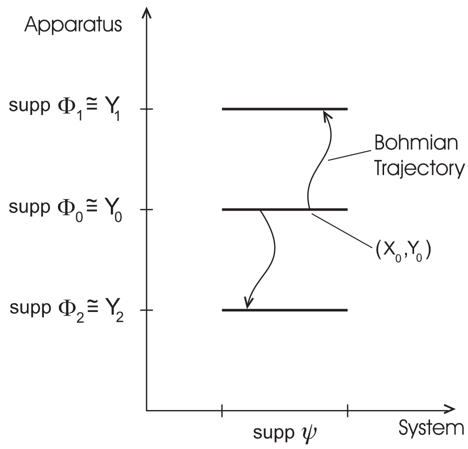

Fig. 3. The apparatus axis represents the configuration space of Y coordinates, the system axis that of the X coordinates. Since the supports of the apparatus wave functions Φ_k are sufficiently disjoint, corresponding, as they do, to macroscopically different pointer positions, only the term in (4) in whose support Y is situated is relevant to the behavior of the system, so that as a practical matter all but one of the terms in (4) may be ignored.

is even more macroscopically entangled, including patterns of ink on paper. Everyone would agree that it would be exceedingly difficult to produce interference which could turn such facts into fakes, but that it is all quantum mechanics, and that it is only because of thermodynamics that these funny things do not happen, that not everyone agrees about. In fact, in the collapse models, discussed by Rimini, these funny things cannot happen. The wave function does collapse by law. But in Bohmian mechanics it is simply a matter of convenience to proceed with the components of the wave function that will guide the particles from now on and forget about the other parts of the wave function. After all, what we really care about is what the particles are doing; this, according to Bohmian mechanics, is what quantum theory is fundamentally about, and it is out of the behavior of the particles that physical phenomena emerge. The collapse is not a physical process, but a matter of pragmatism.[4]

This explains the collapse, but the randomness, where does that come from? That is what I address in the last section.

Decoherence A note on the side: The Schrödinger equation determines the evolution of the wave function of the N-particle system. When this system 'interacts' with another system, of, say, M particles, the wave function for the combined system lives on the configuration space of the $M + N$ particles (as in

[4] At the same time, it is true that the collapse of the wave function can be regarded as a physical process for Bohmian mechanics, but the wave function that collapses is the effective wave function of a subsystem of a larger system, see Definition 1 below.

(3),(4).) This should not be seen as extra requirement supplementing the theoretical description but rather as a consequence of the basic equations (1),(2). I shall return to this point in the next section. In any case, as we just said, the interaction of one system with another brings about the entanglement of their wave functions, and the more complex the entanglement becomes, in particular when the entanglement becomes macroscopic, interference between terms of the resulting superposition becomes practically impossible. In (4) it is certainly the case that interference of the different pointer positions is practically impossible, and this practical impossibility gets stronger with further interactions, since pointers are usually in some room full of radiation, air, flies and all kinds of interacting environments. This aspect of entanglement is now being called 'decoherence'. It is a prerequisite for the collapse to do no harm: For all practical purposes one may forget about macroscopic superpositions. And it is a prerequisite for the formulation of the measurement problem, and in and of itself in no way its solution.

Subsystems Bohmian mechanics is, like all fundamental physical theories, a theory of the Universe – of a Bohmian particle universe, nonrelativistic and so on. So what we are dealing with in everyday life are subsystems of this universe and subsystems of subsystems, and what we want to have is a good description of subsystems, in fact we would like to have Bohmian subsystems, meaning that there is a Schrödinger wave function which guides the particles of the subsystem, at least for some time. What I imply here is indeed that the wave function which enters (1),(2) as defining equations of Bohmian mechanics is the wave function of the largest system conceivable – the universe. What we need to have is a wave function of a hydrogen atom, of a molecule, of many molecules, of pointers, of cats. How can we get such a thing, a wave function for a system of interest, which is a subsystem, almost always, of a larger system?

Let us denote the mother of all wave functions – the wave function of the universe – by $\Psi(q)$ and let us write x for the system-coordinates and y for the environment-coordinates, describing the rest of the universe, everything which is not the system. So we have a splitting $q = (x, y),\ \ Q = (X, Y)$. Recall that the small letters stand for generic variables and the capital letters for the actual configuration of the particles. By inspection of (1), (2) and recalling the last sections, one is quickly lead to the following concept:

Definition 1. Let

$$\Psi(q) = \varphi(x)\Phi(y) + \Phi^{\perp}(q)$$

with Φ and $\Phi^{\perp}$ having (macroscopically) disjoint y-supports and suppose $Y \in$ supp Φ. Then φ is the **effective** wave function of the system.

Easy to check: The effective wave function of a system governs the motion of its configuration $X(t)$. Moreover, by virtue of the decoherence effects discussed in the **decoherence** and **collapse** sections, as long as a system is decoupled from its environment, the evolution of its effective wave function will be governed by the Schrödinger equation (1) for that system. The effective wave function is the precise substitute for the collapsed wave function in orthodox quantum theory, so one should feel comfortable with this notion.

Quantum Equilibrium Hypothesis What comes now is technically very easy but at the same time misunderstood by many. This section is about the foundations of statistical Bohmian mechanics (in fact of the statistical analysis of any physical theory). Where does the randomness come from?

Recall the double slit – or the Stern–Gerlach – experiment. What is striking is that we have regular random outcomes: For the double slit we get relative frequencies of black points on the photo plate which are given by Born's law, and for the Stern Gerlach experiment we can get outcomes as in a coin tossing, i.e., in a long run of sending many particles through the inhomogeneous magnetic field, half go up and half down. How can this lawful random behavior be accounted for? Easy: By the law of large numbers for independent random positions of the particles. Just as one would argue in classical physics: Given some *initial randomness – that is, the statistical ansatz* – the instability of the motion should produce enough independence for a law of large numbers statement to hold (this was made very clear in [15]).

Having said this, a hard question seems to remain: What is responsible for the randomness of the *initial conditions*? Since in a deterministic physical theory there is apparently no room for genuine randomness, it would seem that it must come from 'outside'. Thus the usual answer is: Systems are never closed and from the 'outside' come random disturbances. But after including the 'outside' we are dealing only with a bigger physical system, itself physical and deterministic; thus we have merely shifted the question of the origin of randomness to a bigger system, and so it continues and one hopes that eventually the question will evaporate into empty space. But it does not and only if one continues to ask can one know for sure that the shift to include the 'outside' is a dead end. The biggest system conceivable, the universe, has no outside. It is for the universe that the question of the origin of the randomness must be answered! The answer was given by Boltzmann. It is that there is no need for (and, more to the point, no sense to) randomness in the initial conditions of the universe. Rather, a random pattern of events emerges from the deterministic evolution from a fixed initial condition for the universe – provided that initial condition is typical. I shall explain this a bit more here, but for a deeper understanding the reader should consult [3].

We begin by recalling the well-known fact that the 'quantum equilibrium' distribution $|\psi|^2$ for the configuration of a system has, mathematically, a special status within Bohmian mechanics: It is a quasi-invariant or, more precisely, equivariant distribution, analogous to the microcanonical distribution of statistical mechanics. Consider the continuity equation of the Bohmian flow, analogous to Liouville's equation for a Hamiltonian flow,

$$\partial_t \rho(q,t) + \nabla \cdot \boldsymbol{v}^{\varphi_t} \rho(q,t) = 0. \tag{5}$$

Now $\boldsymbol{v}^{\varphi_t} = \frac{\boldsymbol{j}^{\varphi_t}}{|\varphi_t|^2}$, with $\boldsymbol{j}^{\varphi_t}$ the quantum flux, which satisfies an identity holding for solutions of Schrödinger's equation

$$\partial_t |\varphi_t|^2 + \nabla \cdot \boldsymbol{j}^{\varphi_t} = 0. \tag{6}$$

This is roughly analogous to the fact that a Hamiltonian velocity field is divergence free which – used on the continuity equation for the Hamiltonian flow – yields Liouville's equation, for which the usual stationary solution is obvious.

Here, (6) may be rewritten

$$\partial_t |\varphi_t|^2 + \nabla \cdot \boldsymbol{v}^{\varphi_t} |\varphi_t|^2 = 0 \tag{7}$$

so that $\rho(q,t) = |\varphi(q,t)|^2$ satisfies (5) and thus defines an equivariant distribution: It does not change its form as a functional of φ_t during the evolution ($\equiv$ *equivariance*).

Just as in statistical mechanics, this simple mathematical fact leads naturally to the *Quantum Equilibrium Hypothesis (QEH):*
'If a system has effective wave function φ then its configuration has $|\varphi|^2$.' Much in the spirit of Gibbs, this may be regarded merely as a reasonable statistical ansatz for dealing with subsystems: It is supported by the physical theory, it is simple and it works. As is the common praxis in statistical mechanics, we can think of this distribution as describing an *ensemble of identical subsystems*, obtained by sampling across space and time.

Typicality We are now at a stage analogous to the present status of statistical mechanics. It is known that the rigorous justification of the statistical hypothesis that systems tend to be microcanonically (or Gibbs) distributed is a very hard and unsolved problem. What we would like to have is a rigorous proof that normal thermodynamic behavior emerges for a sufficiently large class of initial conditions for the universe, or, failing that, for the systems of direct interest to us, at least for some initial conditions for these systems. But even the latter is way beyond our reach (see also Goldstein's contribution to this volume).

For Bohmian mechanics, however, one can provide just such a rigorous justification of the QEH; one can show, in fact, that it emerges for 'most' initial conditions for the universe. I will not go through the proof here but rather comment on misconceptions about this statement and its proof.

The statement that the QEH can be justified is a law of large number statement: What we must look at are the empirical distributions of the configuration for ensembles of small subsystems. Small here means compared with the size of the universe. A laboratory is extremely small.

Consider equations (1) and (2) for the evolution of the *universe*! So Q comprises the coordinates of all the particles of a Bohmian universe and Ψ is the mother of all wave functions: the wave function of the universe. Now let

$$\rho_{\text{emp}}^{N,\varphi}(Q,x,t) = \frac{1}{N} \sum_{i=1}^{N} \delta(X_i(Q,t) - x) \tag{8}$$

be the empirical distribution of the configuration for an ensemble of N similar subsystems (subsystem i having configuration X_i), each subsystem having *effective* wave function φ at a given time t (for simplicity I discuss here spatial ensembles at a given time. In general one has time and space ensembles [3]). Here $X_i(Q,t)$ is determined by (2) and the initial conditions Q. We emphasize that ρ_{emp} is a function of Q. Moreover, the set of Q's relevant to (8) is very much constrained by the requirement that the subsystems have effective wave function φ (see the definition of effective wave function).

Suppose now that we could show the following: There exists one initial condition Q, i.e. one evolution of a Bohmian universe, for which[5]

$$\rho_{\text{emp}}^{N,\varphi}(Q,x,t) \approx |\varphi|^2(x). \tag{9}$$

for all appropriate subsystems, wave functions φ, and times t. In other words, that Born's statistical law holds within that special universe. This would be a fantastic result, clearly demonstrating the compatibility of quantum randomness with absolute determinism. Moreover, the empirical data would encourage us to believe that we are in that universe. Nonetheless, such a result would not explain *why* Born's statistical law holds: we would need to know why we should be in that very special universe. Now take the other extreme: Suppose it turned out that for *all* initial Q' (9) is true. That would mean that Born's statistical law is a theorem following from (1) and (2). Then – provided we believe in the correctness of our theory – we must be in one of these universes and purest satisfaction would result.

But Boltzmann's analysis of the origin of the second law of thermodynamics (see the contribution of Goldstein to this volume) should have taught us that the latter is too much to expect. Moreover, it follows from our analysis in [3] that in fact it is impossible – i.e., that there must be bad initial conditions Q for which the QEH fails (see [9]). What is proven in [3], a law of large numbers theorem for (8), thus cannot be much improved upon.[6]

The usual way of formulating a law of large number result is in terms of typicality: Typically the numbers in the interval $[0,1]$ have a decimal expansion in which every digit comes with relative frequency $\approx \frac{1}{10}$. That is: Most of the numbers in the interval $[0,1]$ have a decimal expansion in which every digit comes with relative frequency $\approx \frac{1}{10}$. Now, 'most' is here with respect to the Lebesgue measure. That measure is a measure of typicality, and one may ask what its relevance is. In physics, according to Boltzmann, we are in the fortunate situation that the measure of typicality is dictated by physics: *Typicality must be stationary,* which requires the measure to be equivariant. (The stationarity of

[5] for precise formulations see [3]

[6] In statistical physics the exploration of mixing, the relaxation of a nonequilibrium distribution on the full phase space towards an equilibrium distribution, has become widespread and has led some to the belief that mixing is central to the justification of the equilibrium hypothesis. This belief is particularly attractive, since it is connected with notions like chaoticity and independence. It is nonetheless quite wrong. For more on this see the contribution of Goldstein to this volume.

the notion of typicality makes it equilibrium-like, hence 'Quantum Equilibrium'.) Thus for Bohmian mechanics typicality is defined using the $|\Psi|^2$ distribution. It gives us a notion of 'most' and 'few' in this situation, where counting is not possible.

Warning: *We used $|\varphi|^2$ to formulate the statistical hypothesis, in which it represents an empirical distribution. Now we refer to it (or, rather, to $|\Psi|^2$) to define the measure of typicality, an entirely different concept. Moreover, the wave function Ψ of the universe and the wave function φ of a subsystem are also rather different objects, and it is in a sense an accident that they seem so similar and obey the same equations – and (for good reasons) go by the same name. But while for ensembles of subsystems of one universe empirical statistics are relevant, a statistical ensemble of universes is quite another matter.*

We show that for typical solutions of (2) (the universal wave function is assumed to be given) Born's law (9) is valid: the set of exceptions is assigned very small measure by the $|\Psi|^2$ distribution. The failure to distinguish between $|\varphi|^2$ as an empirical distribution and $|\Psi|^2$ as a measure of typicality leads to the following reformulation of our analysis: Consider a probability distribution of universes with density $|\Psi|^2$. Then the configuration of a subsystem with effective wave function φ is $|\varphi|^2$ distributed.[7] But this is not a very useful result, since sampling across universes is neither possible for nor relevant to humans.

More sensible, but still besides the point, is the following. Consider coin tossing. I make the following model. The outcomes of every toss are independent identically distributed random variables. I take the product measure on the probability space of sequences of outcomes. Then the law of large numbers holds: Typically, the relative frequency of heads is approximately $\frac{1}{2}$. That is 'trivially' true, by construction. It is important to appreciate how this appeal to typicality differs from Boltzmann's and from ours in [3]. Note in particular that there is no simple relationship between the initial configuration Q of the universe and the empirical statistics that emerge from Q via the dynamics (2). Note also that in general we make no claim about empirical statistics at the initial universal time, since it need not be the case that a decomposition into subsystems with the same effective wave function (or any effective wave function at all) is possible at this time.

This result justifies for me the QEH; it explains why Born's law holds – or, what amounts to more or less the same thing, it explains why it should be expected to hold. To go further we would need to analyze just what is meant by explanation; see the contribution of Goldstein to this volume, where this is touched upon.

I remark that the operator formalism of quantum theory emerges (with surprising ease) from statistical Bohmian mechanics [4,17,18].

[7] In [16] this is referred to as the nesting property.

3 Simplicius Simplicissimus

What is typicality? It is a notion for defining the smallness of sets of (mathematically inevitable) exceptions and thus permitting the formulation of law of large numbers type statements. Smallness is usually defined in terms of a measure. What determines the measure? In physics, the physical theory. Typicality is defined by a measure on the set of 'initial conditions' (eventually by the initial conditions of the universe), determined, or at least strongly suggested, by the physical law. Are typical events most likely to happen? No, they happen because they are typical. But are there also atypical events? Yes. They do not happen, because they are unlikely? No, because the are atypical. But in principle they could happen? Yes. So why don't they happen then? Because they are not typical.

Using typicality one may define probability in terms of law of large numbers type statements, i.e., in terms of relative frequencies and empirical statistics. What is meant by: In a Stern–Gerlach experiment the probability for spin up is $\frac{1}{2}$? The same as for heads turning up in a coin tossing. So what is meant? The law of large numbers: That in a long run of repetitions of the experiment the relative frequency of the outcome in which the spin is up or the coin shows heads will typically be close to the value $\frac{1}{2}$.

Is there another sense of probability? Probably there is! Does explanation via appeal to typicality require deeper conceptual analysis? Certainly. Can appeal to typicality be entirely eliminated from scientific explanation? Very unlikely!

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