The essential spectrum of relativistic one-electron ions in the Jansen-Hess model

D. H. Jakubaßa-Amundsen

Mathematics Institute, University of Munich
80333 Munich, Germany

Abstract

It is shown that the essential spectrum of the pseudo-relativistic Dirac operator according to Jansen and Hess which includes the Coulomb potential up to second order, extends from mc^2 to infinity when the nuclear charge is below the critical value $Ze^2 \approx 1.006$. There is also no singular continuous spectrum in that case, and for small Z no embedded eigenvalues. This work is an extension of investigations by Evans, Perry and Siedentop on the Brown-Ravenhall operator which is of first order in the potential. It is based on the fact, recently proven by Brummelhuis, Siedentop and Stockmeyer, that the spectrum of the Jansen-Hess operator is bounded from below for subcritical charges Z.

1 Introduction

The Dirac operator of an electron with mass m and momentum $\mathbf{p}$ in an external field V is given by (in relativistic units, $\hbar = c = 1$)

$$H \;=\; \boldsymbol{\alpha}\mathbf{p} \;+\; \beta m \;+\; V \tag{1.1}$$

where $\boldsymbol{\alpha}$ and β are the Dirac matrices and in the coordinate representation of H, $\mathbf{p}$ has to be identified with $-i\partial/\partial\mathbf{x}$. However H, acting on the Hilbert space $L_2(\mathbb{R}^3) \times \mathbb{C}^4$, is not bounded from below. In the case of a Coulomb potential,

$$V(x) \;=\; -\,\frac{\gamma}{x}, \qquad\qquad \gamma := \; Ze^2 \tag{1.2}$$

(with $x := |\mathbf{x}|$, Z the nuclear charge number of the ion and $e^2 = (137.04)^{-1}$ the fine structure constant), where the exact eigenfunctions are known, this difficulty may be circumvented by introducing the projection operator P_+ onto the positive spectral subspace of H and considering the bounded operator $P_+H\,P_+$ instead which has the same positive-energy eigenstates as H [15].

For noncoulombic central potentials the eigenfunctions and hence P_+ are unknown. An approximation to $P_+H\,P_+$ was considered by Brown and Ravenhall [3] who introduced the operator (see also [15])

$$B := \; \Lambda_+ H\,\Lambda_+, \qquad\qquad \Lambda_+(\mathbf{p}) := \; \frac{1}{2}\left(1 \;+\; \frac{\boldsymbol{\alpha}\mathbf{p} \;+\beta m}{E_p}\right) \tag{1.3}$$

where Λ_+ projects onto the positive spectral subspace of the *free* Dirac operator, E_p being the energy of the electron. By construction, B consists of a zero-order and a first-order term in the potential V, and one may define an operator $b_{0m} + b_{1m}$ acting on the two-dimensional space $L_2(\mathbb{R}^3) \times \mathbb{C}^2$ by using the fact that any four-spinor $\psi \in \mathcal{D}(B)$ can be represented in terms of a Pauli spinor $u \in L_2(\mathbb{R}^3) \times \mathbb{C}^2$. One identifies [6]

$$(\psi, B\,\psi) \;=:\; (u, (b_{0m} + b_{1m})\,u) \tag{1.4}$$

where b_{0m} and b_{1m} denote the corresponding zero- and first-order contributions in V, respectively

$$\begin{aligned} b_{0m} &:= \; E_p := \; \sqrt{p^2+m^2} \\ b_{1m} &:= \; A(p)\;[V \;+\; RVR]\;A(p) \end{aligned} \tag{1.5}$$

with $A(p) := \sqrt{\dfrac{E_p+m}{2E_p}}$, $\quad R := h(p)(\boldsymbol{\sigma}\hat{\mathbf{p}}),\quad h(p) := \dfrac{p}{E_p+m}$
where $\hat{\mathbf{p}} := \mathbf{p}/p$, $p := |\mathbf{p}|$ and $\boldsymbol{\sigma}$ is the vector of the Pauli matrices.

A method to construct an operator which approximates $P_+H\,P_+$ to higher order in V is the Foldy-Wouthuysen transformation technique [7]. It consists of a series of unitary transformations successively applied to H which cast H into a block-diagonal form to any given order in V, leading (within this order) to a decoupling of the positive and negative spectral subspaces of H.

Following Douglas and Kroll [5], the transformed operator is defined by

$$(U_n...U_1U_0)\; H\; (U_n...U_1U_0)^{-1} \;=:\; H^{(n)} \;+\; O(V^{n+1}) \tag{1.6}$$

where $H^{(n)}$ has block-diagonal structure, each block acting on $L_2(\mathbb{R}^3)\times\mathbb{C}^2$. The first transformation, U_0, is the free-particle Foldy-Wouthuysen transformation [2]

$$U_0 \;=\; A(p)\;(1 \;+\; \beta R_0), \qquad\qquad R_0 := \; h(p)\;(\boldsymbol{\alpha}\hat{\mathbf{p}}) \tag{1.7}$$

which casts the zero-order term of H into block-diagonal form while the non-(block)diagonal remainder is of first order in V. The transformations $U_1,...,U_n$ have the form

$$U_i \;=\; (1 \;+\; W_i^2)^{1/2} \;+\; W_i, \qquad\qquad i=1,...,n \tag{1.8}$$

where the anti-hermitean operators W_i are successively constructed from the requirement that the non-(block)diagonal terms of order i vanish. An explicit expression for $H^{(2)}$ was provided by Douglas and Kroll [5], but was later corrected by Jansen and Hess [11]. Its upper block, termed b_m, which corresponds to the positive spectral subspace, agrees with $b_{0m}+b_{1m}$ from (1.4) up to first order in V, but has an additional second-order term

$$b_m \;=\; b_{0m} \;+\; b_{1m} \;+\; b_{2m}$$

$$b_{2m} := \;\frac{1}{2}\;(w_{1m}O_1 \;-\; O_1w_{1m}) \tag{1.9}$$

where $O_1 \;:=\; A(p)\,(RV-VR)\,A(p)$, and w_{1m} is an integral operator linear in V, defined by $w_{1m}E_p + E_pw_{1m} \;=O_1$. It should be noted that due to the particular structure (1.8) of the transformation U_2 combined with the linearity of W_2 in V, b_m is unaffected by U_2.

The Coulomb case (1.2) is well suited for investigating the quality of the above approximations since one can compare with the exact solutions. For this case, the spectral properties of the Brown-Ravenhall operator $b_{0m}+b_{1m}$ were in detail studied by Evans and coworkers [6, 1], and the boundedness from below of the Jansen-Hess operator b_m for subcritical charges γ was recently proven by Brummelhuis et al [4]. In the present work we want to prove, for the Coulomb potential (1.2),

Theorem 1.1 *Let the critical coupling constant* $\gamma_c \approx 1.006$ *be defined as the smaller solution of* $1-\gamma/2\;(\pi/2\;+2/\pi)\;+\gamma^2/8\;(\pi/2\;-2/\pi)^2\;=0$. *If* $\gamma<\gamma_c$,

(i) the essential spectrum of b_m *is given by* $\sigma_{ess}(b_m)\;=\sigma_{ess}(b_{0m})\;=[m,\infty)$,

(ii) the singular continuous spectrum of b_m is empty.

Evans et al [6] proved Theorem 1.1 for the operator $b_{0m} + b_{1m}$, the critical coupling constant being $2/(\pi/2 \ + 2/\pi) \ \approx 124.16 \, e^2$, and Balinsky and Evans [1] showed that above $\max\{m, m(2\gamma - \frac{1}{2})\}$ there are no embedded eigenvalues in the essential spectrum. By using similar techniques we extend their results to the second-order term b_{2m}.

2 Preliminaries

Let $\varphi \in \mathcal{S}(\mathbb{R}^3) \times \mathbb{C}^2$ be a spinor in the Schwartz space of smooth strongly localised functions. Then one can define the energy of the electron as the expectation value of b_m,

$$E_m(\varphi) := \ (\varphi, b_m \varphi) \ = \ \sum_{i=0}^{2} (\varphi, b_{im} \varphi). \tag{2.1}$$

The strategy of this work is to base proofs on the results for the case of massless particles, i.e. for $m = 0$ in (1.1) and in the subsequent equations. (We will drop the index m if reference is made to the $m = 0$ case). When $m = 0$, $\ b_0 = p$, and simple scaling properties are found to hold.

It was proven by Brummelhuis et al [4] that the difference between the energies in the massive and in the massless case is bounded, i.e.

$$|E_m(\varphi) \ - E(\varphi)| \ \leq \ m \, d \, \|\varphi\|^2, \qquad\qquad d > 0. \tag{2.2}$$

From this it is easily shown that b_m is form bounded from below because of the positivity of b, i.e. $(\varphi, b \, \varphi) \ \geq 0$ for $\gamma \leq \gamma_c$ [4]:

$$E_m(\varphi) \ \geq \ -m \, d \, \|\varphi\|^2 \ + \ (\varphi, b \, \varphi) \ \geq \ -m \, d \, \|\varphi\|^2 \tag{2.3}$$

with γ_c from Theorem 1.1. Since b_m is symmetric (being a function of $\mathbf{p}$ and its conjugate $\mathbf{x}$), (2.3) allows for the Friedrichs extension of b_m to a self-adjoint operator on the Hilbert space $L_2(\mathbb{R}^3) \times \mathbb{C}^2$. In the following, the notation b_m will always imply its Friedrichs extension.

Introducing the resolvents $(b_m + \mu)^{-1}$ and $(b_{0m} + \mu)^{-1}$, their boundedness for a suitably chosen $\mu > 1$ follows from the strict positivity of $(b_m + \mu)$ and $(b_{0m} + \mu)$ which is trivial for the latter operator and is a consequence of (2.3) for $b_m + \mu$:

$$(\varphi, (b_m + \mu)\varphi) \ \geq \ -m \, d \, (\varphi, \varphi) \ + \ \mu(\varphi, \varphi) \ > 0 \qquad \text{for } \mu \ \geq m \, d + 1. \tag{2.4}$$

For the proof of Theorem 1.1(i) it suffices to show [6] that the resolvent difference

$$R_m(\mu) := \; (b_m+\mu)^{-1} \; - \; (b_{0m}+\mu)^{-1} \qquad \text{is compact} \tag{2.5}$$

for μ from (2.4). It then follows that the essential spectra of b_m and b_{0m} coincide [14, p.112].

The proof of the compactness (2.5) is based on Lemma 2.6 of Herbst [10] where it is shown that

$$(b_{0m}+\mu)^{-1}\,\frac{1}{\sqrt{x}} \qquad \text{is compact} \tag{2.6}$$

for all $m \geq 0$ and $\mu \geq 1$. Another ingredient of the proof of (2.5) is the b_{0m}-form boundedness of $b_{1m}+b_{2m}$ with relative bound c less than one, i.e.

$$|(\varphi,(b_{1m}+b_{2m})\varphi)| \;\leq\; c\,(\varphi, b_{0m}\varphi) \;+\; C\,(\varphi,\varphi) \tag{2.7}$$

$$\forall\,\varphi \in Q_2, \qquad \text{with}\;\; 0<c<1,\;\; C\in\mathbb{R}$$

where $Q_2 := H_{1/2}(\mathbb{R}^3)\times\mathbb{C}^2$ is the form domain of b_{0m} (with $H_{1/2}(\mathbb{R}^3) := \{\varphi\in L_2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2\,(1+p^2)^{1/2}\,d\mathbf{p} \;<\infty\}$ in the $\mathbf{p}$-space representation). With (2.7), b_m may be defined as a form sum of b_{0m} and $(b_{1m}+b_{2m})$ with coinciding form domain Q_2 for b_m and b_{0m}.

The form boundedness (2.7) can be obtained from the respective form boundedness in the massless case. To show the latter, the following lemma is needed:

Lemma 2.1 *Let* $b := b_0+b_1+b_2$ *the Jansen-Hess operator for* $m=0$. *If* $\gamma \leq 4/\pi$,

$$(\varphi, -b_1\,\varphi) \;\geq\; (\varphi, b_2\,\varphi) \qquad \forall\,\varphi\in Q_2. \tag{2.8}$$

Due to the scaling properties of b in the massless case (discussed in the beginning of section 4), the proof can readily be carried out in Mellin space using the techniques from [4] and is given in Appendix A. It should be noted in passing that a corresponding inequality for the kernels, $-b_1(\mathbf{p},\mathbf{p}') \;\geq\; b_2(\mathbf{p},\mathbf{p}')$ with $b_i(\mathbf{p},\mathbf{p}')$ defined in (3.2) with $m=0$, does not hold; in fact one can show that for $p'=0,\; p\neq 0,\;\; b_2(\mathbf{p},\mathbf{p}')$ is dominating over $-b_1(\mathbf{p},\mathbf{p}')$ for any positive value of the coupling constant γ (while, e.g. for $\mathbf{p}=\mathbf{p}'\neq 0,\;\; -b_1(\mathbf{p},\mathbf{p}')$ is larger than $b_2(\mathbf{p},\mathbf{p}')$).

In the massless case, the form boundedness (2.7) is easily derived with the help of (2.8) and the positivity of b: In the Mellin space representation (A.4) of $(\varphi, b\,\varphi)$ one can show [4] that for $\gamma<\gamma_c,\;\; (b_0^\sharp+\sqrt{2\pi}\,b_{ls}^{(1)\sharp}+\sqrt{2\pi}\,b_{ls}^{(2)\sharp})(t-i/2) \;\geq\; \epsilon>0$ and hence

$$(\varphi, b\varphi) \;\geq\; \epsilon\,(\varphi, b_0\varphi) \tag{2.9}$$

whereas $\epsilon = 0$ for $\gamma = \gamma_c$. Therefore

$$(\varphi, (-b_1 - b_2)\,\varphi) \;\leq\; (1-\epsilon)\,(\varphi, b_0\,\varphi) \tag{2.10}$$

which proves (2.7) in the $m = 0$ case for $\gamma < \gamma_c$.

An additional element in the proof of boundedness of operators necessary to show the compactness (2.5) is the estimate derived by Lieb and Yau [13] which we give in a slightly generalised form:

Let $K(\mathbf{p}, \mathbf{p}') \;= K(\mathbf{p}', \mathbf{p}) \;\geq 0$ *be a symmetric kernel,* $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^3$, *and let* $f(p) > 0$ *for* $p > 0$ *a smooth convergence inducing function. Then*

$$\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{p}\, d\mathbf{p}'\, \overline{\varphi(\mathbf{p})}\, K(\mathbf{p}, \mathbf{p}')\, \psi(\mathbf{p}') \right| \leq \left(\int_{\mathbb{R}^3} d\mathbf{p}\, |\varphi(\mathbf{p})|^2\, I(\mathbf{p}) \int_{\mathbb{R}^3} d\mathbf{p}\, |\psi(\mathbf{p})|^2\, I(\mathbf{p}) \right)^{\frac{1}{2}}$$

$$I(\mathbf{p}) := \int_{\mathbb{R}^3} d\mathbf{p}'\; K(\mathbf{p}, \mathbf{p}')\; \left| \frac{f(p)}{f(p')} \right|^2. \tag{2.11}$$

This estimate relies on Schwarz's inequality. Conventionally, one takes $f(p) := p^\alpha$ with a suitable $\alpha > 0$ to investigate the convergence of I.

For the proof of Theorem 1.1(ii) dilation analyticity is used [6]. One defines for $\theta := e^\xi$ with $|\xi| < \xi_0,\ \xi \in \mathbb{R}$, the unitary dilation group on $L_2(\mathbb{R}^3) \times \mathbb{C}^2$ by

$$d_\theta\; \varphi(\mathbf{p}) :=\; \varphi_\theta(\mathbf{p}) :=\; \theta^{-3/2}\; \varphi(\mathbf{p}/\theta) \tag{2.12}$$

Then one extends θ to the domain $D_0 := \{\theta = e^\xi :\ \xi \in \mathbb{C},\ |\xi| < \xi_0\}$ with $\xi_0 > 0$ to be chosen later, and defines the dilated operators

$$b_{m,\theta} :=\; d_\theta\; b_m\; d_\theta^{-1}. \tag{2.13}$$

For $\theta \in D_0$, d_θ is no longer unitary, neither is $b_{m,\theta}$ self-adjoint.

We have to show that $b_{m,\theta}$ is an analytic operator in D_0. Since we want $b_{m,\theta}$ to be defined as a form sum on $H_{1/2}(\mathbb{R}^3) \times \mathbb{C}^2$, analyticity requires that ([6], [14, p.20])

$$b_{1m,\theta} \;+\; b_{2m,\theta} \qquad \text{is } b_{0m,\theta}\text{-form bounded} \tag{2.14}$$

with relative bound less than one $\quad \forall\, \theta \in D_0$, where $b_{im,\theta} := d_\theta\, b_{im}\, d_\theta^{-1}$, $\quad i = 0, 1, 2$.

Moreover, for $b_{m,\theta}$ defined as an operator on $L_2(\mathbb{R}^3) \times \mathbb{C}^2$, one must show that

$$(b_{0m} + \mu)^{-1/2}\; (b_{1m,\theta} + b_{2m,\theta})\; (b_{0m} + \mu)^{-1/2} \qquad \text{is an analytic family in } D_0. \tag{2.15}$$

Taking $\varphi \in \mathcal{S} \times \mathbb{C}^2$ such that $d_\theta \varphi$ is analytic in D_0, one can extend the formula

$$(\varphi, \frac{1}{b_m - z}\varphi) \;=\; (d_\theta\varphi, \frac{1}{b_{m,\theta} - z}\; d_\theta\varphi), \qquad\qquad z \in \mathbb{C}\backslash\mathbb{R} \tag{2.16}$$

which for $\theta \in D_0 \cap \mathbb{R}$ is based on the unitarity of d_θ, to $D_0 \subset \mathbb{C}$ because analyticity of the r.h.s. allows application of the identity theorem from the theory of complex functions. Since $\mathcal{S}$ is dense in $H_{1/2}(\mathbb{R}^3)$ [8, p.192], (2.16) holds for all $\varphi \in H_{1/2}(\mathbb{R}^3) \times \mathbb{C}^2$ (although $H_{1/2}(\mathbb{R}^3)$ itself is not invariant under complex dilations).

Furthermore we need to show that

$$R_{m,\theta}(\mu) := \; (b_{m,\theta} + \mu)^{-1} \; - \; (b_{0m,\theta} + \mu)^{-1} \qquad \text{is compact} \tag{2.17}$$

for μ defined in (2.4). Then, following the argumentation of [6], one can use Lemma 3 of [14, p.111] together with the strong spectral mapping theorem ([14, p.109]) to prove that the essential spectra of $b_{0m,\theta}$ and $b_{m,\theta}$ coincide. (Note that the additional condition for Lemma 3 to hold, a nonempty resolvent set of $(b_{m,\theta}+\mu)^{-1}$ containing inner points in $\mathbb{C}$, follows from the boundedness of this operator in D_0 as shown in section 5.2). Since $\sigma_{\text{ess}}(b_{0m,\theta})$ and hence $\sigma_{\text{ess}}\,(b_{m,\theta})$ is a curve in the complex plane intersecting $\mathbb{R}$ only in the point m, we have

$$\lim_{\text{Im}\; z \to 0} \; \text{Im}\; (\varphi, \frac{1}{b_m - z}\, \varphi) \; < \; \infty \tag{2.18}$$

by means of (2.16) except when $\text{Re}\; z$ coincides with isolated points of $\mathbb{R}_+$ (namely m or eigenvalues of b_m). (2.18) implies that the singular continuous spectrum of b_m is absent in $[m, \infty)$ ([14, p.137,186]; [10]).

3 Representations of b_m

We start by selecting the momentum representation and define integral operators $b_{im}(\mathbf{p}, \mathbf{p}'),\;\; i = 1, 2$ by means of

$$(\varphi, b_m \varphi) \;=\; \int_{\mathbb{R}^3} d\mathbf{p}\; \overline{\varphi(\mathbf{p})}\; b_{0m}(p)\; \varphi(\mathbf{p}) \tag{3.1}$$

$$+ \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{p}\; d\mathbf{p}'\; \overline{\varphi(\mathbf{p})}\; [b_{1m}(\mathbf{p}, \mathbf{p}') \;+\; b_{2m}(\mathbf{p}, \mathbf{p}')]\; \varphi(\mathbf{p}')$$

where we have indicated explicitly the p-dependence of b_{0m}. These operators were calculated from (1.5) by Evans et al [6] and from (1.9) by Brummelhuis et al [4],

$$b_{1,m}(\mathbf{p}, \mathbf{p}') := \; -\frac{\gamma}{2\pi^2}\; \frac{1}{|\mathbf{p} - \mathbf{p}'|^2}\; [1 \;+\; \boldsymbol{\sigma}\hat{\mathbf{p}}\, \boldsymbol{\sigma}\hat{\mathbf{p}}'\, h(p)\, h(p')]\; A(p)\, A(p')$$

$$b_{2m}(\mathbf{p}, \mathbf{p}') := \; \frac{1}{2}\; \left(\frac{\gamma}{2\pi^2}\right)^2 \int_{\mathbb{R}^3} d\mathbf{p}''\; \frac{1}{|\mathbf{p} - \mathbf{p}''|^2}\; \frac{1}{|\mathbf{p}'' - \mathbf{p}'|^2}$$

$$\cdot \left[\frac{1}{E_{p'} + E_{p''}} \;+\; \frac{1}{E_p + E_{p''}}\right]\; A(p)\, A(p')\, A^2(p'') \tag{3.2}$$

$$\cdot \left[h^2(p'') - \boldsymbol{\sigma}\hat{\mathbf{p}}''\,\boldsymbol{\sigma}\hat{\mathbf{p}}'\,h(p'')\,h(p') - \boldsymbol{\sigma}\hat{\mathbf{p}}''\,\boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p'')\,h(p) + \boldsymbol{\sigma}\hat{\mathbf{p}}\,\boldsymbol{\sigma}\hat{\mathbf{p}}'\,h(p)\,h(p')\right]$$

where the factors of the type $|\mathbf{p}-\mathbf{p}'|^{-2}$ result from the momentum-space representation of the potential (1.2). We note that for $m = 0$, $h(p) = 1$ and $A(p) = \frac{1}{\sqrt{2}}$ while in the general case, $h(p) \in [0,1]$ and $A(p) \in [\frac{1}{\sqrt{2}}, 1]$ are also bounded.

For the proof of the compactness (2.5) it is advantageous to choose an $\mathbf{x}$-space representation. Identifying again $\mathbf{p}$ with $-i\partial/\partial\mathbf{x}$ (and p with $(-\Delta)^{\frac{1}{2}}$), b_{1m} and b_{2m} can be written in the following way

$$b_{1m} \;=\; -\gamma\, A(p)\;\left[\frac{1}{x}\,A(p) \;+\; h(p)\,\boldsymbol{\sigma}\hat{\mathbf{p}}\frac{1}{x}\boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p)\,A(p)\right]$$

$$\begin{aligned}
b_{2m} \;=\; \left(\frac{\gamma}{2\pi}\right)^2\, A(p)\;\Big[&\frac{1}{x}A^2(p)\,h^2(p)\,W_{10,m} \;+\; W_{10,m}A(p)\,h^2(p)\frac{1}{x}A(p)\\
&-\frac{1}{x}A^2(p)\,h(p)\,\boldsymbol{\sigma}\hat{\mathbf{p}}\,W_{11,m} \;-\; W_{11,m}A(p)\frac{1}{x}A(p)\,\boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p)\\
&-\boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p)\frac{1}{x}\boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p)\,A^2(p)\,W_{10,m} \;-\; \boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p)\,W_{11,m}A(p)\frac{1}{x}A(p)\\
&+\boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p)\frac{1}{x}A^2(p)\,W_{11,m} \;+\; \boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p)\,W_{10,m}A(p)\frac{1}{x}A(p)\,\boldsymbol{\sigma}\hat{\mathbf{p}}\,h(p)\Big]. \qquad (3.3)
\end{aligned}$$

In the expression for b_{2m} we have introduced integral operators $W_{10,m}$ and $W_{11,m}$ which are closely related to w_{1m} as defined below (1.9). Since later a factorisation will be used for each term of (3.3), all operators but $1/x$ may be analysed in terms of their momentum representation. In $\mathbf{p}$-space representation, $W_{10,m}$ and $W_{11,m}$ are defined by

$$(W_{10,m}\,\varphi)(\mathbf{p}) := \int_{\mathbb{R}^3} d\mathbf{p}'\;\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\;A(p')\;\frac{1}{E_p+E_{p'}}\;\varphi(\mathbf{p}')$$

$$(W_{11,m}\,\varphi)(\mathbf{p}) := \int_{\mathbb{R}^3} d\mathbf{p}'\;\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\;\boldsymbol{\sigma}\hat{\mathbf{p}}'\cdot h(p')\;A(p')\;\frac{1}{E_p+E_{p'}}\;\varphi(\mathbf{p}') \qquad (3.4)$$

It is readily verified that Fourier transforming b_{1m} and b_{2m} leads to the equations (3.2).

4 The essential spectrum of b_m

In this section we show the b_{0m}-form boundedness (2.7) of $b_{1m}+b_{2m}$ as well as the compactness of $R_m(\mu)$ from (2.5) in order to prove that $\sigma_{\text{ess}}(b_m) = [m,\infty)$ (Theorem 1.1(i)). Many ingredients of these proofs would have to be repeated when dilation analyticity and the compactness of $R_{m,\theta}(\mu)$ is shown. Therefore we formulate the proofs for the generalised operators $b_{m,\theta}$ and consider (2.7) and (2.5) as the special cases for $\theta = 1$.

We start by deriving the scaling properties of $b_{m,\theta}$ defined in (2.13). For $\theta \in \mathbb{R}^+$ we have

$$E_m(\varphi) \;=\; (\varphi, b_m\,\varphi) \;=\; (d_\theta\varphi, (d_\theta\, b_m\, d_\theta^{-1})\; d_\theta\varphi) \tag{4.1}$$

and making in (3.1) the substitution $\mathbf{q} := \theta\mathbf{p}$, $\;\mathbf{q}' := \theta\mathbf{p}'$ one obtains

$$E_m(\varphi) \;=\; \int_{\mathbb{R}^3} d\mathbf{q}\; \theta^{-3/2}\; \overline{\varphi(\mathbf{q}/\theta)}\; b_{0m}(q/\theta)\; \theta^{-3/2}\; \varphi(\mathbf{q}/\theta) \tag{4.2}$$

$$+ \int_{\mathbb{R}^3} d\mathbf{q}\; \theta^{-\frac{3}{2}}\; \overline{\varphi(\mathbf{q}/\theta)} \int_{\mathbb{R}^3} d\mathbf{q}'\; \theta^{-3}\; (b_{1m}(\mathbf{q}/\theta, \mathbf{q}'/\theta) \,+ b_{2m}(\mathbf{q}/\theta, \mathbf{q}'/\theta))\; \theta^{-\frac{3}{2}}\; \varphi(\mathbf{q}'/\theta)$$

Using the definition (2.12) of $d_\theta\varphi$ we obtain upon identification with the r.h.s. of (4.1)

$$b_{0m,\theta}(p) :=\; d_\theta\; b_{0m}(p)\; d_\theta^{-1} \;=\; b_{0m}(p/\theta)$$

$$=\; \sqrt{p^2/\theta^2 \,+ m^2} \;=\; \frac{1}{\theta}\; \sqrt{p^2 \,+ (m\theta)^2} \;=\; \frac{1}{\theta}\; b_{0m\cdot\theta}(p) \tag{4.3}$$

$$b_{im,\theta}(\mathbf{p}, \mathbf{p}') :=\; d_\theta\; b_{im}(\mathbf{p}, \mathbf{p}')\; d_\theta^{-1} \;=\; \theta^{-3}\; b_{im}(\mathbf{p}/\theta, \mathbf{p}'/\theta) \;=\; \frac{1}{\theta}\; b_{im\cdot\theta}(\mathbf{p}, \mathbf{p}'),$$

$i = 1, 2,$ where the last equality results from inspection of the explicit expressions (3.2) for $b_{im}(\mathbf{p}, \mathbf{p}')$, implying that $b_{im\cdot\theta}(\mathbf{p}, \mathbf{p}')$ results from (3.2) by means of the substitutions

$$E_p \;\mapsto E_\theta(p) :=\; \sqrt{p^2 \,+ m^2\theta^2}, \qquad h(p) \;\mapsto h_\theta(p) :=\; \frac{p}{\sqrt{p^2 \,+ m^2\theta^2} \,+ m\theta}$$

$$A(p) \;\mapsto A_\theta(p) :=\; \left(\frac{\sqrt{p^2 \,+ m^2\theta^2} \,+ m\theta}{2\sqrt{p^2 \,+ m^2\theta^2}} \right)^{1/2}. \tag{4.4}$$

The definition (4.3) of the operators $b_{im,\theta}$ and $b_{im\cdot\theta}$ is readily extended to complex $\theta \in D_0$. Note the simple scaling with $1/\theta$ of the corresponding operators $b_{i,\theta}$ $\;(i = 0, 1, 2)\;$ in the massless case which follows from (4.3).

4.1 The b_{0m}-form boundedness

Let us take a general $\theta \in D_0$. Using the scaling property (4.3) we have

$$|(\varphi, (b_{1m,\theta} + b_{2m,\theta})\, \varphi)|$$

$$\leq\; \left|\frac{1}{\theta}\right| \cdot [\; |(\varphi, (b_{1m\cdot\theta} - b_1)\, \varphi)| \;+\; |(\varphi, (b_{2m\cdot\theta} - b_2)\varphi)| \;+\; |(\varphi, (b_1 + b_2)\, \varphi)|\;] \tag{4.5}$$

From the exponential form, $\theta = e^\xi$, one derives the estimate for $|1/\theta|$, valid for $\delta :=\; |\mathrm{Re}\;\xi| \;< 1,$

$$1 - \delta \;\leq\; e^{-\delta} \;\leq\; \left|\frac{1}{\theta}\right| \;=\; e^{-\mathrm{Re}\;\xi} \;\leq\; e^{\delta} \;\leq 1 + 2\delta. \tag{4.6}$$

For $\theta \in D_0$ one has $|\mathrm{Re}\,\xi| \ < \xi_0$, such that by requiring $\xi_0 < 1$ one can replace δ by ξ_0 in (4.6). The same estimates also hold for the inverse, $|\theta|$.

Using the b_0-form boundedness (2.10) of $b_1 + b_2$ we find for the last term of (4.5)

$$\left|\frac{1}{\theta}\right| \cdot |(\varphi, (b_1 + b_2)\,\varphi)| \ \leq \ (1 + 2\xi_0)\ (1 - \epsilon)\ (\varphi, b_0\,\varphi) \tag{4.7}$$

where ξ_0 can be chosen sufficiently small such that $(1 + 2\xi_0)(1 - \epsilon) =: c < 1$. Provided the first two terms in (4.5) are bounded, this proves the b_0-form boundedness of $b_{1m,\theta} + b_{2m,\theta}$ with form bound smaller than 1.

In this section we are concerned with the b_{0m}-form boundedness in the case $\theta = 1$ (the general case being deferred to section 5.1). Then we can use the results of Tix [16, Theorem 1] and Brummelhuis et al [4, Lemma 5] who have shown, by comparing massive and massless operators, the boundedness of the first- and second-order term, respectively

$$|\varphi, (b_{1m} - b_1)\,\varphi)| \ \leq \ (|\varphi|, |b_{1m} - b_1| \cdot |\varphi|) \ \leq \ m\ d_1\ \|\varphi\|^2$$

$$|(\varphi, (b_{2m} - b_2)\,\varphi)| \ \leq \ (|\varphi|, |b_{2m} - b_2| \cdot |\varphi|) \ \leq \ m\ d_2\ \|\varphi\|^2. \tag{4.8}$$

Hence, $b_{1m} + b_{2m}$ is b_0-form bounded. The b_{0m}-form boundedness (2.7) is an immediate consequence since $b_0 \ = p \ \leq \sqrt{p^2 + m^2} \ = b_{0m}$.

4.2 The compactness of the resolvent difference $R_m(\mu)$

We will show later that $(b_{0m,\theta} + \mu)^{-1}$ is bounded for θ in a suitable domain D_0. Then, following Evans et al [6] we use the second resolvent identity to write

$$(b_{m,\theta} + \mu)^{-1} \ - \ (b_{0m,\theta} + \mu)^{-1} \ = \ -(b_{0m,\theta} + \mu)^{-1}(b_{1m,\theta} + b_{2m,\theta})\ (b_{m,\theta} + \mu)^{-1}$$

$$= \ - \left[(b_{0m,\theta} + \mu)^{-1}(b_{0m} + \mu)\right] \cdot \{(b_{0m} + \mu)^{-1}(b_{1m,\theta} + b_{2m,\theta})\ (b_{0m} + \mu)^{-1/2}\}$$

$$\cdot \left[(b_{0m} + \mu)^{1/2}(b_{m,\theta} + \mu)^{-1}\right] \tag{4.9}$$

We will show that the operator in curly brackets is compact while the two operators in square brackets are bounded. Then the product of all three operators is compact.

For the case $\theta = 1$ the left operator in square brackets is unity. For the proof of the boundedness of the rightmost operator for $\theta = 1$ we use $b_m + \mu > 0$ to define the bounded square root operator $(b_m + \mu)^{-1/2}$ and we make the decomposition

$$(b_{0m} + \mu)^{1/2}(b_m + \mu)^{-1} \ = \ (b_{0m} + \mu)^{1/2}(b_m + \mu)^{-1/2}(b_m + \mu)^{-1/2}. \tag{4.10}$$

Let $\varphi \in H_{1/2}(\mathbb{R}^3) \times \mathbb{C}^2$ and define $\psi := (b_m + \mu)^{-1/2}\varphi$. Then, making use of the self-adjointness of $(b_m + \mu)^{1/2}$ (by means of its Friedrichs extension), the requirement for the boundedness of $(b_{0m}+\mu)^{1/2}(b_m+\mu)^{-1/2}$ can be expressed in the following way

$$\| \,(b_{0m}+\mu)^{1/2}(b_m+\mu)^{-1/2}\varphi\,\|^2 = \|\,(b_{0m}+\mu)^{1/2}\psi\,\|^2 = \;(\psi,(b_{0m}+\mu)\,\psi)$$
$$\le\; c_0\;\|\,\varphi\,\|^2 = \;c_0\;(\psi,(b_m+\mu)\,\psi) \tag{4.11}$$

for a suitable constant $c_0>0$.

This means that the required boundedness is proven provided the following inequality holds

$$c_0\;(\psi,(b_{0m}+b_{1m}+b_{2m}+\mu)\,\psi)\;-\;(\psi,(b_{0m}+\mu)\,\psi)\;\ge 0 \tag{4.12}$$

However, recalling that from the b_{0m}-form boundedness (2.7) one has the estimate

$$(\psi,(b_{1m}+b_{2m})\,\psi)\;\ge\;-c\;(\psi,b_{0m}\,\psi)\;-\;C\;(\psi,\psi) \tag{4.13}$$

with $1>c:=1-\epsilon$, and noting that b_{0m} and μ are nonnegative, (4.12) holds true if one chooses $c_0\ge\;1/\epsilon$ and $\mu>\max(1,\,c_0C/(c_0-1))$.

Next we show the compactness of the operator in curly brackets from (4.9). The compactness of the first-order term $(b_{0m}+\mu)^{-1}b_{1m,\theta}\;(b_{0m}+\mu)^{-1/2}$ was already shown by Evans et al [6]. So we concentrate on the second-order term

$$(b_{0m}+\mu)^{-1}b_{2m,\theta}\;(b_{0m}+\mu)^{-1/2}\;=:\frac{1}{\theta}\cdot\left(\frac{\gamma}{2\pi}\right)^2\sum_{i=1}^{8}\beta_{im\cdot\theta} \tag{4.14}$$

where $\beta_{im\cdot\theta}$ is obtained with the help of (3.3) and the scaling (4.3),

$$\sum_{i=1}^{8}\beta_{im\cdot\theta} = (b_{0m}+\mu)^{-1}A_\theta(p)\Big[\frac{1}{x}A_\theta^2(p)h_\theta^2(p)W_{10,m\cdot\theta}+W_{10,m\cdot\theta}A_\theta(p)h_\theta^2(p)\frac{1}{x}A_\theta(p)$$
$$-\;\frac{1}{x}A_\theta^2(p)\,h_\theta(p)\,\boldsymbol{\sigma}\hat{\mathbf{p}}\,W_{11,m\cdot\theta}\;-\;W_{11,m\cdot\theta}A_\theta(p)\frac{1}{x}A_\theta(p)\,\boldsymbol{\sigma}\hat{\mathbf{p}}\,h_\theta(p) \tag{4.15}$$
$$-\;\boldsymbol{\sigma}\hat{\mathbf{p}}\,h_\theta(p)\frac{1}{x}\boldsymbol{\sigma}\hat{\mathbf{p}}\,h_\theta(p)\;A_\theta^2(p)\;W_{10,m\cdot\theta}\;-\;\boldsymbol{\sigma}\hat{\mathbf{p}}\,h_\theta(p)\;W_{11,m\cdot\theta}A_\theta(p)\frac{1}{x}A_\theta(p)$$
$$+\boldsymbol{\sigma}\hat{\mathbf{p}}\,h_\theta(p)\frac{1}{x}A_\theta^2(p)W_{11,m\cdot\theta}+\boldsymbol{\sigma}\hat{\mathbf{p}}\,h_\theta(p)W_{10,m\cdot\theta}A_\theta(p)\frac{1}{x}A_\theta(p)\boldsymbol{\sigma}\hat{\mathbf{p}}\,h_\theta(p)\Big](b_{0m}+\mu)^{-\frac{1}{2}}$$

with $W_{10,m\cdot\theta}$ and $W_{11,m\cdot\theta}$ from (3.4) with the replacements (4.4). First we note that all eight terms $\beta_{im\cdot\theta}$ from (4.15) contain the coordinate $\mathbf{x}$ in the form $1/x$ and differ in their momentum dependence only by the bounded operators $\boldsymbol{\sigma}\hat{\mathbf{p}}$, $A_\theta(p)$ or $h_\theta(p)$ (their boundedness is shown in the next section). We shall only present the proof of compactness for the first term $\beta_{1m\cdot\theta}$ in detail. One can readily carry through the proof for the other seven terms using the same techniques, together with $\|\boldsymbol{\sigma}\hat{\mathbf{p}}\,\varphi\|^2\;=(\varphi,(\boldsymbol{\sigma}\hat{\mathbf{p}})^2\varphi)\;=\|\varphi\|^2$. In particular, each of the terms is found to contain the compact operator $(b_{0m}+\mu)^{-1}\, x^{-1/2}$ from (2.6) which assures compactness provided the remaining factors are bounded.

We use the commutativity of multiplication operators depending only on momentum $\mathbf{p}$ (such as $A_\theta,\ h_\theta,\ b_{0m}+\mu$) to decompose $\beta_{1m\cdot\theta}$ into

$$\beta_{1m\cdot\theta} \;=\; A_\theta(p)\;\{(b_{0m}+\mu)^{-1}\frac{1}{\sqrt{x}}\}\cdot\left[\frac{1}{\sqrt{x}}A_\theta^2(p)\,h_\theta^2(p)\;(b_{0m}+\mu)^{-1/2}\right]$$
$$\cdot\left[(b_{0m}+\mu)^{1/2}\,W_{10,m\cdot\theta}\;(b_{0m}+\mu)^{-1/2}\right] \tag{4.16}$$

For $\theta=1$, the prefactor $A(p)$ is bounded by 1. We are left to prove that the two operators in square brackets are bounded.

Let us concentrate on the first operator. Then, defining $\psi :=\ A_\theta^2(p)h_\theta^2(p)$ $(b_{0m}+\mu)^{-1/2}\varphi$ we obtain, using the inequality of Kato [12, p.307], $1/x\le\frac{\pi}{2}\,p$, and the self-adjointness of $1/x$

$$\|\,\frac{1}{\sqrt{x}}A_\theta^2(p)\,h_\theta^2(p)\;(b_{0m}+\mu)^{-1/2}\,\varphi\,\|^2 =\;(\psi,\frac{1}{x}\psi)\;\le\;\frac{\pi}{2}\;(\psi,p\,\psi)$$
$$\le\;\frac{\pi}{2}\;(\psi,(b_{0m}+\mu)\,\psi)\;=\;\frac{\pi}{2}\;\|\,A_\theta^2(p)\;h_\theta^2(p)\;\varphi\,\|^2$$
$$\le\;\frac{\pi}{2}\;\|\,A_\theta^2(p)\,\|^2\cdot\|\,h_\theta^2(p)\,\|^2\cdot\|\varphi\|^2\;=:c_1\;\|\varphi\|^2 \tag{4.17}$$

with μ from (2.4). For $\theta=1,\;\; h(p)$ is bounded by 1 such that $c_1<\infty$.

Next we show the boundedness of the operator

$$W_{10,m\cdot\theta}^\lambda :=\;(b_{0m}+\mu)^\lambda\;W_{10,m\cdot\theta}\;(b_{0m}+\mu)^{-\lambda} \tag{4.18}$$

for $\lambda=-1,\ -\frac{1}{2}$ and $\frac{1}{2}$ which are our cases of interest (actually, boundedness can be shown for $|\lambda|\,<\,\frac{3}{2}$). Defining the nonnegative kernel

$$\tilde{K}(\mathbf{p},\mathbf{p}') :=\;(b_{0m}(p)+\mu)^\lambda\;\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\;|A_\theta(p')|\;\left|\frac{1}{E_\theta(p)+E_\theta(p')}\right|\;(b_{0m}(p')+\mu)^{-\lambda} \tag{4.19}$$

and the convolution $K(\mathbf{p}',\mathbf{p}'') := \int_{\mathbb{R}^3} d\mathbf{p}\,\tilde{K}(\mathbf{p},\mathbf{p}')\cdot\tilde{K}(\mathbf{p},\mathbf{p}'')$ which is symmetric in $\mathbf{p}'$ and $\mathbf{p}''$ and also nonnegative, we obtain with the help of the Lieb and Yau formula (2.11) (with $\psi:=\varphi$ and the choice $f(p):=p^\alpha$)

$$\|\,W_{10,m\cdot\theta}^\lambda\,\varphi\,\|^2 = \int_{\mathbb{R}^3} d\mathbf{p}\;\left|(b_{0m}(p)+\mu)^\lambda\int_{\mathbb{R}^3} d\mathbf{p}'\;\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\;A_\theta(p')\right.$$
$$\left.\cdot\frac{1}{E_\theta(p)+E_\theta(p')}\;(b_{0m}(p')+\mu)^{-\lambda}\;\varphi(\mathbf{p}')\right|^2$$
$$\le\int_{\mathbb{R}^3} d\mathbf{p}\left(\int_{\mathbb{R}^3} d\mathbf{p}'\,\tilde{K}(\mathbf{p},\mathbf{p}')\,|\varphi(\mathbf{p}')|\right)^2\;\le\;\int_{\mathbb{R}^3} d\mathbf{p}'\;|\varphi(\mathbf{p}')|^2\cdot I(\mathbf{p}') \tag{4.20}$$

$$I(\mathbf{p}') := \int_{\mathbb{R}^3} d\mathbf{p}''\ K(\mathbf{p}',\mathbf{p}'')\ \left(\frac{p'^{\alpha}}{p''^{\alpha}}\right)^2$$

Thus $W^{\lambda}_{10,m\cdot\theta}$ is bounded if $I(\mathbf{p}')$ is finite for all $\mathbf{p}' \in \overline{\mathbb{R}}^3$ with a suitably chosen α.

Let us turn to the case $\theta = 1$ again. Since the integrand of $I(\mathbf{p}')$ is nonnegative, we can estimate $\tilde{K}(\mathbf{p},\mathbf{p}')$ and hence $I(\mathbf{p}')$ from above by replacing $A(p')$ by 1. In addition we need the estimates (with $\mu > 1$ and $\lambda > 0$)

$$\frac{1}{E_p + E_{p'}} \ = \ \frac{1}{\sqrt{p^2+m^2} + \sqrt{p'^2+m^2}} \ \leq \ \frac{1}{p+p'}\ ,$$

$$\begin{aligned}
(b_{0m}(p) + \mu)^{-\lambda} \ &= \ (\sqrt{p^2+m^2} + \mu)^{-\lambda} \ \leq \ (p+\mu)^{-\lambda} \ \leq \ (p+1)^{-\lambda}, \\
(b_{0m}(p) + \mu)^{\lambda} \ &\leq \ (p+m+\mu)^{\lambda} \ \leq \ ((p+1) \ + \ m+\mu+1)^{\lambda} \\
\leq \ ((p+1) \ + \ (p+1)\,(m+\mu+1))^{\lambda} \ &= \ (p+1)^{\lambda}\,(m+\mu+2)^{\lambda}
\end{aligned} \tag{4.21}$$

since $p+1 \geq 1$. With this we estimate $\tilde{K}(\mathbf{p},\mathbf{p}')$

$$\tilde{K}(\mathbf{p},\mathbf{p}') \ \leq \ (m+\mu+2)^{|\lambda|}\ (p+1)^{\lambda}\ \frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\ \frac{1}{p+p'}\ (p'+1)^{-\lambda} \tag{4.22}$$

Apart from a finite constant, the r.h.s. of (4.22) is just the corresponding kernel in the massless case (since for $m = 0$, $E_p = b_0(p) = p$, and one can take $\mu = 1$). This means that the integral $I(\mathbf{p}')$ from (4.20) can be estimated by the corresponding integral in the massless case. The finiteness of $I(\mathbf{p}')$ for $m = 0$ is shown in Appendix B.

5 Dilation analyticity

In the massless case, dilation analyticity is trivial because from (4.3), dilation of b_m is equivalent to multiplication by the bounded, analytic factor $1/\theta$. For the $m \neq 0$ case, we start by showing that the operators $A_\theta(p)$, $h_\theta(p)$, $(E_\theta(p) + E_\theta(p'))^{-1}$ are bounded for $\theta \in D_0$ with D_0 a neighbourhood of unity in the complex plane (defined below (2.12)), and we derive bounds which are related to the respective operators for $\theta = 1$. Such bounds were given by Evans et al [6] for $E_\theta(p)$. Indeed, from the estimate of $|\theta|$ given below (4.6), $1 - \xi_0 \ \leq \ |\theta| \ \leq 1 + 2\xi_0$, we get for $|\mathrm{Im}\ \xi| \ \leq \xi_0 \ \leq 1/2$ (using $1 - 2x^2 \ \leq \cos 2x \ \leq 1$)

$$(1-\xi_0)\ E_p \ \leq \ |E_\theta(p)| \ \leq \ (1+2\xi_0)\ E_p. \tag{5.1}$$

We demonstrate the techniques in the case of $E_\theta(p) + m\theta$ which is needed to estimate $A_\theta(p)$ and $h_\theta(p)$. The upper estimate is obtained from

$$|E_\theta(p) + m\theta| \ \leq \ |E_\theta(p)| \ + \ m|\theta| \ \leq \ (1+2\xi_0)\ E_p \ + \ m\,(1+2\xi_0)$$

$$= (1+2\xi_0)\,(E_p+m) \tag{5.2}$$

For the lower bound we define the phases φ_p and φ_0 of $E_\theta(p) = \sqrt{p^2+m^2\theta^2}$ and $\theta = e^\xi$, respectively

$$\varphi_p := \frac{1}{2}\arctan\frac{m^2 e^{2\mathrm{Re}\;\xi}\sin(2\;\mathrm{Im}\;\xi)}{p^2+m^2e^{2\mathrm{Re}\;\xi}\cos(2\;\mathrm{Im}\;\xi)}, \qquad \varphi_0 := \;\mathrm{Im}\;\xi \tag{5.3}$$

With the restriction $|\mathrm{Im}\;\xi| < \pi/4$ we assure $\cos(2\;\mathrm{Im}\;\xi) > 0$ i.e. positivity of the denominator of φ_p. Since arctan is an odd, monotonically increasing function we can estimate $|\varphi_p|$ from above by dropping p^2 in the denominator, $|\varphi_p| \leq \frac{1}{2}\arctan(\tan 2|\mathrm{Im}\;\xi|) = |\mathrm{Im}\;\xi|$. We therefore get $|\varphi_0-\varphi_p| \leq 2|\mathrm{Im}\;\xi|$ and hence $\cos(\varphi_0-\varphi_p) \geq \cos(2\;\mathrm{Im}\;\xi) \geq 1-2|\mathrm{Im}\;\xi|^2 \geq (1-\xi_0)^2$ for $|\mathrm{Im}\;\xi| < \xi_0 \leq 1/2$. Thus we estimate

$$\begin{aligned}|E_\theta(p)+m\theta| &= \Big||E_\theta(p)| + m|\theta|e^{i(\varphi_0-\varphi_p)}\Big| \geq |E_\theta(p)| + m|\theta|\cos(\varphi_0-\varphi_p)\\ &\geq (1-\xi_0)\,E_p + m\,(1-\xi_0)^3 \geq (1-\xi_0)^3\,(E_p+m)\end{aligned} \tag{5.4}$$

where in the first inequality we have dropped the imaginary part and used that its r.h.s. is nonnegative. From this we find, using the definition (4.4) of $A_\theta(p)$ and $h_\theta(p)$

$$\begin{aligned}\frac{(1-\xi_0)^3}{1+2\xi_0}\,A^2(p) &\leq |A_\theta(p)|^2 \leq \frac{1+2\xi_0}{1-\xi_0}\,A^2(p)\\ \frac{1}{1+2\xi_0}\,h(p) &\leq |h_\theta(p)| \leq \frac{1}{(1-\xi_0)^3}\,h(p)\end{aligned} \tag{5.5}$$

In a similar way we find

$$\frac{1}{|E_\theta(p)+E_\theta(p')|} \leq \frac{1}{(1-\xi_0)^3}\,\frac{1}{E_p+E_{p'}} \leq \frac{1}{(1-\xi_0)^3}\,\frac{1}{p+p'} \tag{5.6}$$

Taking $\xi_0 \leq 1/2$ assures that (5.5) and (5.6) are valid for all $\theta \in D_0$.

5.1 The $b_{0m,\theta}$-form boundedness

Refering to (4.5) and (4.7) it remains to prove the boundedness of $|(\varphi,(b_{im\cdot\theta}-b_i)\,\varphi)|$, $i=1,2$, as well as the estimate of b_0 by $b_{0m,\theta}$ since (4.7) only provides the b_0-form boundedness.

In order to show the second item we start by estimating the real part of $b_{0m\cdot\theta}$

$$\begin{aligned}\mathrm{Re}\sqrt{p^2+m^2\theta^2} &= \left[(p^2+m^2e^{2\mathrm{Re}\xi}\cos(2\,\mathrm{Im}\,\xi))^2+(m^2e^{2\mathrm{Re}\xi}\sin(2\,\mathrm{Im}\,\xi))\right]^{\frac{1}{4}}\cos\varphi_p\\ &\geq p\cos\varphi_p \geq p\cos(\mathrm{Im}\;\xi) \geq p\,(1-|\mathrm{Im}\;\xi|) \geq p\,(1-\xi_0)\end{aligned} \tag{5.7}$$

for $|\mathrm{Im}\ \xi| \ < \xi_0 \leq \pi/4$, where φ_p is defined in (5.3) and we have followed the argumentation below (5.3) and used that $\cos x \ \geq 1 - \ |x|$ for $|x| \ \leq 1$. Then with (5.7)

$$|(\varphi, b_{0m,\theta}\, \varphi)| \ = \ \left|\frac{1}{\theta}\right| \cdot |(\varphi, b_{0m\cdot\theta}\, \varphi)| \ \geq \ (1-\xi_0) \cdot |\mathrm{Re}\ (\varphi, b_{0m\cdot\theta}\, \varphi)|$$

$$\geq \ (1-\xi_0)^2\ (\varphi, p\, \varphi) \ = \ (1-\xi_o)^2\ (\varphi, b_0\, \varphi) \tag{5.8}$$

Hence, the r.h.s. of (4.7) can be estimated by $c_0\, |(\varphi, b_{0m,\theta}\, \varphi)|$ with $c_0 := (1+2\xi_0)\ (1-\epsilon)\ (1-\xi_0)^{-2} \ < 1$ for sufficiently small ξ_0, which provides the $b_{0m,\theta}$-form boundedness with form bound < 1.

The first item is proven by estimating every term of $|(\varphi, (b_{im\cdot\theta} - b_i)\, \varphi)|$ by the corresponding term in the $\theta = 1$ case. Since all these terms (for $\theta = 1$) have separately been shown to be bounded by Brummelhuis et al [4] and Tix [16] during the course of their proofs of (4.8), we are done.

We use the partial wave expansion (A.2) of the energy $E_m(\varphi)$ and note that $b_{im\cdot\theta}$ and $b_{lsm\cdot\theta}$ are obtained from b_{im} and b_{lsm}, respectively, by attaching to every occuring m the multiplication factor θ. With the explicit form of $b_{lsm}^{(1)}$ from (A.2) we get

$$|(\varphi, (b_{1m\cdot\theta} - b_1)\, \varphi)| \ \leq \ \frac{\gamma}{2\pi} \sum_\nu \int_0^\infty dp\ |a_\nu(p)| \ \int_0^\infty dp'\ |a_\nu(p')|$$

$$\cdot \left\{ \left| \sqrt{1 + \frac{m\theta}{E_\theta(p)}}\ \sqrt{1 + \frac{m\theta}{E_\theta(p')}} \ - 1 \right| \ q_l(p/p') \right.$$

$$\left. + \ \left| \sqrt{1 - \frac{m\theta}{E_\theta(p)}}\ \sqrt{1 - \frac{m\theta}{E_\theta(p')}} \ - 1 \right| \ q_{l+2s}(p/p') \right\} \tag{5.9}$$

where for each term of $b_{1m\cdot\theta} - b_1$, the triangle inequality was used. We now define $f_\pm(m) := \sqrt{1 \pm \frac{m\theta}{E_\theta(p)}} \cdot \sqrt{1 \pm \frac{m\theta}{E_\theta(p')}}$ and use the mean value theorem in the following form (it is applicable to the real functions Re f_+ and Im f_+)

$$|f_+(m) \ - \ f_+(0)| \ = \ \sqrt{\mathrm{Re}^2(f_+(m) - f_+(0)) \ + \ \mathrm{Im}^2(f_+(m) - f_+(0))}$$

$$= \ \sqrt{\left(m \cdot \mathrm{Re}\ \frac{df_+}{dm}(\tilde{m}_1)\right)^2 \ + \ \left(m \cdot \mathrm{Im}\ \frac{df_+}{dm}(\tilde{m}_2)\right)^2} \tag{5.10}$$

$$\leq \ m \sqrt{\left|\frac{df_+}{dm}(\tilde{m}_1)\right|^2 \ + \ \left|\frac{df_+}{dm}(\tilde{m}_2)\right|^2} \ \leq \ m \ \left[\ \left|\frac{df_+}{dm}(\tilde{m}_1)\right| \ + \ \left|\frac{df_+}{dm}(\tilde{m}_2)\right| \ \right]$$

with $\tilde{m}_1, \tilde{m}_2$ some values between 0 and m. With $z := m \cdot \theta$ we obtain

$$\frac{df_+}{dm} \ = \ \theta\, \frac{df_+}{dz} \ = \ \theta\, \frac{d}{dz} \sqrt{1 + \frac{z}{\sqrt{p^2 + z^2}}}\ \sqrt{1 + \frac{z}{\sqrt{p'^2 + z^2}}} \tag{5.11}$$

$$= \theta \sqrt{1 + \frac{z}{\sqrt{p^2+z^2}}} \;\frac{1}{2}\;\frac{p'^2}{p'^2+z^2}\;\frac{1}{\sqrt{(p'^2+z^2)\; + z\sqrt{p'^2+z^2}}} \;+\; (p \leftrightarrow p')$$

where the symbol $(p \leftrightarrow p')$ stands for the first term with p and p' interchanged, and df_-/dz is given by (5.11) with z replaced by $-z$. With the estimates (4.6) and (5.1) for $|\theta|$ and $|E_\theta(p)|$, as well as (5.2) and (5.4) we deduce

$$\left|\frac{df_+}{dm}(\tilde{m})\right| \;\leq\; (1+\tilde{\epsilon})\;\left(\left|\frac{df_+^{(1)}}{dm}(\tilde{m})\right|_{\theta=1} \;+\; \left|\frac{df_+^{(2)}}{dm}(\tilde{m})\right|_{\theta=1}\right) \tag{5.12}$$

for a suitable $\tilde{\epsilon}$, where $df_+^{(1)}/dm$ denotes the first term in the last line of (5.11) and $df_+^{(2)}/dm$ is this term with p and p' interchanged.

We now follow Tix [16] to estimate (5.12) by an expression proportional to the inverse momentum and independent of $\tilde{m}$ by using that for $\theta = 1,\;\; z = \tilde{m} \geq 0$

$$\left|\frac{df_+^{(1)}}{dm}(\tilde{m})\right|_{\theta=1} \;\leq\; \sqrt{2}\cdot\frac{1}{2}\cdot\frac{p'^2}{p'^2}\cdot\frac{1}{\sqrt{p'^2+\tilde{m}^2}} \;\leq\; \frac{1}{p'} \tag{5.13}$$

and likewise $\left|\frac{df_+^{(2)}}{dm}(\tilde{m})\right|_{\theta=1} \;\leq\; \frac{1}{p}$.

Upon substitution of (5.10) with (5.12) and (5.13) into (5.9), one obtains integrals which Tix [16] has proven to be finite with the help of the formula (2.11) of Lieb and Yau.

The same method, i.e. the modified mean value theorem (5.10) together with an estimate of type (5.12) to each of the terms appearing in the derivative of the corresponding function given in [4], can be applied to show boundedness of $b_{2m\cdot\theta} - b_2$ by relying on the respective proof by Brummelhuis et al [4] for the $\theta = 1$ case.

5.2 Analyticity of the operator $(b_{0m}+\mu)^{-1/2}\,(b_{1m,\theta}+b_{2m,\theta})\,(b_{0m}+\mu)^{-1/2}$

In this subsection we show that $T_2(\theta) :=\;\; (b_{0m}+\mu)^{-1/2}\,b_{2m,\theta}\,(b_{0m}+\mu)^{-1/2}$ is an analytic family. For the first-order term $T_1(\theta)$ relating to $b_{1m,\theta}$ this was already proven by Evans et al [6]. According to [14, p.14] the following items are required:

(i) $\;\;T_2(\theta)$ *is closed for* $\theta \in D_0$.

This requires the proof of boundedness of $T_2(\theta)$ because $T_2(\theta)$ is defined on the Hilbert space $L_2(\mathbb{R}^3) \times \mathbb{C}^2$. For a complete domain, boundedness implies closure.

The boundedness of $T_2(\theta)$ is shown by the same means as compactness of $(b_{0m}+\mu)^{-1}\, b_{2m,\theta}\, (b_{0m}+\mu)^{-1/2} \;= (b_{0m}+\mu)^{-1/2}\, T_2(\theta)$. Refering to section 4.2 the latter requires the proof of compactness of the eight operators $\beta_{im\cdot\theta}$ of type (4.16). For $i=1$ and $A_\theta(p)$ and $h_\theta(p)$ estimated by the bounded operators $A(p)$ and $h(p)$, cf. (5.5), the l.h.s. of (4.17) is bounded also for $\theta \neq 1$. In the proof of the boundedness of $(b_{0m}+\mu)^{1/2}\, W_{10,m\cdot\theta}\, (b_{0m}+\mu)^{-1/2}$ for $\theta \in D_0$, we can by means of the estimates (5.5) and (5.6) for the θ-dependent quantities provide an upper bound for the kernel $\tilde{K}(\mathbf{p},\mathbf{p'})$ from (4.19), which is proportional to the corresponding kernel in the $m=0$ case. Boundedness of $(b_{0m}+\mu)^{1/2}\, W_{10,m\cdot\theta}\; (b_{0m}+\mu)^{-1/2}$ then follows from Appendix B in the same way as for the $\theta=1$ case. The very same tools also yield compactness for the other $\beta_{im\cdot\theta},\;\; i=2,...,8$.

For the boundedness of $T_2(\theta)$ we define $\tilde{\beta}_{im\cdot\theta} := \;\; (b_{0m}+\mu)^{1/2}\beta_{im\cdot\theta}$. The basic difference is that in the decompositions of type (4.16) the compact operator $(b_{0m}+\mu)^{-1}\, x^{-1/2}$ is now replaced by $(b_{0m}+\mu)^{-1/2}\, x^{-1/2}$ and that $(b_{0m}+\mu)^{-1}\, W_{10,m\cdot\theta}\, (b_{0m}+\mu)$ is replaced by $(b_{0m}+\mu)^{-1/2}\, W_{10,m\cdot\theta}\, (b_{0m}+\mu)^{1/2}$. However, we have already included the case $\lambda = -1/2$ in the earlier proof of boundedness of $W^{\lambda}_{10,m\cdot\theta}$ from (4.18), so it only remains to prove boundedness of $(b_{0m}+\mu)^{-1/2}\, x^{-1/2}$. This is done by means of Kato's inequality [12, p.307] in the inverse form, $1/p \;\leq \frac{\pi}{2}\, x$, and introducing $\psi := \;\dfrac{1}{\sqrt{x}}\,\varphi$ one has

$$\begin{aligned} \|\, (b_{0m}+\mu)^{-1/2}\, \frac{1}{\sqrt{x}}\,\varphi\, \|^2 &= \; (\psi, (b_{0m}+\mu)^{-1}\, \psi) \;\leq\; (\psi, \frac{1}{p}\,\psi) \\ &\leq\; \frac{\pi}{2}\,(\psi, x\,\psi) \;=\; \frac{\pi}{2}\,(\varphi,\varphi) \end{aligned} \tag{5.14}$$

(ii) $\;\;T(\theta) := \; T_1(\theta) + T_2(\theta)$ *has a nonempty resolvent set* $\varrho(T(\theta))$ *for each* $\theta \in D_0$.

Since the (operator-)boundedness of $T(\theta)$ implies its form boundedness because of $|(\varphi, T(\theta)\,\varphi)| \;\;\leq \|\varphi\|\cdot\, \|\; T(\theta)\,\varphi\;\|$, the expectation value of $T(\theta)$ and hence the spectrum of $T(\theta)$ is bounded, i.e. $\neq \mathbb{C}$. This means that $\varrho(T(\theta)) \neq \emptyset$.

(iii) $\;\;$*For every* $\theta_0 \in D_0$, $\;T(\theta)$ *is an analytic function of* θ *in a neighbourhood* $U(\theta_0)$ *of* θ_0.

This is true because D_0 is open and $b_{1m,\theta},\;\, b_{2m,\theta}$ depend analytically on θ for $\theta \in D_0$. (Note that $b_{im,\theta} \;\;= (1/\theta) b_{im\cdot\theta},\;\;\; i=1,2,$ and as seen from (3.2) and (1.5), the $m\cdot\theta$-dependence enters analytically through $E_\theta(p)$ and $E_\theta(p) + m\cdot\theta$ the moduli of which are bounded away from zero for $\theta \in D_0$ and $m \neq 0$ according to (5.1) and (5.4)). To obtain $T(\theta),\;\;\; b_{1m,\theta} + b_{2m,\theta}$ is only multiplied by bounded factors which are independent of θ, hence $T(\theta)$ is analytic in D_0.

(iv) $\;\;$*For every* $\theta_0 \in D_0$ *there is a* $\lambda_0 \in \varrho(T(\theta_0))$ *which is also in the resolvent set of the 'neighbouring' operator* $T(\theta)$ *for* $\theta \in U(\theta_0)$.

This follows from the θ-independence of the form bound of $T(\theta)$ for all $\theta \in D_0$. For $|(\varphi, T(\theta)\,\varphi)| \le M(\varphi,\varphi)$, every z with $|z| > M$ is in the resolvent set of $T(\theta)$ for all $\theta \in D_0$, hence also for $T(\theta_0)$ and for $T(\theta)$ with $\theta \in U(\theta_0)$.

5.3 The compactness of $R_{m,\theta}(\mu)$

Our starting point is (4.9) for $\theta \in D_0$. The compactness of the operator in curly brackets was proven in the last section, and the boundedness of the first factor in square brackets is easily shown. With m replaced by μ in (5.4) and with $\mu > 1$ we have

$$|(b_{0m,\theta}+\mu)^{-1}\,(b_{0m}+\mu)| \;=\; \left|\theta\,\frac{\sqrt{p^2+m^2}+\mu}{E_\theta(p)+\mu\theta}\right|$$

$$\le\; |\theta|\,\left|\frac{\sqrt{p^2+m^2}+\mu}{(1-\xi_0)^3\,(E_p+\mu)}\right| \;\le\; \frac{1+2\xi_0}{(1-\xi_0)^3} \tag{5.15}$$

which proves its boundedness (and simultaneously the boundedness of $(b_{0m,\theta}+\mu)^{-1}$ since $E_p+\mu > 1$ for $\mu > 1$).

It remains to show the boundedness of $(b_{0m}+\mu)^{1/2}\;(b_{m,\theta}+\mu)^{-1}$. The boundedness of $(b_{m,\theta}+\mu)^{-1}$ for $\theta \in D_0$ follows from the positivity (2.4) of this operator for $\theta = 1$ and the analyticity of $b_{m,\theta}$ in D_0. This assures that there is a neighbourhood of 1 in $\mathbb{C}$ such that Re $(b_{m,\theta}+\mu) > 0$ and hence $|b_{m,\theta}+\mu| > 0$. For ξ_0 sufficiently small, D_0 is a subset of this neighbourhood.

Further we note that the operator $(b_{0m,\theta}+\mu)^{1/2}$ exists because for $\theta \in D_0$, Re $(b_{0m,\theta}+\mu) >$ Re $b_{0m,\theta} \ge m\,(1-\xi_0) \ge 0$. This follows from a (5.7)-type sequence of inequalities because for $b_{0m,\theta} = \sqrt{p^2/\theta^2+m^2}$ only p and m have to be interchanged in (5.7), a sign reversal of ξ playing no role for the real part of this operator. With this we decompose

$$(b_{0m}+\mu)^{\frac12}(b_{m,\theta}+\mu)^{-1} = \left[(b_{0m}+\mu)^{\frac12}(b_{0m,\theta}+\mu)^{-\frac12}\right]\left[(b_{0m,\theta}+\mu)^{\frac12}(b_{m,\theta}+\mu)^{-1}\right] \tag{5.16}$$

The boundedness of the first operator in square brackets follows from (5.15). In order to prove boundedness of the second operator in square brackets let us first take $\theta \in \mathbb{R}^+ \cap D_0$. Then from unitarity of the dilation operator d_θ one has

$$\|\;(b_{0m}+\mu)^{1/2}\;(b_m+\mu)^{-1}\,\varphi\;\| = \|\;d_\theta\,(b_{0m}+\mu)^{1/2}\,d_\theta^{-1}\cdot d_\theta\;(b_m+\mu)^{-1}\;d_\theta^{-1}\cdot d_\theta\varphi\;\|$$

$$= \|\;(b_{0m,\theta}+\mu)^{1/2}\;(b_{m,\theta}+\mu)^{-1}\;\varphi_\theta\;\| \tag{5.17}$$

With the choice of $\varphi \in \mathcal{S}\times\mathbb{C}^2$, i.e. φ_θ an analytic vector in $\theta \in D_0$ and using that $b_{m,\theta}$ (as well as $b_{0m,\theta}$) is analytic in D_0, the r.h.s. of (5.17) is analytic in D_0. From the identity theorem we infer that (5.17) holds for all $\theta \in D_0$.

However, the l.h.s. of (5.17) is bounded by, say, c_1 as shown in section 4.2. Hence

$$\| \, (b_{0m,\theta} + \mu)^{1/2} \, (b_{m,\theta} + \mu)^{-1} \, \varphi_\theta \, \| \le \, c_1 \, (\varphi, \varphi) \, = c_1 \, (\varphi_\theta, \varphi_\theta) \tag{5.18}$$

which proves the desired boundedness. Note that the last equality in (5.18) also is a consequence of the identity theorem.

6 Absence of embedded eigenvalues

We conclude this work by showing that for $m \neq 0$ there is an m-dependent bound above which there are no eigenvalues of b_m embedded in the essential spectrum. For $m = 0$ we prove the absence of eigenvalues.

Theorem 6.1 *Let the critical coupling constant γ_c as in Theorem 1.1. If $m = 0$ and $\gamma < \gamma_c$, the spectrum of b is absolutely continuous.*

For the proof we only have to show that b has no eigenvalues. Then the spectrum is given by $\sigma(b) \, = \sigma_{\text{ess}}(b) \, = \sigma_{\text{ac}} \, (b)$ because $\sigma_{\text{sc}}(b) = \emptyset$ as stated in Theorem 1.1 (ac = absolute continuous and sc = singular continuous). Following Evans et al [6] we proceed in two steps

(i) Assume $E \neq 0$ is an eigenvalue of b, i.e. there exists $\varphi \in H_{1/2}(\mathbb{R}^3) \times \mathbb{C}^2$ such that $b\,\varphi = E\varphi$. As demonstrated in [6] for the operator $b_0 + b_1$ this leads to a contradiction since for each $\theta \in D_0 \cap \mathbb{R}^+$, $d_\theta \varphi$ is an eigenfunction of b to the eigenvalue θE because of the scaling of b with θ (in the massless case), $\theta \cdot (d_\theta b \, d_\theta^{-1}) \, d_\theta \varphi \, = \theta \cdot b/\theta \, (d_\theta \varphi) \, = \theta \cdot E \, d_\theta \varphi$. However, the existence of an uncountable set of (orthonormal) eigenvectors of a (self-adjoint) operator in the Hilbert space $H_{1/2}(\mathbb{R}^3) \times \mathbb{C}^2$ contradicts separability of the Hilbert space.

(ii) Assume $E = 0$ is an eigenvalue of b, i.e. there exists $\varphi \neq 0$ such that $b\,\varphi = 0$. Using the partial wave decomposition of b and φ as introduced in Appendix A we have from (A.4) in Mellin space

$$0 \, = \, (\varphi, b\,\varphi) \, = \, E(\varphi) \, = \, \sum_\nu \int_{-\infty}^{\infty} dt \; |a_\nu^\sharp(t + i/2)|^2 \; b_{ls}^\sharp(t - i/2) \tag{6.1}$$

where $b_{ls}^\sharp \, := \, b_0^\sharp + \sqrt{2\pi} \; b_{ls}^{(1)\sharp} + \sqrt{2\pi} \; b_{ls}^{(2)\sharp}$. However, positivity of b or equivalently, of $b_{ls}^\sharp(t - i/2)$ for $\gamma < \gamma_c$ implies that the r.h.s. of (6.1) can only be zero if for each ν,

$$|a_\nu^\sharp(t + i/2)| \, = \, 0 \qquad \text{almost everywhere for } t \in \mathbb{R}. \tag{6.2}$$

If $\varphi \in \mathcal{S} \times \mathbb{C}^2$ then $a_\nu^\sharp$ is an analytic function of τ in the strip $\{\tau \in \mathbb{C} : -\infty < t = \mathrm{Re}\,\tau < \infty,\ 0 \leq \mathrm{Im}\,\tau \leq \frac{1}{2}\}$. From the identity theorem it follows that $|a_\nu^\sharp(t)| = 0$ and unitarity of the Mellin transform gives

$$0 \;=\; \int_{-\infty}^{\infty} dt\; |a_\nu^\sharp(t)|^2 \;=\; \int_0^\infty dp\; |a_\nu(p)|^2 \tag{6.3}$$

hence $a_\nu(p) = 0$ in $\mathbb{R}^+$ and thus $\varphi = 0$. However, since $\mathcal{S}$ is dense in $H_{1/2}(\mathbb{R}^3)$ we have $\varphi = 0$ in $H_{1/2}(\mathbb{R}^3) \times \mathbb{C}^2$ which is a contradiction to our assumption $\varphi \neq 0$.

For the $m \neq 0$ case, we have

Theorem 6.2 *Let $\gamma < \gamma_c$ with γ_c as in Theorem 1.1. Then the eigenvalues λ of b_m are confined to $\lambda \leq m(1 + s(\gamma))$ with*

$$s(\gamma) := \max\{0,\ s_0(m_1\gamma - m_0 + m_2\gamma^2)\}$$

where $s_0 := 5$, $m_0 := 0.3058$, $m_1 := \frac{2}{5}$ and $m_2 := 2.253$. In particular, for $\gamma < 0.29$ (i.e. $Z < 40$) the essential spectrum of b_m has no embedded eigenvalues.

The proof proceeds along the lines provided by Balinsky and Evans [1] in the case of the Brown-Ravenhall operator $b_{0m} + b_{1m}$. However, a refinement of the estimates is mandatory to show the absence of embedded eigenvalues for small coupling constants.

Starting point is the virial theorem [1, Lemma 2.1]. If φ is an eigenfunction to b_m with eigenvalue λ and use is made of the scaling property (4.3) of $b_{m,\theta}$ with θ, the virial theorem takes the form

$$\lim_{\theta \to 1} (\varphi_\theta,\ \frac{b_{m\cdot\theta} - b_m}{\theta - 1}\,\varphi) \;=\; \lambda\,\|\varphi\|^2 \tag{6.4}$$

for $\theta \in \mathbb{R}^+$ and $\varphi_\theta = d_\theta \varphi$ from (2.12). In order to interchange the limit $\theta \to 1$ with the integration, the uniform absolute convergence of the form on the l.h.s. of (6.4) is needed.

Since $m \cdot \theta \in \mathbb{R}^+$, the proofs [16, 4] of form boundedness of $\left|\frac{db_{im\cdot\theta}}{dm\cdot\theta}\right|$, $i = 1, 2,$ also guarantee boundedness for the off-diagonal form if use is made of the generalised Lieb and Yau formula (2.11),

$$(|\varphi_\theta|,\ \left|\frac{db_{m\cdot\theta}}{dm\cdot\theta}\right|\ |\varphi|)| \;\leq\; c\,\|\varphi_\theta\| \cdot \|\varphi\| \;=\; c\,\|\varphi\|^2, \qquad c \in \mathbb{R}. \tag{6.5}$$

Hence from the mean value theorem, with ξ some value between $\min\{m\cdot\theta, m\}$ and $\max\{m\cdot\theta, m\}$,

$$\left|(\varphi_\theta,\ \frac{b_{m\cdot\theta} - b_m}{\theta - 1}\,\varphi)\right| \;\leq\; (|\varphi_\theta|,\ m\left|\frac{db_{m\cdot\theta}}{dm\cdot\theta}(\xi)\right|\ |\varphi|) \;\leq\; mc\,\|\varphi\|^2 \tag{6.6}$$

such that the dominated convergence theorem applies. We therefore obtain from (6.4)

$$\lambda\,\|\varphi\|^2 \;=\; m^2\int_{\mathbb{R}^3} d\mathbf{p}\,|\varphi(\mathbf{p})|^2\,\frac{1}{E_p}$$
$$+\,m\int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}d\mathbf{p}'\;\overline{\varphi(\mathbf{p})}\;\left(\frac{db_{1m}(\mathbf{p},\mathbf{p}')}{dm}\;+\;\frac{db_{2m}(\mathbf{p},\mathbf{p}')}{dm}\right)\,\varphi(\mathbf{p}') \tag{6.7}$$

Due to the self-adjointness of b_{1m} and hence of db_{1m}/dm, the interchange of $\mathbf{p}$ and $\mathbf{p}'$ in the expectation value leads to complex conjugation. Therefore the term linear in the coupling constant can be written in the following way

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}\,d\mathbf{p}'\,\overline{\varphi(\mathbf{p})}\;\frac{db_{1m}(\mathbf{p},\mathbf{p}')}{dm}\;\varphi(\mathbf{p}')$$
$$=\;\;\mathrm{Re}\int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}\,d\mathbf{p}'\;\overline{\varphi(\mathbf{p})}\;\left(\frac{1}{E_p}\;-\;\frac{m}{E_p^2}\right)\,b_{1m}(\mathbf{p},\mathbf{p}')\,\varphi(\mathbf{p}')$$
$$+\frac{\gamma}{2\pi^2}\int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}d\mathbf{p}'\overline{\varphi(\mathbf{p})}\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}A(p)A(p')\boldsymbol{\sigma}\hat{\mathbf{p}}\boldsymbol{\sigma}\hat{\mathbf{p}}'h(p)\,h(p')\left(\frac{1}{E_p}+\frac{1}{E_p'}\right)\varphi(\mathbf{p}'). \tag{6.8}$$

Following [1], the first term in (6.8) carrying the negative sign of b_{1m} is eliminated with the help of the eigenvalue equation in the form

$$(\psi,b_m\varphi)=\int_{\mathbb{R}^3} d\mathbf{p}\overline{\psi(\mathbf{p})}E_p\varphi(\mathbf{p})+\int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}d\mathbf{p}'\overline{\psi(\mathbf{p})}\,[b_{1m}(\mathbf{p},\mathbf{p}')+b_{2m}(\mathbf{p},\mathbf{p}')]\,\varphi(\mathbf{p}')$$
$$=\;(\psi,\lambda\varphi)$$
$$\psi(\mathbf{p}) := \;\left(\frac{1}{E_p}\;-\;\frac{m}{E_p^2}\right)\;\varphi(\mathbf{p}). \tag{6.9}$$

This procedure of eliminating a negative first-order term at the expense of additional zero-order terms (for which no further estimate is needed) and second-order terms (which are small for small γ) is mandatory for the desired estimate on the eigenvalue λ. With (6.8) and (6.9), (6.7) results in

$$\frac{\lambda}{m}\,\|\varphi\|^2=\int_{\mathbb{R}^3} d\mathbf{p}\,|\varphi(\mathbf{p})|^2\left(\frac{m}{E_p}+(\frac{\lambda}{E_p}-1)(1-\frac{m}{E_p})\right)+\frac{\gamma}{2\pi^2}\int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}d\mathbf{p}'\,\overline{\varphi(\mathbf{p})}$$
$$\cdot A(p)\,A(p')\left[\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\boldsymbol{\sigma}\hat{\mathbf{p}}\,\boldsymbol{\sigma}\hat{\mathbf{p}}'h(p)\,h(p')\left(\frac{1}{E_p}+\frac{1}{E_{p'}}\right)+\frac{\gamma}{4\pi^2}\,T_2(\mathbf{p},\mathbf{p}')\right]\varphi(\mathbf{p}') \tag{6.10}$$

where the lengthy expression for $T_2(\mathbf{p},\mathbf{p}')$ is given in Appendix C. Applying the Lieb and Yau formula (2.11) with $\varphi := \psi := \varphi A h$ and φA, respectively, to the first-order and second-order term, one obtains with $|\boldsymbol{\sigma}\hat{\mathbf{p}}\boldsymbol{\sigma}\hat{\mathbf{p}}'|\;\leq 1$

$$\left(\frac{\lambda}{m}-1\right)\int_{\mathbb{R}^3} d\mathbf{p}\,|\varphi(\mathbf{p})|^2\left(1-\frac{m}{E_p}+\frac{m^2}{E_p^2}\right)\leq-\int_{\mathbb{R}^3} d\mathbf{p}\,|\varphi(\mathbf{p})|^2\,\frac{(E_p-m)(2E_p-m)}{E_p^2}$$

$$+\ \frac{\gamma}{2\pi^2}\int_{\mathbb{R}^3} d\mathbf{p}\,|\varphi(\mathbf{p})|^2\,A(p)^2\ \left\{h^2(p)\int_{\mathbb{R}^3} d\mathbf{p}'\,\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\ \left(\frac{1}{E_p}\ +\ \frac{1}{E_{p'}}\right)\ \left|\frac{f(p)}{f(p')}\right|^2\right.$$

$$\left.+\ \frac{\gamma}{4\pi^2}\int_{\mathbb{R}^3} d\mathbf{p}'\ |T_2(\mathbf{p},\mathbf{p}')|\ \left|\frac{f(p)}{f(p')}\right|^2\right\} \tag{6.11}$$

The last term in (6.11) can be further estimated by breaking $T_2(\mathbf{p},\mathbf{p}')$ into its constituents and estimating each contribution separately. Note that the convergence inducing function can be chosen differently for each integral. Apart from the conventional choice $f(p)\ =\ p^{3/4}$ [1, 4], we also allow for functions of the type $f(p)\ =\ p^{3/4}\,(\frac{p}{\epsilon(p)})^{1/2}$ with $\epsilon(p)\in\{E_p, E_p+m, p+m\}$ to optimise the estimates. Further, the following estimate is used in the evaluation of the integrals over p',

$$\frac{1}{\sqrt{(qp')^2+1}\ +c}\ \le\ \left\{\begin{array}{ll}\frac{1}{1+c}, & p'\le 1/q\\[2ex] \frac{1}{qp'}, & p'>1/q\end{array}\right.\ ,\qquad c\ge 0,\ q\ge 0 \tag{6.12}$$

An outline of the evaluation of the second-order term in γ is given in Appendix C. Defining $\mathbf{q}:=\mathbf{p}/m$, denoting the estimate of $\int\limits_{\mathbb{R}^3} d\mathbf{p}'\,|T_2(\mathbf{p},\mathbf{p}')|\,|f(p)/f(p')|^2$ by $(4\pi^2)^2q^2M_2(q)$, and taking $f(p):=p^{3/4}$ in the term linear in γ, such that (with (B.1), the substitution $q':=p'/mq$ and (6.12))

$$\int_{\mathbb{R}^3} d\mathbf{p}'\,\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\,\frac{1}{E_p'}\,\left(\frac{p}{p'}\right)^{3/2}\ \le\ 4\pi^2\,\alpha(q) \tag{6.13}$$

$$\alpha(q):=\ 1+\frac{1}{\pi}\,\left(2\sqrt{q}\,\ln\left|\frac{1+q}{1-q}\right|\ +\ 2\,(q-1)\,\arctan\frac{1}{\sqrt{q}}\ -\ (q+1)\,\ln\left|\frac{1+\sqrt{q}}{1-\sqrt{q}}\right|\ \right)$$

we arrive at the following estimate

$$0\ \le\ m^3\int_{\mathbb{R}^3} d\mathbf{q}\,|\varphi(m\mathbf{q})|^2\,\left(1\ -\ \frac{1}{\sqrt{q^2+1}}\ +\ \frac{1}{q^2+1}\right)\ \left(1\ -\ \frac{\lambda}{m}\ +\ \phi(q)\right) \tag{6.14}$$

$$\phi(q):=\ \frac{q^2}{q^2+2-\sqrt{q^2+1}}\ \frac{1}{f_0(q)}\ \left(-g_0(q)\ +\ \gamma g_1(q)\ +\ \gamma^2 g_2(q)\right)\ f_0(q)$$

where

$$g_0(q):=\ \frac{2\sqrt{q^2+1}-1}{\sqrt{q^2+1}+1},\qquad g_1(q):=\ \frac{q+\alpha(q)\sqrt{q^2+1}}{\sqrt{q^2+1}+1}$$

$$g_2(q):=\ (q^2+1+\sqrt{q^2+1})\,M_2(q),\qquad f_0(q):=\ \frac{q+c}{aq+b} \tag{6.15}$$

are nonnegative bounded functions. The auxiliary function f_0 with $a,b,c>0$ has been introduced to improve on the estimate of ϕ. It follows from (6.14) that for $\phi < 0$, $\lambda < m$ since the factor multiplying the last bracket is nonnegative. With $m_0 := \min g_0 f_0$, $m_1 := \sup g_1 f_0$ and $m_2 := \max g_2 f_0$ for $0 \leq q < \infty$, this condition is fulfilled for $-m_0 + m_1\gamma + m_2\gamma^2 < 0$, i.e. $\gamma < \gamma_0$, say. For $a := 5$, $b := \frac{1}{5}$, $c := 1.1$, we obtain $m_0 = 0.3058$, $m_1 = \frac{2}{5}$, $m_2 = 2.253$, and hence $\gamma_0 = 0.29$. This improves on the value $\gamma_0 = 0.159$ obtained for $f_0 = 1$ (where $m_0 = \frac{1}{2}$, $\sup g_1 = 2$, $\sup g_2 = \frac{29}{4}$). Denoting by s_0 the supremum of the prefactor of $\phi(q)$ in $q \in \mathbb{R}^+$, $s_0 := \sup q^2/(q^2 + 2 - \sqrt{q^2+1})\, f_0^{-1}(q) = 5$, we can estimate $\phi(q)$ for $\gamma > \gamma_0$ to obtain from (6.14)

$$\lambda \leq m\,(1 + \phi(q)) \leq m\,(1 + s_0(m_1\gamma - m_0 + m_2\gamma^2)). \tag{6.16}$$

In the Brown-Ravenhall case ($g_2 \equiv 0$) $\phi(q)$ can be written as $\gamma - g_0/g_1(q)$ multiplied by a nonnegative factor, and one obtains for the eigenvalues $\tilde{\lambda}$ of $b_{0m} + b_{1m}$ the estimate $\tilde{\lambda} < m$ for $\gamma < \tilde{\gamma}_0 := 0.973$ which is the minimum of g_0/g_1 in $\mathbb{R}^+$. This covers the whole range of boundedness (from below) of the Brown-Ravenhall operator ($\gamma < 2/(\pi/2 + 2/\pi) = 0.906$ [6]) and improves on the result of Balinsky and Evans [1] ($\gamma \leq 3/4$ obtained for $\alpha(q) = 1$.)

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Appendix A

Proof of Lemma 2.1

It is convenient to introduce the partial wave expansions [6]

$$\varphi(\mathbf{p}) = \sum_\nu p^{-1}\, a_\nu(p)\, \Omega_\nu(\hat{\mathbf{p}}) \qquad \nu = \{l, M, s\}$$

$$\frac{1}{|\mathbf{p}-\mathbf{p}'|^2} = \frac{2\pi}{pp'} \sum_{lM} q_l(\frac{p}{p'})\, \overline{Y_{lM}(\hat{\mathbf{p}})}\, Y_{lM}(\hat{\mathbf{p}}') \tag{A.1}$$

where $\Omega_\nu(\hat{\mathbf{p}})$ are the Dirac angular momentum eigenstates (the vector spherical hamonics [2]), $Y_{lM}(\hat{\mathbf{p}})$ are spherical harmonics, $l = 0, 1, ...,$ $M = -l - \frac{1}{2}, -l + \frac{1}{2}, ..., l + \frac{1}{2}$, $s = -\frac{1}{2}, \frac{1}{2}$, and $q_l(x)$ is related to the Legendre function $Q_l(x)$ of the second kind by $q_l(x) := Q_l(\frac{1}{2}x + \frac{1}{2x})$. Then the energy (3.1) can be written in the following way [6, 4], making use of orthonormality of the set $\Omega_\nu(\hat{\mathbf{p}})$ and likewise of $Y_{lM}(\hat{\mathbf{p}})$,

$$E_m(\varphi) \;=\; \sum_\nu \int_0^\infty dp\, \overline{a_\nu(p)} \int_0^\infty dp'\; b_{lsm}(p,p')\; a_\nu(p')$$

$$b_{lsm}(p,p') := \; b_{0m}\, \delta(p-p') \;+\; b^{(1)}_{lsm}(p,p') \;+\; b^{(2)}_{lsm}(p,p') \tag{A.2}$$

where

$$b^{(1)}_{lsm}(p,p') \;=\; -\frac{\gamma}{\pi}\;[q_l(p/p') \;+\; h(p)\,h(p')\; q_{l+2s}(p/p')]\;\; A(p)\,A(p')$$

$$b^{(2)}_{lsm}(p,p') \;=\; \frac{1}{2}\,\left(\frac{\gamma}{\pi}\right)^2 \int_0^\infty dp''\; \left[\frac{1}{E_{p'}+E_{p''}} \;+\; \frac{1}{E_p+E_{p''}}\right]\; A(p)\,A(p')\; A^2(p'')$$
$$\cdot\,[q_l(p''/p)\; h(p'') \;-\; q_{l+2s}(p''/p)\; h(p)]\;\; [q_l(p'/p'')\; h(p'') \;-\; q_{l+2s}(p'/p'')\; h(p')]$$

As a next step, the Mellin space representation is introduced because in the $m=0$ case, it offers an integral representation of $E(\varphi)$ with a positive integrand. For a function $f \in L_2(\mathbb{R}^+)$ the Mellin transform $f^\sharp \in L_2(\mathbb{R})$ is defined as

$$f^\sharp(t) := \;\frac{1}{\sqrt{2\pi}} \int_0^\infty dp\; f(p)\; p^{-it-1/2} \tag{A.3}$$

Since the Mellin transform is unitary, $E(\varphi) \;=\; \sum_{i=0}^{2} (\varphi, b_i\varphi)$ is invariant and can be cast into the following form [4]

$$E(\varphi) \;=\; \sum_\nu \int_{-\infty}^\infty dt\; \overline{a^\sharp_\nu(t)}\;\; \left(\int_0^\infty dp'\; b_{ls}(\cdot,p')\; a_\nu(p')\right)^\sharp(t) \tag{A.4}$$
$$=\; \sum_\nu \int_{-\infty}^\infty dt\; \left|a^\sharp_\nu(t+\frac{i}{2})\right|^2\; \left(b^\sharp_0 \;+\; \sqrt{2\pi}\, b^{(1)\sharp}_{ls} \;+\; \sqrt{2\pi}\, b^{(2)\sharp}_{ls}\right)(t-\frac{i}{2})$$

with

$$b^\sharp_0(t-\frac{i}{2}) \;=1, \qquad b^{(1)\sharp}_{ls}(t-\frac{i}{2}) \;=\; -\frac{\gamma}{2\pi}\left[q^\sharp_l(t-i/2) \;+\; q^\sharp_{l+2s}(t-i/2)\right]$$

$$b^{(2)\sharp}_{ls}(t-\frac{i}{2}) \;=\; \frac{\sqrt{2\pi}}{2}\;\left(\frac{\gamma}{2\pi}\right)^2\;\left[q^\sharp_l(t-i/2) \;-\; q^\sharp_{l+2s}(t-i/2)\right]^2$$

Since $q^\sharp_l(t-i/2) \;\geq\; 0 \quad \forall\; l \in \mathbb{N}_0$ [4] one has $-b^{(1)\sharp}_{ls}(t-i/2) \;\; \geq\; 0$ and $b^{(2)\sharp}_{ls}(t-i/2) \;\geq 0$ and therefore also $-b_1 \geq 0$ and $b_2 \geq 0$. We show that

$$-b^{(1)\sharp}_{ls}(t-\frac{i}{2}) \;-\; b^{(2)\sharp}_{ls}(t-\frac{i}{2}) \;\geq 0 \tag{A.5}$$

which thus proves $-b_1 - b_2 \geq 0$, i.e. Lemma 2.1. We proceed in two steps. First we show the existence of a sufficiently large l_1 such that (A.5) holds for $l \geq l_1$ and $s = \pm\frac{1}{2}$. Subsequently we use a recurrence relation to prove that if (A.5) holds for a given l it also holds for $l-1$.

(i) From [9, p.937]

$$\lim_{|z|\to\infty} \left| \frac{\Gamma(z+a)}{\Gamma(z)}\, z^{-a} \right|^2 \;=\; 1 \qquad \text{for } z \in \mathbb{C}\backslash(\mathbb{Z}^- \cup \{0\}),\;\; a \in \mathbb{R} \tag{A.6}$$

We take $a := -\frac{1}{2},\;\; z := \frac{l}{2} + 1 - \frac{it}{2}$ and introduce the explicit expression for $q_l^\sharp(t - i/2)$ in terms of Gamma functions [4]

$$q_l^\sharp(t - \frac{i}{2}) \;=\; \frac{\sqrt{\pi}}{2\sqrt{2}} \left| \frac{\Gamma(\frac{l}{2} + \frac{1}{2} - \frac{it}{2})}{\Gamma(\frac{l}{2} + 1 - \frac{it}{2})} \right|^2 \tag{A.7}$$

Then from (A.6) follows the existence of $l_0 \in \mathbb{N}$ such that for any given ϵ with $0 < \epsilon < 1$,

$$(1-\epsilon)\, \frac{1}{|\frac{l}{2} + 1 - \frac{it}{2}|} \;<\; 2\sqrt{\frac{2}{\pi}}\, q_l^\sharp(t - \frac{i}{2}) \;<\; (1+\epsilon)\, \frac{1}{|\frac{l}{2} + 1 - \frac{it}{2}|} \tag{A.8}$$

for $l > l_0$. From this it follows that the upper and lower bounds of $q_l^\sharp(t - i/2)$ and hence of $b_{ls}^{(1)\sharp}(t - i/2)$ decrease as l^{-1} for $l \to \infty$, while the bounds of $b_{ls}^{(2)\sharp}(t - i/2)$ are of order $O(l^{-2})$ making that term negligible with respect to $b_{ls}^{(1)\sharp}(t - i/2)$ for sufficiently large l. Hence there exists $l_1 \in \mathbb{N}$ such that

$$(-b_{ls}^{(1)\sharp} - b_{ls}^{(2)\sharp})(t - \frac{i}{2}) \;\geq 0 \qquad \text{for all } l \geq l_1,\;\; s = \pm\frac{1}{2}. \tag{A.9}$$

(ii) It was shown by Brummelhuis et al [4] that for $\gamma \leq 4/\pi$ (and hence for $\gamma < \gamma_c$)

$$1 \;+\; \sqrt{2\pi}\,(b_{l-1,1/2}^{(1)\sharp} \;+\; b_{l-1,1/2}^{(2)\sharp}) \;\leq\; 1 \;+\; \sqrt{2\pi}\,(b_{l,1/2}^{(1)\sharp} \;+\; b_{l,1/2}^{(2)\sharp}) \tag{A.10}$$

$\forall\, l \in \mathbb{N}_0$, which by means of $b_{l+1,-1/2}^{(i)\sharp} = b_{l,1/2}^{(i)\sharp},\;\; i = 1,2,\; l \in \mathbb{N}_0$ (which follows from (A.4)) holds also for $s = -1/2$. Here and in the following the argument $(t - i/2)$ of $b_{ls}^{(i)\sharp}$ is suppressed. Hence

$$-b_{ls}^{(1)\sharp} \;-\; b_{ls}^{(2)\sharp} \;\leq\; -b_{l-1,s}^{(1)\sharp} \;-\; b_{l-1,s}^{(2)\sharp}, \qquad (l,s) \;\in\; \{(\mathbb{N}_0, \frac{1}{2}) \;\cup\; (\mathbb{N}, -\frac{1}{2})\} \tag{A.11}$$

Setting in (A.11) $l = 0$ for $s = \frac{1}{2}$ and $l = 1$ for $s = -\frac{1}{2}$ and continuing the chain of inequalities to the left until l_1 is reached, proves that $-b_{ls}^{(1)\sharp} - b_{ls}^{(2)\sharp} \;\geq 0$ for all l, s.

Appendix B

Proof of the boundedness of W_{10}^{λ} in the massless case

In order to prove the finiteness of $I(\mathbf{p}')$ as defined in (4.20) we start by showing that $I(\mathbf{p}')$ only depends on p'. Choosing spherical coordinates for $\mathbf{p}'$, the angular integrations can be performed by means of [9, p.58]

$$\int_{S^2} d\omega' \, \frac{1}{|\mathbf{p}-\mathbf{p}'|^2} \;=\; \frac{2\pi}{pp'} \, \ln\left|\frac{p+p'}{p-p'}\right| \;=\; \frac{2\pi}{pp'} \begin{cases} \frac{2p}{p'} \;+\; O(p^2), & p\to 0 \\ \\ \frac{2p'}{p} \;+\; O(\frac{1}{p^2}), & p\to\infty \end{cases} \tag{B.1}$$

With $A(p')\le 1$ and $m=0$ (and the choice $\mu := 1$) in the definition (4.19) for $\tilde{K}(\mathbf{p},\mathbf{p}')$ we have

$$I(\mathbf{p}') \;=\; \int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}''\, d\mathbf{p}\; \tilde{K}(\mathbf{p},\mathbf{p}')\; \tilde{K}(\mathbf{p},\mathbf{p}'')\; \frac{p'^{2\alpha}}{p''^{2\alpha}}$$

$$\le \int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}'' d\mathbf{p} (p+1)^{2\lambda} \frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\frac{1}{p+p'}(p'+1)^{-\lambda}\frac{1}{|\mathbf{p}-\mathbf{p}''|^2}\frac{1}{p+p''}(p''+1)^{-\lambda}\frac{p'^{2\alpha}}{p''^{2\alpha}}$$

$$=:\; 4\pi^2 \tilde{I}(p') \tag{B.2}$$

In order to separate the variables we perform scaling transformations $q := p/p'$ and $q'' := p''/qp'$ of the variables p and p'', respectively [4], to obtain

$$\tilde{I}(p') \;=\; (p'+1)^{-\lambda} \int_0^\infty dq\; q^{1-2\alpha}\; (qp'+1)^{2\lambda}\; \frac{1}{q+1}\; \ln\left|\frac{q+1}{q-1}\right|$$

$$\cdot \int_0^\infty dq''\; (q'')^{1-2\alpha}\; (qp'q''+1)^{-\lambda}\; \frac{1}{1+q''}\; \ln\left|\frac{1+q''}{1-q''}\right| \tag{B.3}$$

showing that, apart from the factors $(qp'+1)^{2\lambda}$ and $(qp'q''+1)^{-\lambda}$ the two integrals are alike and decouple.

Case (a): $p'\to\infty$

This case gives the most severe restrictions to the exponent α of the convergence inducing function. Assume $q\neq 0$, $q''\neq 0$ (and note that if e.g. $q\to 0$ would be taken before the limit $p'\to\infty$ was carried out, the q-integrand would behave like $q^{2-2\alpha}$ near zero, leading to the restriction $2-2\alpha > -1$, i.e. $\alpha < 3/2$ for convergence). Then

$$(p'+1)^{-\lambda}\; (qp'+1)^{2\lambda}\; (qp'q''+1)^{-\lambda} \;\longrightarrow\; q^{\lambda}\, q''^{-\lambda} \tag{B.4}$$

which makes $\tilde{I}(p')$ independent of p' in the limit $p'\to\infty$ and leads to a splitting into the product of two integrals, $\lim\limits_{p'\to\infty} \tilde{I}(p') \;=: I_\infty(\lambda)\, I_\infty(-\lambda)$. We thus have to find α such that $I_\infty(\pm\lambda)$ is finite for the two cases of interest, $\lambda = \frac{1}{2}, 1$. The integrand of $I_\infty(\lambda)$ behaves like

$$q^{1+\lambda-2\alpha}\ \frac{1}{q+1}\ \ln\left|\frac{q+1}{q-1}\right| \ \longrightarrow\ \begin{cases} 2q^{2+\lambda-2\alpha}, & q\to 0 \\ 2q^{-1+\lambda-2\alpha}, & q\to\infty \end{cases} \tag{B.5}$$

Convergence at the lower limit requires $2+\lambda-2\alpha\ >-1$ and at the upper limit $-(1-\lambda+2\alpha)\ <-1$ such that finiteness of $\tilde{I}(p')$ for $p'\to\infty$ is achieved if

$$\pm\frac{\lambda}{2}\ <\alpha\ <\frac{3}{2}\pm\frac{\lambda}{2} \tag{B.6}$$

where either all upper or all lower signs must be taken. For $\lambda = 1$, one obtains the interval $\{\frac{1}{2}<\alpha<2\}\ \cap\ \{-\frac{1}{2}<\alpha\ <1\}\ =\ \{\frac{1}{2}<\alpha<1\}$ while for $\lambda = 1/2$ one gets $\{\frac{1}{4}<\alpha\ <\frac{7}{4}\}\ \cap\ \{-\frac{1}{4}<\alpha\ <\frac{5}{4}\}\ =\{\frac{1}{4}<\alpha<\frac{5}{4}\}$. Both values of λ are covered by the interval

$$\{\alpha\in\mathbb{R}^+ :\ \frac{1}{2}<\alpha<1\} \tag{B.7}$$

In particular, (B.7) satisfies the condition $\alpha < 3/2$ imposed above (B.4).

Case (b): $p' = 0$

This case renders $\tilde{I}(p')$ independent of λ, given by the product of identical integrals $\tilde{I}(p'=0) =\ (I_\infty(0))^2$. From (B.6) one obtains finiteness for $0<\alpha<3/2$ whereof (B.7) is a subset.

Case (c): $0 < p' < \infty$

Let us first consider the second integral. It behaves like $(q'')^{2-2\alpha}$ for $q''\to 0$ which requires $\alpha < 3/2$. For $q''\to\infty$, it behaves like $q^{-\lambda}\cdot(q'')^{-1-2\alpha-\lambda}$ for $q\neq 0$ and like $(q'')^{-1-2\alpha}$ for $q=0$, resulting in the restrictions $\alpha > -\lambda/2$ and $\alpha > 0$, respectively. Hence, for the values of λ under consideration, one finds $\frac{1}{2}<\alpha<\frac{3}{2}$ in total.

Turning to the q-integral, it behaves for $q=0$ like $q^{2-2\alpha}$ as noted before, and for $q\to\infty$ like $q^{-1-2\alpha+\lambda}$ (where we have included the factor $q^{-\lambda}$ contained in the second integral) leading to $\alpha > \lambda/2$. In total we obtain the permitted interval $\frac{1}{4}<\alpha<\frac{3}{2}$. Since $\{\frac{1}{2}<\alpha<\frac{3}{2}\}\ \cap\ \{\frac{1}{4}<\alpha<\frac{3}{2}\}\ \supseteq\{\frac{1}{2}<\alpha<1\}$, the cases (b) and (c) give no further restriction on the interval (B.7). Hence we have proven that $I(\mathbf{p}')$ is finite for e.g. $\alpha = 3/4$ when $\lambda\in\{\pm\frac{1}{2},1\}$.

Appendix C

We present guidelines to the proof of the boundedness from above of the point spectrum of b_m.

The operator T_2 defined in (6.10) is given explicitly by

$$T_2(\mathbf{p},\mathbf{p}') := \int_{\mathbb{R}^3} d\mathbf{p}'' \, \frac{1}{|\mathbf{p}-\mathbf{p}''|^2} \, \frac{1}{|\mathbf{p}''-\mathbf{p}'|^2} \tag{C.1}$$

$$\Bigg\{ - \frac{p''^2}{2E_{p''}(E_{p''}+m)} \left[\frac{m}{E_{p''}} \left(\frac{1}{E_{p'}} \frac{1}{E_{p'}+E_{p''}} + \frac{1}{E_p} \frac{1}{E_p+E_{p''}} \right) \right.$$
$$\left. + \left(\frac{1}{E_{p'}+E_{p''}} + \frac{1}{E_p+E_{p''}} \right) \left(\frac{1}{E_{p''}} + \frac{m}{E_{p''}^2} \right) \right]$$
$$+ \boldsymbol{\sigma}\hat{\mathbf{p}}'' \, \boldsymbol{\sigma}\hat{\mathbf{p}}' \, \frac{p'p''}{2E_{p''}(E_{p'}+m)} \left[\frac{m}{E_{p''}} \left(\frac{1}{E_{p'}} \frac{1}{E_{p'}+E_{p''}} + \frac{1}{E_p} \frac{1}{E_p+E_{p''}} \right) \right.$$
$$\left. + \left(\frac{1}{E_{p'}+E_{p''}} + \frac{1}{E_p+E_{p''}} \right) \left(\frac{1}{E_{p'}} + \frac{m}{E_{p''}^2} \right) \right]$$
$$+ \boldsymbol{\sigma}\hat{\mathbf{p}}'' \, \boldsymbol{\sigma}\hat{\mathbf{p}} \, \frac{pp''}{2E_{p''}(E_p+m)} \left[\frac{m}{E_{p''}} \left(\frac{1}{E_{p'}(E_{p'}+E_{p''})} + \frac{1}{E_p(E_p+E_{p''})} \right) \right.$$
$$\left. + \left(\frac{1}{E_{p'}+E_{p''}} + \frac{1}{E_p+E_{p''}} \right) \left(\frac{1}{E_p} + \frac{m}{E_{p''}^2} \right) \right]$$
$$- \boldsymbol{\sigma}\hat{\mathbf{p}} \, \boldsymbol{\sigma}\hat{\mathbf{p}}' \, \frac{pp'(E_{p''}+m)}{2E_{p''}(E_p+m)(E_{p'}+m)} \left[\frac{m}{E_{p''}} \left(\frac{1}{E_{p'}} \frac{1}{E_{p'}+E_{p''}} + \frac{1}{E_p} \frac{1}{E_p+E_{p''}} \right) \right.$$
$$\left. + \left(\frac{1}{E_{p'}+E_{p''}} + \frac{1}{E_p+E_{p''}} \right) \left(\frac{1}{E_p} + \frac{1}{E_{p'}} - \frac{1}{E_{p''}} + \frac{m}{E_{p''}^2} \right) \right] \Bigg\}$$

We demonstrate the procedure of estimating the integral over $T_2(\mathbf{p},\mathbf{p}')$ introduced in (6.11) for one particular term,

$$I := \int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{p}' d\mathbf{p}'' \, |\boldsymbol{\sigma}\hat{\mathbf{p}}'' \, \boldsymbol{\sigma}\hat{\mathbf{p}}'| \, \frac{p'p''}{2E_{p''}(E_{p'}+m)} \, \frac{m}{E_{p''}E_p} \, \frac{1}{E_p+E_{p''}}$$
$$\cdot \frac{1}{|\mathbf{p}-\mathbf{p}''|^2} \, \frac{1}{|\mathbf{p}''-\mathbf{p}'|^2} \, \left| \frac{f(p)}{f(p')} \right|^2 . \tag{C.2}$$

We take $f^2(p) := p^{5/2}/(E_p+m)$ and make the substitutions $\mathbf{q}'' := \dfrac{\mathbf{p}''}{mq}$ and $\mathbf{q}' := \dfrac{\mathbf{p}'}{mqq''}$ for $\mathbf{p}''$ and $\mathbf{p}'$, respectively. With $\mathbf{q} := \mathbf{p}/m$ we perform the angular integrations with the help of (B.1) and estimate $|\boldsymbol{\sigma}\hat{\mathbf{p}}'' \, \boldsymbol{\sigma}\hat{\mathbf{p}}'|$ by 1 such that

$$I \;\le\; 2\pi^2 q^4 \, \frac{1}{\sqrt{q^2+1}} \, \frac{1}{\sqrt{q^2+1}+1} \, \int_0^\infty dq' \, \frac{1}{q'^{\frac{1}{2}}} \, \ln\left| \frac{1+q'}{1-q'} \right|$$

$$\cdot \int_0^\infty dq'' q''^{\frac{3}{2}} \ \ln\left|\frac{1+q''}{1-q''}\right| \ \frac{1}{(qq'')^2+1} \ \frac{1}{\sqrt{q^2+1}+\sqrt{(qq'')^2+1}} \tag{C.3}$$

We estimate the last factor with the help of $\sqrt{(qq'')^2+1} \ \geq 1$ and then use the estimate (6.12) to obtain

$$\begin{aligned} I \ &\leq \ 4\pi^3 q^4 \frac{1}{\sqrt{q^2+1}} \frac{1}{(\sqrt{q^2+1}+1)^2} \left[\int_0^{1/q} dq'' q''^{\frac{3}{2}} \ \ln\left|\frac{1+q''}{1-q''}\right| \right. \\ &\left. + \frac{1}{q^2} \int_{1/q}^\infty dq'' \frac{1}{q''^{\frac{1}{2}}} \ln\left|\frac{1+q''}{1-q''}\right| \right] \\ &= \ 8\pi^3 q^2 \frac{1}{\sqrt{q^2+1}} \frac{1}{(\sqrt{q^2+1}+1)^2} \left[\pi \ - \ \frac{4}{5q^{\frac{1}{2}}} \ln\left|\frac{1+q}{1-q}\right| \right. \\ &\left. + \left(\frac{2q^2}{5} - 2\right) \arctan\frac{1}{\sqrt{q}} \ + \ \left(1 \ - \ \frac{q^2}{5}\right) \ln\left|\frac{1+\sqrt{q}}{1-\sqrt{q}}\right| \ + \ \frac{4}{15}\sqrt{q} \right] \end{aligned} \tag{C.4}$$

Due to the above choice of $f(p)$, the r.h.s. of (C.4)$\sim q^{\frac{5}{2}}$ for $q \to 0,$ assuring that its contribution to $M_2(q)$ defined below (6.12) is finite. The integrals occuring here and in the remaining contributions to T_2 can be found in [9, p.205,206] after substitutions of the type $x := 1/q, \ x := q^{\frac{1}{2}}$.

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