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The single-particle pseudorelativistic Jansen–Hess operator with magnetic field

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Abstract

The pseudorelativistic no-pair Jansen–Hess operator is derived for the case where in addition to the Coulomb potential an external magnetic field $\mathbf{B}$ is permitted. With some restrictions on the vector potential, it is shown that this operator is positive provided the strength γ of the Coulomb potential is below a critical value ($\gamma_c \leqslant 0.35$, depending on the magnetic field energy E_f). Moreover, for $\gamma < 0.32$ and for $\mathbf{B}$ tending asymptotically to zero in a weak sense, the essential spectrum is given by $[m, \infty) + E_f$.

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1. Introduction

The spectral properties of the Dirac operator and its nonrelativistic limit, the Pauli operator, describing an atom in an external magnetic field, are a topic of current interest (see the comprehensive review by Erdös [8]). The Dirac operator for an electron in an electric field V and a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, acting in the Hilbert space $L_2(\mathbb{R}^3) \otimes \mathbb{C}^4$, is given by [3, section 1.3]

$$\begin{aligned} H &= D_A + V + E_f \\ D_A &:= \boldsymbol{\alpha}\mathbf{p}_A + \beta m, \qquad \mathbf{p}_A := \mathbf{p} - e\mathbf{A}. \end{aligned} \tag{1.1}$$

D_A is the free Dirac operator with $\boldsymbol{\alpha}$ and β Dirac matrices, m is the electron mass, $V = -\gamma/x$ is the Coulomb field generated by a nucleus of charge Z fixed at the origin ($\gamma = Ze^2$ with $e^2 \approx 1/137.04$ the fine structure constant). In (1.1) the (classical) field energy E_f is included:

$$E_f := \frac{1}{8\pi}\int_{\mathbb{R}^3} B^2(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{1}{8\pi}\|\mathbf{B}\|^2 \tag{1.2}$$

where $\|\cdot\|$ denotes the L_2-norm, $\mathbf{x}$ is the coordinate and $\mathbf{p} = -\mathrm{i}\nabla$ the momentum of the electron. Relativistic units ($\hbar = c = 1$) are used and $|\mathbf{x}| = x$. There is a simple relation to the Pauli operator, $\frac{1}{2m}(\boldsymbol{\sigma}\mathbf{p}_A)^2 = \frac{1}{2m}[(\mathbf{p}_A)^2 - e\boldsymbol{\sigma}\mathbf{B}]$, where $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices

[3, section 1.4],

$$D_A^2 = (\mathbf{p} - e\mathbf{A})^2 - e\boldsymbol{\sigma}\mathbf{B} + m^2. \tag{1.3}$$

We need regularity conditions on the vector potential $\mathbf{A}$ to assure that H is well defined and self-adjoint. First, we require that

$$\nabla \cdot \mathbf{A} = 0, \qquad \|\mathbf{B}\| < \infty. \tag{1.4}$$

These conditions imply the commutation relation $\mathbf{pA} = \mathbf{Ap}$ [19, p 438] and $\mathbf{A} \in L_6(\mathbb{R}^3)$ which results from a Sobolev inequality [9]. The condition $\mathbf{B} \in L_2(\mathbb{R}^3)$ renders E_f finite. If, in addition to $\nabla \cdot \mathbf{A} = 0$, $\mathbf{A}$ is a C^1-function, it was shown ([17], based on [13]) that $(\mathbf{p}_A)^2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^2$. Later, $\mathbf{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^3)$ was established as the weakest possible condition for this property to be true [1], [5, p 9]. As a second condition, we require therefore that $\mathbf{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^3)$. Let the magnetic field satisfy

$$N_B(\mathbf{x}) := \int_{|\mathbf{x}-\mathbf{y}|\leqslant 1} |\mathbf{B}(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y} \leqslant C \tag{1.5}$$

with a constant $C \in \mathbb{R}$ independent of $\mathbf{x}$ ((1.5) holds for any $\mathbf{B} \in L_2(\mathbb{R}^3)$). This guarantees the essential self-adjointness of the Pauli operator. The proof is based on the work of Udim [32, theorem 4.2], showing that a consequence of (1.5) is the $(\mathbf{p}_A)^2$-boundedness of $e\boldsymbol{\sigma}\mathbf{B}$ with bound zero. This property establishes the required essential self-adjointness according to the Kato–Rellich theorem [28, theorem X.12].

From the symmetry of $\boldsymbol{\sigma}\mathbf{p}_A$, we have $(\psi, (\boldsymbol{\sigma}\mathbf{p}_A)^2\psi) = \|\boldsymbol{\sigma}\mathbf{p}_A\psi\|^2 \geqslant 0$ for $\psi \in C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^2$. Thus, $(\boldsymbol{\sigma}\mathbf{p}_A)^2$ is a non-negative, self-adjoint operator (by means of closure). It follows [18, theorem 3.35, p 281] that this is also true for

$$E_A := |D_A| = \sqrt{(\boldsymbol{\sigma}\mathbf{p}_A)^2 + m^2} \geqslant m \tag{1.6}$$

which is the kinetic energy term of the pseudorelativistic operator that will be introduced in section 2.

Due to the positron degrees of freedom, the Dirac operator H has a spectrum which is unbounded from below. However, in the spectroscopy of static or slowly moving ions, pair creation plays no role. One of the current techniques, used in the field-free case ($\mathbf{A} = 0$), to construct from H an operator which solely describes the electronic states is the application of a unitary transformation scheme to H (see, e.g., [7, 15, 30]). A perturbative expansion in the central field strength γ provides pseudorelativistic operators which are block diagonal in the free (i.e., $Z = 0$) electronic positive and negative spectral subspaces up to a given order in γ. The zero- plus first-order term in this series, the Brown–Ravenhall operator, has obtained widespread interest because it is simply the restriction of H to the positive spectral subspace. The terms up to second order, comprising the Jansen–Hess operator, provide, however, a much better representation of the bound-state energies [35]. This operator has been proven to be positive with essential spectrum $\sigma_{\mathrm{ess}} = [m, \infty)$ for sufficiently small γ [4, 12, 14].

If $\mathbf{A} \neq 0$, investigations are scarce. It is known that in the absence of the Coulomb field V, the Dirac operator can be block diagonalized by means of a Foldy–Wouthuysen transformation U_0 [6, section 3.1],

$$\begin{aligned} U_0 D_A U_0^{-1} &= \beta E_A \\ U_0 &:= \left(\frac{m + E_A}{2E_A}\right)^{\frac{1}{2}} + \frac{\beta\boldsymbol{\alpha}\mathbf{p}_A}{(2E_A(m + E_A))^{\frac{1}{2}}}. \end{aligned} \tag{1.7}$$

U_0^{-1} is obtained from U_0 by replacing $\beta\boldsymbol{\alpha}\mathbf{p}_A$ by $\boldsymbol{\alpha}\mathbf{p}_A\beta = -\beta\boldsymbol{\alpha}\mathbf{p}_A$. For later use, we note that E_A commutes with U_0, $[E_A, U_0] = 0$, because $[\beta\boldsymbol{\alpha}\mathbf{p}_A, E_A] = [\beta, E_A]\,\boldsymbol{\alpha}\mathbf{p}_A + \beta[\boldsymbol{\alpha}\mathbf{p}_A, E_A]$ vanishes (the first commutator being zero since E_A is block diagonal). There are also a few studies of the 'magnetic' Brown–Ravenhall operator showing that this operator is either unbounded from below (if $\mathbf{A}$ is disregarded in the projector onto the positive spectral subspace [10]) or that it is positive for $\gamma < \frac{2}{\pi}$ (if $\mathbf{A}$ is not disregarded) which assures stability of relativistic matter in this model [22, 23].

The aim of the present work is to derive the 'magnetic' Jansen–Hess operator $H^{(2)}$ from the corresponding transformation scheme (section 2), to show under which conditions it is positive (theorem 1, section 4) and to provide criteria for $\sigma_{\mathrm{ess}} = [m, \infty) + E_f$ to hold (theorem 3, section 6). An auxiliary step is the invariance of the essential spectrum upon removal of the Jansen–Hess potential (theorem 2, section 5). Consequently, theorem 3 also holds for the 'magnetic' Brown–Ravenhall operator (which results from dropping the second-order term in γ). The basic difference from the $\mathbf{A} = 0$ case in constructing and analysing $H^{(2)}$ is due to the fact that the kinetic energy operator E_A is no longer a multiplicator in momentum space (as is $E_{A=0} =: E_P = \sqrt{p^2 + m^2}$). Hence, formal techniques have to replace Fourier analysis (sections 2 and 3). Moreover, in contrast to the 'magnetic' Brown–Ravenhall operator, the required bounds on γ for self-adjointness and positivity depend nontrivially on the magnetic field. Therefore, these bounds are inferior to the $\mathbf{A} = 0$ case. With $\gamma \to 0$ for $B \to \infty$, our analysis makes the Jansen–Hess operator an unlikely candidate for stability of matter. However, for laboratory magnetic fields up to 10^{12} G this operator should be superior to the 'magnetic' Brown–Ravenhall operator regarding electron spectroscopy.

2. The transformed Dirac operator

Let us define the projector onto the positive magnetic spectral subspace of the electron (defined by switching off V but fully including $\mathbf{A}$),

$$\Lambda_{A,+} := \frac{1}{2}\left(1 + \frac{D_A}{|D_A|}\right). \tag{2.1}$$

For any $\varphi_+ \in \mathcal{H}_{+,1} := \Lambda_{A,+}(H_1(\mathbb{R}^3) \otimes \mathbb{C}^4)$ (where the Sobolev space $H_1(\mathbb{R}^3) \otimes \mathbb{C}^4$ is the domain of D_A), we have trivially $\Lambda_{A,+}\varphi_+ = \varphi_+$ and $D_A\varphi_+ = E_A\varphi_+$, and one easily verifies that with $\psi := \binom{u}{0}$, $u \in H_1(\mathbb{R}^3) \otimes \mathbb{C}^2$, φ_+ can be expressed as

$$\varphi_+ = U_0^{-1}\,\psi \tag{2.2}$$

(namely using (1.7), $D_A\big(U_0^{-1}\psi\big) = U_0^{-1}\beta E_A\psi = U_0^{-1}E_A(\beta\psi) = U_0^{-1}E_A\psi = E_A U_0^{-1}\psi$).

Let $H_V := D_A + V$. We construct a unitary transformation U such that the transformed Dirac operator decouples the magnetic spectral subspaces of the electron,

$$U^{-1}HU = \Lambda_{A,+}(U^{-1}H_V U)\Lambda_{A,+} + \Lambda_{A,-}(U^{-1}H_V U)\Lambda_{A,-} + E_f, \tag{2.3}$$

with $\Lambda_{A,+}$ from (2.1) and $\Lambda_{A,-} = 1 - \Lambda_{A,+}$. The choice of the projector $\Lambda_{A,+}$ in (2.3) preserves the gauge invariance of the transformed operator [22]. The field energy E_f is a constant which is not affected by U. If one defines P_+ as the projector onto the positive spectral subspace of the Dirac operator H_V, then (2.3) is equivalent to the condition

$$U^{-1}P_+U = \Lambda_{A,+}. \tag{2.4}$$

If, in addition, the Foldy–Wouthuysen transformation U_0 is applied, the desired block-diagonal operator is obtained as a consequence of $U_0\Lambda_{A,+}U_0^{-1} = \frac{1}{2}(1+\beta)$ (see (1.7) and the discussion below):

$$\begin{aligned} M &= \frac{1}{2}(1+\beta)M\frac{1}{2}(1+\beta) + \frac{1}{2}(1-\beta)M\frac{1}{2}(1-\beta) =: \begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix} \\ M &:= U_0\,U^{-1}H_V\,U\,U_0^{-1} \end{aligned} \tag{2.5}$$

where h, g are matrices in $\mathbb{C}^{2,2}$.

Rather than solving (2.4) for U (which was recently achieved in the field-free case [29, 30]), we start from (2.3) and apply a technique [15] which is equivalent to the Douglas–Kroll transformation scheme [7, 16]. We formally expand $U = \exp\left(\mathrm{i}\sum_{k=1}^{\infty} B_k\right)$, where B_k is an operator which contains the potential V to kth order, and we are interested in the transformed operator which is block diagonal up to second order in the potential strength γ. Denoting by $H^{(2)}$ the second-order solution of (2.3) restricted to $\mathcal{H}_{+,1}$ (the 'magnetic' Jansen–Hess operator) we have, in analogy to the $\mathbf{A} = 0$ case,

$$H^{(2)} := \Lambda_{A,+}\left\{D_A + V + \frac{\mathrm{i}}{2}[W_1, B_1] + E_f\right\}\Lambda_{A,+} \tag{2.6}$$

with $W_1 := \Lambda_{A,+}V\Lambda_{A,-} + \Lambda_{A,-}V\Lambda_{A,+}$ being the off-diagonal part of V. B_1 is determined from the condition

$$W_1 = -\mathrm{i}[D_A, B_1]. \tag{2.7}$$

Alternatively, we can obtain B_1 from (2.4). Using the integral representation of P_+ [18, chapter II.1.4] and expanding P_+ in terms of V by means of the second resolvent identity, we have

$$\begin{aligned} P_+ &= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{D_A + V + \mathrm{i}\eta} \\ &= \Lambda_{A,+} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{D_A + \mathrm{i}\eta} V \frac{1}{D_A + V + \mathrm{i}\eta} = \Lambda_{A,+} + F_A + R \end{aligned} \tag{2.8}$$

where $\Lambda_{A,+}$ and F_A are the zero- and first-order terms, respectively, while the remainder R is of higher order in V. Defining $\tilde{D}_A := D_A/|D_A|$ and solving (2.4) up to first order in V, we get

$$2F_A - \mathrm{i}B_1\tilde{D}_A + \mathrm{i}\tilde{D}_A B_1 = 0. \tag{2.9}$$

Multiplication of (2.9) by $\tilde{D}_A$ from the left and, respectively, from the right and addition of the resulting equations provides the useful relation

$$F_A\tilde{D}_A = -\tilde{D}_A F_A. \tag{2.10}$$

Whereas (2.9) is also only an implicit equation for B_1, a trial for B_1 can be found from the formal solution U of (2.4) which is completely analogous to the field-free case [29], $U^{-1} = [1 + (\Lambda_{A,+} - \Lambda_{A,-})(P_+ - \Lambda_{A,+})](1 - (P_+ - \Lambda_{A,+})^2)^{-\frac{1}{2}})$. An expansion of this formal solution up to first order in V leads to

$$B_1 = \mathrm{i}\tilde{D}_A F_A. \tag{2.11}$$

With the help of (2.10), it is easily verified that (2.11) is indeed a solution to (2.9). Insertion into (2.6) finally results in

$$\begin{aligned} H^{(2)} &= \Lambda_{A,+}\{D_A + V + B_{2m} + E_f\}\Lambda_{A,+} \\ B_{2m} &:= \tfrac{1}{4}[VF_A\tilde{D}_A + \tilde{D}_A F_A V + \tilde{D}_A V F_A + F_A V \tilde{D}_A]. \end{aligned} \tag{2.12}$$

3. Relative form boundedness of the Jansen–Hess potential

In order to establish self-adjointness of $H^{(2)}$, the form boundedness of the potential contributions to $H^{(2)}$ (restricted to the 'positive' space $\mathcal{H}_{+,1}$) relative to the kinetic energy operator E_A is needed. We have to fix the potential strength γ such that this bound becomes smaller than one. We start by showing the relative boundedness of the linear term (in γ) V, then we prove the boundedness of the operator B_1 (introduced by the transformation U) and subsequently the relative boundedness of the quadratic term. The resulting form boundedness of the Jansen–Hess potential relative to E_A is stated in lemma 1, and the condition for $H^{(2)}$ being self-adjoint is part of theorem 1.

3.1. E_A-boundedness of V and boundedness of B_1

A basic ingredient is the inequality $\big(\varphi, \exp\big(-\mathbf{p}_A^2 t\big)\varphi\big) \leqslant (\varphi, \exp(-p^2 t)\varphi)$, valid for $t \geqslant 0$ and $\mathbf{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^3)$ ([1] and references therein). Making use of $(\varphi, p^2\varphi) = -\lim_{t\to 0} \big(\varphi, \frac{\exp(-tp^2)-1}{t}\varphi\big)$ [24], one derives

$$(\varphi, (\mathbf{p} - e\mathbf{A})^2\varphi) \geqslant (\varphi, p^2\varphi) \tag{3.1}$$

which is known as diamagnetic inequality (see also earlier work [13] for the related inequality $(|\varphi|, p^2|\varphi|) \leqslant (\varphi, (\mathbf{p} - e\mathbf{A})^2\varphi))$. A consequence is

$$|\mathbf{p} - e\mathbf{A}| \geqslant p. \tag{3.2}$$

Further, let $\mathcal{O}_- := \frac{1}{2}(|\mathcal{O}| - \mathcal{O}) \geqslant 0$ be the negative part of an operator $\mathcal{O}$ and $\mathrm{tr}\,\mathcal{O}_-$ its trace (i.e., the sum over the absolute values of the negative eigenvalues of $\mathcal{O}$ times the spin degrees of freedom). Then by means of (3.1) and the Lieb–Thirring inequality [21, 23] for any $\mu > 0$ and $d > 0$ one has

$$\mathrm{tr}[\mu(\mathbf{p} - e\mathbf{A})^2 + e\boldsymbol{\sigma}\mathbf{B}]_-^d \leqslant \mu^d\,\mathrm{tr}\left[p^2 + \frac{e\boldsymbol{\sigma}\mathbf{B}}{\mu}\right]_-^d \leqslant 2\mu^d L_{d,3}\int_{\mathbb{R}^3}\left(\frac{e|\mathbf{B}|}{\mu}\right)^{d+\frac{3}{2}} \mathrm{d}\mathbf{x} \tag{3.3}$$

with constants $L_{\frac{1}{2},3} \leqslant 0.060\,03$ and $L_{1,3} \leqslant 0.0403$.

Then, following [23] we get the form estimate for $\varphi \in H_1(\mathbb{R}^3) \otimes \mathbb{C}^4$, $\|\varphi\| = 1$, using Kato's inequality $\frac{1}{x} \leqslant \frac{\pi}{2}p$ and (3.2) as well as the trace inequality for non-negative, self-adjoint operators, $\mathrm{tr}(\mathcal{O}_1 - \mathcal{O}_2)_- \leqslant \mathrm{tr}\big(\mathcal{O}_1^2 - \mathcal{O}_2^2\big)_-^{\frac{1}{2}}$,

$$\begin{aligned}(\varphi, E_A\varphi) - \left(\varphi, \frac{\gamma_0}{x}\varphi\right) &\geqslant \left(\varphi, \sqrt{E_A^2 - m^2}\varphi\right) - \frac{\gamma_0\pi}{2}(\varphi, |\mathbf{p} - e\mathbf{A}|\varphi)\\ &\geqslant -\mathrm{tr}\left[\big(E_A^2 - m^2\big) - \left(\frac{\gamma_0\pi}{2}|\mathbf{p} - e\mathbf{A}|\right)^2\right]_-^{\frac{1}{2}}\\ &\geqslant -2L_{\frac{1}{2},3}\frac{e^2}{[1 - (\gamma_0\pi/2)^2]^{\frac{3}{2}}}\|\mathbf{B}\|^2\end{aligned} \tag{3.4}$$

for $\gamma_0 < \frac{2}{\pi}$.

Moreover, using $\mathrm{tr}\,\mathcal{O}_- \leqslant \big(\mathrm{tr}\,\mathcal{O}_-^{\frac{1}{2}}\big)^2$ [27, p 210] and Hardy's inequality $\frac{1}{x^2} \leqslant 4p^2$,

$$\begin{aligned}\|E_A\varphi\|^2 - \left\|\frac{\gamma_1}{x}\varphi\right\|^2 &\geqslant \big(\varphi, \big[\big(1 - 4\gamma_1^2\big)(\mathbf{p} - e\mathbf{A})^2 - e\boldsymbol{\sigma}\mathbf{B}\big]\varphi\big)\\ &\geqslant -\big(\mathrm{tr}\,\big[\big(1 - 4\gamma_1^2\big)(\mathbf{p} - e\mathbf{A})^2 - e\boldsymbol{\sigma}\mathbf{B}\big]_-^{\frac{1}{2}}\big)^2\\ &\geqslant -\left[2L_{\frac{1}{2},3}\frac{e^2}{\big[1 - 4\gamma_1^2\big]^{\frac{3}{2}}}\|\mathbf{B}\|^2\right]^2\end{aligned} \tag{3.5}$$

for $\gamma_1 < \frac{1}{2}$. Thus, we obtain the E_A-boundedness of the potential V in the form and in the norm [28, p 162],

$$|(\varphi, V\varphi)| \leqslant \frac{\gamma}{\gamma_0}(\varphi, E_A\varphi) + \gamma c_B(\varphi, \varphi), \qquad c_B := \frac{2}{\gamma_0}L_{\frac{1}{2},3}\frac{e^2}{[1 - (\gamma_0\pi/2)^2]^{\frac{3}{2}}}\|\mathbf{B}\|^2 \tag{3.6}$$

and

$$\|V\varphi\| \leqslant \frac{\gamma}{\gamma_1}\|E_A\varphi\| + \gamma d_B\|\varphi\|, \qquad d_B := \frac{2}{\gamma_1} L_{\frac{1}{2},3} \frac{e^2}{\left[1-4\gamma_1^2\right]^{\frac{3}{2}}}\|\mathbf{B}\|^2. \tag{3.7}$$

The boundedness of B_1 is a consequence of the boundedness of F_A, since

$$\|B_1\| \leqslant \|\tilde{D}_A\|\|F_A\| = \|F_A\|. \tag{3.8}$$

With (3.6) at hand, the boundedness of F_A is easy to show. Following the proof of [30, lemma 1], we have for $\varphi_+, \psi_+ \in \mathcal{H}_{+,1}$ from (2.8)

$$\begin{aligned}
\|F_A\| &= \frac{1}{2\pi}\left\|\int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{D_A + \mathrm{i}\eta} V \frac{1}{D_A + \mathrm{i}\eta}\right\| \\
&\leqslant \frac{\gamma}{2\pi} \sup_{\|\varphi_+\|=\|\psi_+\|=1} \int_{-\infty}^{\infty} \mathrm{d}\eta \left|\left(\varphi_+, \frac{1}{D_A + \mathrm{i}\eta}\frac{1}{x^{1/2}} \cdot \frac{1}{x^{1/2}}\frac{1}{D_A + \mathrm{i}\eta}\psi_+\right)\right| \\
&\leqslant \frac{\gamma}{2\pi} \sup_{\|\varphi_+\|=\|\psi_+\|=1} \int_{-\infty}^{\infty} \mathrm{d}\eta \left\|\frac{1}{x^{1/2}}\frac{1}{D_A - \mathrm{i}\eta}\varphi_+\right\| \cdot \left\|\frac{1}{x^{1/2}}\frac{1}{D_A + \mathrm{i}\eta}\psi_+\right\|.
\end{aligned} \tag{3.9}$$

An application of the Schwarz inequality leads to

$$\|F_A\| \leqslant \frac{\gamma}{2\pi} \sup_{\|\varphi_+\|=\|\psi_+\|=1} \left(\int_{-\infty}^{\infty} \mathrm{d}\eta \left\|\frac{1}{x^{1/2}}\frac{1}{D_A - \mathrm{i}\eta}\varphi_+\right\|^2\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{\infty} \mathrm{d}\eta \left\|\frac{1}{x^{1/2}}\frac{1}{D_A + \mathrm{i}\eta}\psi_+\right\|^2\right)^{\frac{1}{2}}. \tag{3.10}$$

Setting $\varphi := \frac{1}{D_A - \mathrm{i}\eta}\varphi_+$ (note that $D_A^2 > 0$ for $m \neq 0$ such that $(D_A - \mathrm{i}\eta)^{-1}$ is bounded for $\eta \in \mathbb{R}$), we have from (3.6)

$$\left\|\frac{1}{x^{\frac{1}{2}}}\frac{1}{D_A - \mathrm{i}\eta}\varphi_+\right\|^2 = \left(\varphi, \frac{1}{x}\varphi\right) \leqslant \frac{1}{\gamma_0}(\varphi, E_A\varphi) + c_B(\varphi, \varphi) \tag{3.11}$$

and thus we get for the two (equal) integrals in (3.10), using $D_A\varphi_+ = E_A\varphi_+$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}\eta \left\|\frac{1}{x^{1/2}}\frac{1}{D_A - \mathrm{i}\eta}\varphi_+\right\|^2 &\leqslant \frac{1}{\gamma_0}\left(\varphi_+, \int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{E_A + \mathrm{i}\eta} E_A \frac{1}{E_A - \mathrm{i}\eta}\varphi_+\right) \\
&\quad + c_B\left(\varphi_+, \int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{E_A^2 + \eta^2}\varphi_+\right) = \frac{1}{\gamma_0} \cdot \pi\|\varphi_+\|^2 + c_B\pi\left(\varphi_+, \frac{1}{E_A}\varphi_+\right).
\end{aligned} \tag{3.12}$$

We estimate $E_A \geqslant m$ and finally obtain the boundedness of $\|F_A\|$:

$$\|F_A\| \leqslant \frac{\gamma}{2\gamma_0}\left(1 + c_B\frac{\gamma_0}{m}\right). \tag{3.13}$$

We note that due to the existence of zero modes [25] the lower bound m of E_A is sharp: there is a field $\mathbf{B}_0 = \nabla \times \mathbf{A}_0 \in L_2(\mathbb{R}^3)$, satisfying (1.4) and hence (1.5), and a function $\psi_0 \in H_1(\mathbb{R}^3)\backslash\{0\} \otimes \mathbb{C}^2$ such that

$$\boldsymbol{\sigma}(\mathbf{p} - e\mathbf{A}_0)\psi_0 = 0. \tag{3.14}$$

From this it follows that the 4-spinor $\binom{\psi_0}{0}$ obeys $D_{A_0}\binom{\psi_0}{0} = m\binom{\psi_0}{0}$, i.e. it lies in the positive magnetic spectral subspace of the electron, and m is the lowest positive eigenvalue of D_{A_0}.

3.2. Relative boundedness of the Jansen–Hess potential

From (2.12) we get for $\psi_+ \in \mathcal{H}_{+,1}$, with $\|\Lambda_{A,+}\| = 1$,

$$\begin{aligned}\|(4\Lambda_{A,+}B_{2m}\Lambda_{A,+})\psi_+\| &\leqslant \|4B_{2m}\psi_+\| \\ &\leqslant \|VF_A\tilde{D}_A\psi_+\| + \|\tilde{D}_AF_AV\psi_+\| + \|\tilde{D}_AVF_A\psi_+\| + \|F_AV\tilde{D}_A\psi_+\|.\end{aligned} \tag{3.15}$$

We shall estimate each of these four terms separately, using the boundedness (3.13) of F_A and the relative boundedness (3.7) of V. First, we show

$$[D_A, F_A] = \tfrac{1}{2}[\tilde{D}_A, V]. \tag{3.16}$$

We multiply (2.7) with $\tilde{D}_A$ and insert B_1 from (2.11). This gives

$$\tilde{D}_AW_1 = -\mathrm{i}\tilde{D}_A(\mathrm{i}D_A\tilde{D}_AF_A - \mathrm{i}\tilde{D}_AF_AD_A) = [D_A, F_A]. \tag{3.17}$$

Inserting for W_1 (below (2.6)) results in (3.16).

Using that $\|\tilde{D}_A\| = 1$ and $\tilde{D}_A\psi_+ = \psi_+$, (3.15) gives

$$\|4B_{2m}\psi_+\| \leqslant 2\|VF_A\psi_+\| + 2\|F_A\|\|V\psi_+\|. \tag{3.18}$$

With (3.7) and (3.16), defining $F_A\psi_+ =: \varphi$, we estimate the first term by

$$\begin{aligned}\|VF_A\psi_+\| &\leqslant \frac{\gamma}{\gamma_1}\||D_A|\varphi\| + \gamma d_B\|\varphi\| \leqslant \frac{\gamma}{\gamma_1}\|D_AF_A\psi_+\| + \gamma d_B\|F_A\|\|\psi_+\| \\ &\leqslant \frac{\gamma}{\gamma_1}\left\{\|F_A\|\|D_A\psi_+\| + \frac{1}{2}\|\tilde{D}_A\|\|V\psi_+\| + \frac{1}{2}\|V\psi_+\|\right\} + \gamma d_B\|F_A\|\|\psi_+\|.\end{aligned} \tag{3.19}$$

Thus, we get

$$\|B_{2m}\psi_+\| \leqslant \frac{\gamma}{\gamma_1}\left(\frac{\gamma}{2\gamma_1} + \|F_A\|\right)\|D_A\psi_+\| + \gamma d_B\left(\frac{\gamma}{2\gamma_1} + \|F_A\|\right)\|\psi_+\|. \tag{3.20}$$

Using (3.13) this results in

$$\begin{aligned}&\|B_{2m}\psi_+\| \leqslant c\|E_A\psi_+\| + C\|\psi_+\| \\ &c := \frac{\gamma^2}{2\gamma_1}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_0} + \frac{c_B}{m}\right), \qquad C := \gamma^2\frac{d_B}{2}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_0} + \frac{c_B}{m}\right).\end{aligned} \tag{3.21}$$

Note that both constants, c and C, depend on the field energy through $\|\mathbf{B}\| = (8\pi E_f)^{\frac{1}{2}}$.

From the E_A-boundedness of B_{2m} follows the E_A-form boundedness of B_{2m} with the same relative bound c [28, p 168]. Thus, we have proven

Lemma 1. *Let $H^{(2)} = D_A + V + B_{2m} + E_f$ be the 'magnetic' Jansen–Hess operator acting on $\mathcal{H}_{+,1}$. Then $V + B_{2m}$ is E_A-form bounded,*

$$|(\psi_+, (V + B_{2m})\psi_+)| \leqslant \left(\frac{\gamma}{\gamma_0} + c\right)(\psi_+, E_A\psi_+) + \tilde{C}(\psi_+, \psi_+), \tag{3.22}$$

with $c = \frac{\gamma^2}{2\gamma_1}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_0} + \frac{c_B}{m}\right)$, where c_B and d_B are defined in (3.6) and (3.7), and $\tilde{C}$ is some $\|\mathbf{B}\|$-dependent constant. (The parameters $\gamma_0 < \frac{2}{\pi}$ and $\gamma_1 < \frac{1}{2}$ can be chosen arbitrarily.)

4. Positivity of $H^{(2)}$

Let $\delta > 0$ and recall that $E_A \geqslant m$ is bounded below. If in (3.21), the δE_A-bound $\frac{c}{\delta}$ of B_{2m} is smaller than unity, then according to [18, theorem 4.11, p 291] $\delta E_A + B_{2m}$ is also bounded below by means of

$$(\psi_+, (\delta E_A + B_{2m})\psi_+) \geqslant \left(\delta m - \max\left\{\frac{C}{1-c/\delta}, C + cm\right\}\right)(\psi_+, \psi_+), \qquad (4.1)$$

where the constants c and C are defined in (3.21).

Using the above results, we can estimate

$$\begin{aligned}(\psi_+, H^{(2)}\psi_+) &= (\psi_+, E_A\psi_+) - |(\psi_+, V\psi_+)| + (\psi_+, B_{2m}\psi_+) + E_f(\psi_+, \psi_+) \\ &\geqslant \left(\psi_+, \left(\left(1 - \frac{\gamma}{\gamma_0}\right)E_A + B_{2m}\right)\psi_+\right) - \gamma c_B(\psi_+, \psi_+) + E_f(\psi_+, \psi_+) \\ &\geqslant \left(\left(1 - \frac{\gamma}{\gamma_0}\right)m - \max\left\{\frac{C}{1 - c/(1-\gamma/\gamma_0)}, C + cm\right\} - \gamma c_B + E_f\right)(\psi_+, \psi_+).\end{aligned} \qquad (4.2)$$

This results in

Theorem 1. *Let $H^{(2)} = D_A + V + B_{2m} + E_f$ be the 'magnetic' Jansen–Hess operator acting on $\mathcal{H}_{+,1}$. If the E_A-form bound of $V + B_{2m}$ is smaller than unity,*

$$\frac{\gamma}{\gamma_0} + c < 1, \qquad (4.3)$$

then $H^{(2)}$ is bounded below and thus extends to a self-adjoint operator on $\Lambda_{A,+}(L_2(\mathbb{R}^3)\otimes\mathbb{C}^4)$. If in addition

$$\left(1 - \frac{\gamma}{\gamma_0}\right)m - \gamma c_B - \max\left\{\frac{C(1-\gamma/\gamma_0)}{1-\gamma/\gamma_0 - c}, C + cm\right\} + E_f > 0, \qquad (4.4)$$

then $H^{(2)}$ is positive. This restricts the potential strength to $\gamma < \gamma_c$ where $\gamma_c \leqslant 0.353$ depending on the magnetic field $\mathbf{B}$.

In order to derive the conditions on the bound for γ which are required for theorem 1, we first consider the case $\mathbf{B} = 0$. Then, we can set $\gamma_0 = \frac{2}{\pi}$ and $\gamma_1 = \frac{1}{2}$, and both inequalities, (4.3) and (4.4), are satisfied for $\gamma < \gamma_c^{(0)}$, where $\gamma_c^{(0)} = 0.353 (Z \leqslant 48)$ is a solution of

$$\frac{\gamma}{\gamma_0} + c = \gamma\frac{\pi}{2} + \gamma^2\left(2 + \frac{\pi}{2}\right) = 1. \qquad (4.5)$$

This is considerably smaller than the critical γ obtained earlier for the field-free case ($\gamma_c^{(0)} = 1.006$ [4]), where one is able to work in momentum space and to use Mellin transform techniques.

When $\mathbf{B}$ is turned on, the bound on γ from the self-adjointness condition decreases slowly. For example, if $\|\mathbf{B}\| = 2.5$, then by optimizing γ_0 and γ_1 one gets from (4.3) $\gamma_c = 0.335 (\gamma_0 = 0.6, \gamma_1 = 0.498)$, whereas positivity is guaranteed for $\gamma < 0.316 (\gamma_0 = 0.6, \gamma_1 = 0.47)$. The relativistic ground-state binding energy of an electron, $|E_g - m| := m\left|\sqrt{1-\gamma^2} - 1\right| = 0.0644$ (in units where $m = 1$, using $\gamma = \gamma_c^{(0)}$), may be used as a reference value with which to compare the field energy E_f. Even for quite large fields[1], e.g. $\|\mathbf{B}\| = 10$ (where $E_f \approx 60|E_g - m|$), the critical potential strength (with

[1] In conventional units, $B = 1m^2e^3c/\hbar^3 = 2.35 \times 10^9$ G.

$\gamma_0 = 0.54$, $\gamma_1 = 0.499$) has only slightly decreased, $\gamma_c = 0.299$ ($Z < 41$) while $H^{(2)} > 0$ for $\gamma < 0.275$ ($\gamma_0 = 0.54$, $\gamma_1 = 0.45$).

However, when $\|\mathbf{B}\|$ becomes extremely large (but still is finite), our estimates (resulting in (4.4)) no longer guarantee positivity because C is of fourth order in $\|\mathbf{B}\|$ and eventually dominates E_f. In order to remedy this deficiency, different estimates for the E_A-boundedness of the potential V are required.

For the magnetic fields which are $(\mathbf{p}_A)^2$-bounded with bound $\kappa \to 0$ (and hence also $(\mathbf{p}_A)^2$-form bounded with the same bound), we have from (1.3)

$$\begin{aligned}(\varphi, |\mathbf{B}|\varphi) &\leqslant \kappa\big(\varphi, \mathbf{p}_A^2\,\varphi\big) + C_\kappa(\varphi,\varphi)\\ &\leqslant \kappa\big(\varphi, E_A^2\,\varphi\big) + \kappa e(\varphi, |\mathbf{B}|\varphi) + C_\kappa(\varphi,\varphi)\end{aligned} \tag{4.6}$$

proving the E_A^2-form boundedness of $|\mathbf{B}|$ with bound $\kappa/(1-\kappa e)$. It can be shown [34, proof of theorem 10.17] that the constant C_κ depends linearly on $\sup_{\mathbf{x}\in\mathbb{R}}(N_B(\mathbf{x}))^{\frac{1}{2}}$ which in turn can be estimated above by $\|\mathbf{B}\|$. So, we get from Hardy's inequality and (3.1)

$$(\varphi, V^2\,\varphi) \leqslant 4\gamma^2\big(\varphi, \mathbf{p}_A^2\,\varphi\big) \leqslant 4\gamma^2\big(\varphi, E_A^2\,\varphi\big) + 4\gamma^2 e(\varphi, |\mathbf{B}|\,\varphi). \tag{4.7}$$

Using (4.6), we eventually obtain the estimate

$$\|V\varphi\| \leqslant 2\gamma\left(1 + \frac{\kappa e}{1-\kappa e}\right)^{\frac{1}{2}}\|E_A\varphi\| + 2\gamma\left(\frac{e}{1-\kappa e}\right)^{\frac{1}{2}} C_\kappa^{\frac{1}{2}}\,\|\varphi\| \tag{4.8}$$

in place of (3.7). Note that κ can be taken arbitrarily close to 0 such that the E_A-bound of V agrees with the one in (3.7). However, the last term in (4.8) increases only $\sim \|\mathbf{B}\|^{\frac{1}{2}}$. A similar estimate replaces (3.6) for the E_A-form boundedness of V.

In order to get explicit constants, let us for the moment assume that $\mathbf{B}$ is bounded with $\|\mathbf{B}\|_\infty \leqslant \|\mathbf{B}\|$. Then, the last term in (4.7) is estimated by $4\gamma^2 e\|\mathbf{B}\|_\infty(\varphi,\varphi) \leqslant 4\gamma^2 e\|\mathbf{B}\|(\varphi,\varphi)$, giving $\|V\varphi\| \leqslant 2\gamma\|E_A\varphi\| + 2\gamma(e\|\mathbf{B}\|)^{\frac{1}{2}}\|\varphi\|$. For the form bound, using Kato's inequality, one gets

$$\begin{aligned}|(\varphi, V\varphi)| &\leqslant \gamma\frac{\pi}{2}(\varphi, |\mathbf{p} - e\mathbf{A}|\,\varphi) \leqslant \gamma\frac{\pi}{2}\Big(\varphi, \sqrt{E_A^2 + e|\mathbf{B}|}\,\varphi\Big)\\ &\leqslant \gamma\frac{\pi}{2}(\varphi, E_A\,\varphi) + \gamma\frac{\pi}{2}(e\|\mathbf{B}\|)^{\frac{1}{2}}(\varphi,\varphi).\end{aligned} \tag{4.9}$$

When (3.6) and (3.7) are replaced by these two inequalities in the subsequent estimates, conditions (4.3) and (4.4) of theorem 1 now read

$$1 - \gamma\frac{\pi}{2} - c_1 > 0 \tag{4.10}$$

and

$$\left(1 - \gamma\frac{\pi}{2}\right)m - \gamma\frac{\pi}{2}(e\|\mathbf{B}\|)^{\frac{1}{2}} - \max\left\{\frac{C_1(1-\gamma\pi/2)}{1-\gamma\pi/2 - c_1}, C_1 + c_1 m\right\} + E_f > 0 \tag{4.11}$$

where c_1 and C_1 are the changed bounds for B_{2m}, replacing (3.21),

$$c_1 := \gamma^2\left(2 + \frac{\pi}{2} + \frac{\pi}{2}\frac{(e\|\mathbf{B}\|)^{\frac{1}{2}}}{m}\right), \qquad C_1 := \gamma^2\left(\left[2 + \frac{\pi}{2}\right](e\|\mathbf{B}\|)^{\frac{1}{2}} + \frac{\pi}{2}\frac{e\|\mathbf{B}\|}{m}\right). \tag{4.12}$$

In condition (4.11) for the positivity of $H^{(2)}$ the leading term in $\|\mathbf{B}\|$ is now E_f, guaranteeing positivity for sufficiently large $\|\mathbf{B}\|$. For example, for $\|\mathbf{B}\| = 10$, (4.10) and (4.11) hold for $\gamma < 0.304$, this limit already exceeding the corresponding one from (4.3).

We close this section by showing that a $\mathbf{B}$-dependent constant in the form boundedness of V (which in turn leads to a $\mathbf{B}$-dependent condition (4.3) for self-adjointness of $H^{(2)}$) cannot be avoided [36].

It was proven [2] that for a homogeneous magnetic field **B**, the ground-state energy of the Pauli operator in a central Coulomb field of any given strength $Z_0\, e^2$ diverges logarithmically with B. This leads to the estimate

$$\frac{1}{2m}\left(\varphi, \left(E_A^2 - m^2\right)\varphi\right) - \left(\varphi, \frac{Z_0\, e^2}{x}\,\varphi\right) \geqslant -c_0(\ln B)^2\,(\varphi,\varphi), \tag{4.13}$$

with a suitable (Z_0-dependent) constant c_0 and sufficiently large B. The estimate is sharp since (4.13) turns into an equality if φ is the ground-state function. Let $Z_0 = Z/2$. Then, (4.13) is written in the following way:

$$\begin{aligned}\left(\varphi, \frac{2Z_0\, e^2}{x}\,\varphi\right) &= |(\varphi, V\varphi)| \\ &\leqslant c_3(\varphi, E_A\varphi) + \left(\varphi, E_A\left(\frac{E_A}{m} - c_3\right)\varphi\right) + \left[2c_0(\ln B)^2 - m\right](\varphi,\varphi),\end{aligned} \tag{4.14}$$

where $0 < c_3 < 1$ is an arbitrary real number. Since $E_A \geqslant m$, the second term in (4.14) is positive and cannot compensate the B-dependence of the third term for $B \to \infty$. The fact that a homogeneous **B**-field violates our requirement $\|\mathbf{B}\| < \infty$ is no serious problem, since the strong localization of the ground-state function in all three spatial directions [2, 26] allows for the replacement of the homogeneous **B** by an L_2-field (by smoothly cutting off at very large distances) without changing the ground-state energy.

5. Relative compactness of the perturbation

The aim of this section is to prove

Theorem 2. *Let $H^{(2)} = H_0 + W$ be the 'magnetic' Jansen–Hess operator with $H_0 := \Lambda_{A,+}(D_A + E_f)\Lambda_{A,+}$ and $W := \Lambda_{A,+}(V + B_{2m})\Lambda_{A,+}$. Then, we have for $\gamma < \tilde{\gamma}_c$*

$$\sigma_{\text{ess}}(H^{(2)}) = \sigma_{\text{ess}}(H_0). \tag{5.1}$$

The critical potential strength is $\tilde{\gamma}_c \leqslant \tilde{\gamma}_c^{(0)} = 0.319$ and depends on the magnetic field **B**.

Equivalently [18, problem 5.38, p 244], we have to prove the compactness of the difference R_μ of the resolvents of $H^{(2)}$ and H_0,

$$R_\mu := \frac{1}{H^{(2)} + \mu} - \frac{1}{H_0 + \mu} = -\frac{1}{H_0 + \mu}\Lambda_{A,+}(V + B_{2m})\Lambda_{A,+}\frac{1}{H^{(2)} + \mu}, \tag{5.2}$$

where the second resolvent identity is used, and $\mu > 0$ has to be chosen suitably. We decompose

$$\begin{aligned} R_\mu &=: R_\mu(V) + R_\mu(B_{2m}) \\ &= -\left\{\frac{1}{H_0 + \mu}(\Lambda_{A,+}V\Lambda_{A,+} + \Lambda_{A,+}B_{2m}\Lambda_{A,+})\frac{1}{(H_0+\mu)^\lambda}\right\}\left[(H_0+\mu)^\lambda\frac{1}{H^{(2)} + \mu}\right]\end{aligned} \tag{5.3}$$

where $\lambda \in \left\{\frac{1}{2}, 1\right\}$, and we will show that the two operators in curly brackets are compact while the factor in square brackets is bounded. This will prove the compactness of R_μ.

5.1. Relative compactness of $V^{\frac{1}{2}}$

For the proof of the above assertion we need, with $V = -\gamma/x$, the following lemma.

Lemma 2. *Let $H_0 = \Lambda_{A,+}(D_A + E_f)\Lambda_{A,+}$ with D_A from (1.1) and $\Lambda_{A,+}$ from (2.1). Then, the operator*

$$\frac{1}{x^{\frac{1}{2}}}\Lambda_{A,+}\frac{1}{H_0+\mu} \tag{5.4}$$

is compact for $\mu > 0$.

According to [18, theorem 4.10, p 159], its adjoint $(H_0+\mu)^{-1}\Lambda_{A,+}x^{-\frac{1}{2}}$ is then compact too.

Proof. We start by showing the boundedness of $x^{-\frac{1}{2}}(|D_A|+\mu)^{-\frac{1}{2}}$ on $L_2(\mathbb{R}^3)\otimes\mathbb{C}^4$. From (3.6), we get

$$\left\|\frac{1}{x^{\frac{1}{2}}}\frac{1}{(|D_A|+\mu)^{\frac{1}{2}}}\psi\right\|^2 \leqslant \frac{1}{\gamma_0}\left\||D_A|^{\frac{1}{2}}\frac{1}{(|D_A|+\mu)^{\frac{1}{2}}}\psi\right\|^2 + c_B\left\|\frac{1}{(|D_A|+\mu)^{\frac{1}{2}}}\psi\right\|^2. \tag{5.5}$$

Since $(|D_A|+\mu)^{-\frac{1}{2}}$ is bounded for $\mu > 0$ and since $|D_A|(|D_A|+\mu)^{-1} \leqslant 1$, the rhs of (5.5) is bounded. This implies the relative boundedness of $x^{-\frac{1}{2}}$ with respect to $|D_A|$ with form bound $a = 0$. In fact, using [28, p 340, problem 19],

$$a = \lim_{\mu\to\infty}\left\|\frac{1}{x^{\frac{1}{2}}}(|D_A|+\mu)^{-1}\right\|, \tag{5.6}$$

we have from (5.5), with $|D_A| \geqslant m$,

$$\left\|x^{-\frac{1}{2}}(|D_A|+\mu)^{-1}\psi\right\| \leqslant \left\|x^{-\frac{1}{2}}(|D_A|+\mu)^{-\frac{1}{2}}\right\|\left(\frac{1}{m+\mu}\right)^{\frac{1}{2}}\|\psi\|$$

which proves $a = 0$.

Following [31, lemma 11.5], we define a smooth function $\chi_0 \in C_0^\infty(\mathbb{R}^3)$ mapping to [0, 1] by means of

$$\chi_0(\mathbf{x}) := \begin{cases} 1, & x < R \\ 0, & x \geqslant R+1 \end{cases} \tag{5.7}$$

with some $R > 0$, such that $\operatorname{supp}(1-\chi_0) \subset \mathbb{R}^3\backslash B_R(0)$, where $B_R(0)$ is a ball of radius R centred at the origin. Further, let $(\psi_n)_{n\in\mathbb{N}}$ be a normalized sequence in $H_1(\mathbb{R}^3)\otimes\mathbb{C}^4$ weakly converging to zero. We prove the compactness of (5.4) by showing that $\left\|x^{-\frac{1}{2}}\Lambda_{A,+}(H_0+\mu)^{-1}\psi_n\right\| \to 0$ for $n\to\infty$. We decompose

$$\left\|\frac{1}{x^{\frac{1}{2}}}\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\| \leqslant \left\|(1-\chi_0)\frac{1}{x^{\frac{1}{2}}}\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\| + \left\|\frac{1}{x^{\frac{1}{2}}}\chi_0\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\|. \tag{5.8}$$

For the first term, we have

$$\left\|(1-\chi_0)\frac{1}{x^{\frac{1}{2}}}\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\| \leqslant \frac{1}{R^{\frac{1}{2}}}\left\|(1-\chi_0)\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\| \leqslant \frac{c}{R^{\frac{1}{2}}} \tag{5.9}$$

with some constant c. Thus, it can be made smaller than $\epsilon/2$ if $R > (2c/\epsilon)^2$.

For the second term, we define $\tilde{\psi}_n := \chi_0\Lambda_{A,+}(H_0+\mu)^{-1}\psi_n$ and use the $|D_A|$-boundedness of $x^{-\frac{1}{2}}$ with bound $a \to 0$,

$$\left\|\frac{1}{x^{\frac{1}{2}}}\chi_0\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\| \leqslant a\||D_A|\tilde{\psi}_n\| + b\|\tilde{\psi}_n\|, \tag{5.10}$$

with some constant b. In order to establish that $\||D_A|\tilde{\psi}_n\|$ is finite (such that $a\||D_A|\tilde{\psi}_n\|$ can be dropped), we consider

$$\mathcal{O} := [D_A, \chi_0] = \boldsymbol{\alpha}(\mathbf{p}\chi_0) \tag{5.11}$$

which is bounded because χ_0 is a C_0^∞-function. Thus, we can decompose

$$\chi_0 |D_A|^2 \chi_0 = \chi_0 D_A \cdot D_A \chi_0 = D_A \chi_0^2 D_A - \mathcal{O}\chi_0 D_A + D_A \chi_0 \mathcal{O} - \mathcal{O}^2 \tag{5.12}$$

and estimate

$$\begin{aligned} \||D_A|\tilde{\psi}_n\|^2 &= \left(\psi_n, \frac{1}{H_0+\mu}\Lambda_{A,+}\big(D_A\chi_0^2 D_A - \mathcal{O}\chi_0 D_A + D_A\chi_0\mathcal{O} - \mathcal{O}^2\big)\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right) \\ &\leqslant \|\chi_0\|_\infty^2 \left\| D_A\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\|^2 + \left\|\mathcal{O}\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\| \\ &\quad \times \|\chi_0\|_\infty \left\| D_A\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\| \cdot 2 + \left\|\mathcal{O}\Lambda_{A,+}\frac{1}{H_0+\mu}\psi_n\right\|^2. \end{aligned} \tag{5.13}$$

Since $D_A\Lambda_{A,+}(H_0+\mu)^{-1} = \Lambda_{A,+}D_A\Lambda_{A,+}(\Lambda_{A,+}D_A\Lambda_{A,+}+\Lambda_{A,+}E_f\Lambda_{A,+}+\mu)^{-1} \leqslant 1$, all terms on the rhs of (5.13) are bounded.

Concerning the last term of (5.10), we will establish the compactness of the operator $K := \chi_0\Lambda_{A,+}(H_0+\mu)^{-1}$. Then $\|\tilde{\psi}_n\| \to 0$ for $n \to \infty$. Collecting results, this shows that the second term of (5.8) can be made smaller than $\epsilon/2$ for n sufficiently large and thus proves the desired compactness of the operator (5.4).

The strategy to show the compactness of K is to start with the operator $K_1 := \chi_0(p^2 + m^2)^{-\frac{1}{2}}$ which is compact as a product of bounded functions $f(\mathbf{x})$, $g(\mathbf{p})$, each of which tending to zero as x, respectively p, go to infinity (see, e.g., [31, lemma 7.10]). Then, bounded operators $\mathcal{O}_1, \mathcal{O}_2$ are constructed such that $K_1 \cdot \prod \mathcal{O}_i = K$.

Let $\mathcal{O}_1 := \sqrt{p^2+m^2}D_A^{-1}$. For showing the boundedness of $\mathcal{O}_1$ let $\psi := D_A^{-1}\varphi$. Then from the diamagnetic inequality and (4.6),

$$\begin{aligned} \|\mathcal{O}_1\varphi\|^2 &= (\psi, (p^2+m^2)\psi) \leqslant \big(\psi, \big(E_A^2 + e|\mathbf{B}|\big)\psi\big) \\ &\leqslant \left(1 + \frac{\kappa e}{1-\kappa e}\right)\|\varphi\|^2 + \frac{eC_\kappa}{1-\kappa e}\big\|D_A^{-2}\big\|\|\varphi\|^2, \end{aligned} \tag{5.14}$$

the rhs being obviously bounded.

With $\mathcal{O}_2 := D_A\Lambda_{A,+}(H_0+\mu)^{-1} \leqslant 1$ (as shown above), we arrive at $K_1 \cdot \mathcal{O}_1 \cdot \mathcal{O}_2 = K$. □

We remark that in the same way the compactness of $x^{-\frac{1}{2}}(|D_A|+\mu)^{-1}$ on $L_2(\mathbb{R}^3)\otimes\mathbb{C}^4$ can be shown. The only additional ingredient is the boundedness of $D_A(|D_A|+\mu)^{-1}$ in the equation corresponding to (5.13), which follows from $\|D_A(|D_A|+\mu)^{-1}\psi_n\|^2 = (\psi_n, |D_A|^2(|D_A|+\mu)^{-2}\psi_n) \leqslant \|\psi_n\|^2$.

5.2. *Boundedness of* $(H_0+\mu)^\lambda(H^{(2)}+\mu)^{-1}$

Let first $\lambda = 1$. From (3.7) and (3.21), we have the relative form boundedness of the potential for $\psi \in H_1(\mathbb{R}^3)\otimes\mathbb{C}^4$ and $\psi_+ := \Lambda_{A,+}\psi$,

$$\|\Lambda_{A,+}(V+B_{2m})\Lambda_{A,+}\psi\| \leqslant \|V\psi_+\| + \|B_{2m}\psi_+\| \leqslant a_0\|D_A\psi_+\| + b_0\|\psi_+\| \tag{5.15}$$

with

$$a_0 := \frac{\gamma}{\gamma_1} + \frac{\gamma^2}{2\gamma_1}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_0} + \frac{c_B}{m}\right), \qquad b_0 := \gamma d_B + C, \tag{5.16}$$

where c_B is defined in (3.6). We have to restrict $\gamma < \tilde{\gamma}_c$ such that $a_0 < 1$. $\tilde{\gamma}_c$ depends on $\mathbf{B}$, its maximum value (for $\mathbf{B} = 0$) being $\tilde{\gamma}_c^{(0)} = 0.319\,(Z \leqslant 43)$, obtained as solution to $2\gamma + \gamma^2\big(2+\frac{\pi}{2}\big) = 1$.

Let $\epsilon := 1 - a_0$ with $0 < \epsilon < 1$. With $\psi := (H^{(2)} + \mu)^{-1}\psi_+$, we want to show

$$\left\| (H_0 + \mu)\frac{1}{H^{(2)} + \mu}\psi_+ \right\|^2 = \|(H_0 + \mu)\psi\|^2 \overset{!}{\leqslant} c_1^2\|\psi_+\|^2 = c_1^2\|(H^{(2)} + \mu)\psi\|^2 \tag{5.17}$$

for a suitable $c_1 > 0$. We estimate, using $\|D_A\psi_+\| = \|\Lambda_{A,+}D_A\Lambda_{A,+}\psi\| \leqslant \|H_0\psi\|$ and $\|\psi_+\| \leqslant \|\Lambda_{A,+}\|\|\psi\|$,

$$\begin{aligned} c_1\|(H^{(2)} + \mu)\psi\| &\geqslant c_1\|(H_0 + \mu)\psi\| - c_1\|\Lambda_{A,+}(V + B_{2m})\Lambda_{A,+}\psi\| \\ &\geqslant c_1\|(H_0 + \mu)\psi\| - c_1\{a_0\|H_0\psi\| + b_0\|\psi\|\} \\ &\overset{!}{\geqslant} c_1\|(H_0 + \mu)\psi\| + (1 - c_1)(\|H_0\psi\| + \mu\|\psi\|) \geqslant \|(H_0 + \mu)\psi\|. \end{aligned} \tag{5.18}$$

Condition (5.18) is satisfied if $-c_1a_0 \geqslant 1 - c_1$ as well as $-c_1b_0 \geqslant (1 - c_1)\mu$, requiring the choice $c_1 \geqslant 1/\epsilon$ and $\mu \geqslant c_1b_0/(c_1 - 1)$.

For $\lambda = \frac{1}{2}$, the bound on γ can be improved by working with quadratic forms. From (3.22), we have

$$(\psi, \Lambda_{A,+}(V + B_{2m})\Lambda_{A,+}\psi) \geqslant -a_1(\psi, \Lambda_{A,+}D_A\Lambda_{A,+}\psi) - \tilde{C}(\psi, \psi) \tag{5.19}$$

with $a_1 := a_0 - \gamma\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_0}\right)$. Trivially, we have $(H_0 + \mu)^{\frac{1}{2}}(H^{(2)} + \mu)^{-1} = (H_0 + \mu)^{\frac{1}{2}}(H^{(2)} + \mu)^{-\frac{1}{2}} \cdot (H^{(2)} + \mu)^{-\frac{1}{2}}$ where the last factor is bounded. For the boundedness of the other factor, we use the strategy of (5.17) to require $\|(H_0 + \mu)^{\frac{1}{2}}\psi\|^2 \leqslant c_2\|(H^{(2)} + \mu)^{\frac{1}{2}}\psi\|^2$ which is satisfied if $c_2 \geqslant 1/(1 - a_1)$ and $\mu \geqslant c_2\tilde{C}/(c_2 - 1)$. The necessary condition for $c_2 > 0$ is $a_1 < 1$, i.e. inequality (4.3). The corresponding maximum value for γ is $\gamma_c^{(0)} = 0.353$.

5.3. *Compactness of* $R_\mu(V)$

We take $\lambda = \frac{1}{2}$ and decompose

$$-\frac{1}{H_0 + \mu}\Lambda_{A,+}V\Lambda_{A,+}\frac{1}{(H_0 + \mu)^{\frac{1}{2}}} = \gamma\left\{\frac{1}{H_0 + \mu}\Lambda_{A,+}\frac{1}{x^{\frac{1}{2}}}\right\}\left[\frac{1}{x^{\frac{1}{2}}}\Lambda_{A,+}\frac{1}{(H_0 + \mu)^{\frac{1}{2}}}\right]. \tag{5.20}$$

The factor in square brackets is bounded according to a (5.5)-type estimate by using that $\Lambda_{A,+}|D_A|\Lambda_{A,+} \leqslant H_0 + \mu$. Together with lemma 2 and the result of section 5.2, this proves the compactness of $R_\mu(V)$ for $\gamma < \gamma_c$ determined from (4.3).

5.4. *Compactness of* $R_\mu(B_{2m})$

According to the four contributions of B_{2m} from (2.12), we define

$$\frac{1}{H_0 + \mu}(\Lambda_{A,+}B_{2m}\Lambda_{A,+})\frac{1}{(H_0 + \mu)^\lambda} =: -\frac{\gamma}{4}\sum_{i=1}^{4}\mathcal{O}_i(\lambda). \tag{5.21}$$

For $i = 1$, we take $\lambda = 1$ and decompose

$$\mathcal{O}_1(1) = \left\{\frac{1}{H_0 + \mu}\Lambda_{A,+}\frac{1}{x^{\frac{1}{2}}}\right\} \cdot \left[\frac{1}{x^{\frac{1}{2}}}F_A\tilde{D}_A\Lambda_{A,+}\frac{1}{H_0 + \mu}\right]. \tag{5.22}$$

In order to show the boundedness of the operator in square brackets, we use a (5.5)-type estimate for $x^{-\frac{1}{2}}$ and note that $F_A\tilde{D}_A\Lambda_{A,+}(H_0 + \mu)^{-1}$ is bounded. It then remains to show the boundedness of $M := |D_A|^{\frac{1}{2}}F_A\tilde{D}_A\Lambda_{A,+}(H_0 + \mu)^{-1}$. We estimate for $\varphi \in L_2(\mathbb{R}^3) \otimes \mathbb{C}^4$, noting that $F_A\tilde{D}_A = -\tilde{D}_AF_A$,

$$\begin{aligned} \|M\varphi\|^2 &= \left(\tilde{D}_AF_A\Lambda_{A,+}\frac{1}{H_0 + \mu}\varphi, |D_A|\tilde{D}_AF_A\Lambda_{A,+}\frac{1}{H_0 + \mu}\varphi\right) \\ &\leqslant \left\|\tilde{D}_AF_A\Lambda_{A,+}\frac{1}{H_0 + \mu}\right\| \|\varphi\| \cdot \left\|D_AF_A\Lambda_{A,+}\frac{1}{H_0 + \mu}\varphi\right\|. \end{aligned} \tag{5.23}$$

We use (3.16) to commute D_A with F_A, being left with two terms involving the potential V. In turn, these terms can be estimated according to (3.7) by replacing V with $|D_A|$ plus a bounded remainder. For example, we get

$$\left\| \frac{1}{2} \tilde{D}_A V \Lambda_{A,+} \frac{1}{H_0+\mu} \varphi \right\| \leqslant \frac{\gamma}{2\gamma_1} \| \tilde{D}_A \| \left\| |D_A| \Lambda_{A,+} \frac{1}{H_0+\mu} \varphi \right\| + \frac{\gamma d_B}{2} \| \tilde{D}_A \| \left\| \Lambda_{A,+} \frac{1}{H_0+\mu} \right\| \| \varphi \| \tag{5.24}$$

which obviously is bounded.

For $i = 2$, we take $\lambda = \frac{1}{2}$ and decompose

$$\mathcal{O}_2\left(\frac{1}{2}\right) = \frac{1}{H_0+\mu} \Lambda_{A,+} \tilde{D}_A F_A (|D_A| + \mu) \cdot \left\{ \frac{1}{|D_A|+\mu} \frac{1}{x^{\frac{1}{2}}} \right\} \cdot \left[\frac{1}{x^{\frac{1}{2}}} \Lambda_{A,+} \frac{1}{(H_0+\mu)^{\frac{1}{2}}} \right]. \tag{5.25}$$

Referring to our previous considerations, it remains to show the boundedness of the adjoint of the first term, $|D_A| F_A \tilde{D}_A \Lambda_{A,+} (H_0+\mu)^{-1}$, since $\mu (H_0+\mu)^{-1} \Lambda_{A,+} \tilde{D}_A F_A$ is trivially bounded (and since any bounded operator has a bounded adjoint). With $|D_A| F_A \tilde{D}_A = -D_A F_A$, we arrive at the last term of (5.23), the boundedness of which has just been shown.

For $i = 3$, we take again $\lambda = 1$. Then,

$$\begin{aligned} \mathcal{O}_3(1) &= \frac{1}{H_0+\mu} \Lambda_{A,+} \tilde{D}_A \frac{1}{x} F_A \Lambda_{A,+} \frac{1}{H_0+\mu} \\ &= \tilde{D}_A \cdot \left\{ \frac{1}{H_0+\mu} \Lambda_{A,+} \frac{1}{x^{\frac{1}{2}}} \right\} \cdot \left[\frac{1}{x^{\frac{1}{2}}} F_A \Lambda_{A,+} \frac{1}{H_0+\mu} \right], \end{aligned} \tag{5.26}$$

of which the first factor is compact and the second factor bounded. For the factor in square brackets we estimate according to (5.5), and further

$$\left\| |D_A|^{\frac{1}{2}} F_A \Lambda_{A,+} \frac{1}{H_0+\mu} \varphi \right\|^2 \leqslant \left\| F_A \Lambda_{A,+} \frac{1}{H_0+\mu} \varphi \right\| \cdot \left\| |D_A| F_A \Lambda_{A,+} \frac{1}{H_0+\mu} \varphi \right\|. \tag{5.27}$$

Since $\| |D_A| \tilde{\varphi} \|^2 = (\tilde{\varphi}, D_A^2 \tilde{\varphi}) = \| D_A \tilde{\varphi} \|^2$, the second factor agrees with the one from (5.23).

For $i = 4$ and $\lambda = 1$, we have $\mathcal{O}_4(1) = (H_0+\mu)^{-1} \Lambda_{A,+} F_A \frac{1}{x} \tilde{D}_A \Lambda_{A,+} (H_0+\mu)^{-1} = \mathcal{O}_3(1)^*$. Together with the result from section 5.2, this proves compactness of $R_\mu(B_{2m})$ for $\gamma < \tilde{\gamma}_c$ defined below (5.16).

6. The essential spectrum

For the Schrödinger operator with purely magnetic field, $\mathbf{p}_A^2$, it was shown, following the work of Jörgens [17], that its essential spectrum is given by

$$\sigma_{\mathrm{ess}}\left(\mathbf{p}_A^2\right) = [0, \infty) \tag{6.1}$$

provided $\mathbf{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^3)$ and

$$N_A(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}| \leqslant 1} |\mathbf{A}(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y} \longrightarrow 0 \tag{6.2}$$

as $\mathbf{x} \to \infty$ ([20], see also [33]). In particular, condition (6.2) is satisfied if $\mathbf{B} \to 0$ as $\mathbf{x} \to \infty$ [20]. It is, however, easy to show that it is sufficient that $N_B(\mathbf{x}) \to 0$ (as $\mathbf{x} \to \infty$) for (6.2) to hold. We use the relation between $\mathbf{A}$ and $\mathbf{B}$ introduced in [11],

$$\mathbf{A}(\mathbf{y}) = \int_0^1 t \, \mathrm{d}t \, \mathbf{B}(\mathbf{x} + t(\mathbf{y}-\mathbf{x})) \wedge (\mathbf{y}-\mathbf{x}), \tag{6.3}$$

which satisfies $\nabla \times \mathbf{A} = \mathbf{B}$ (since $\nabla \cdot \mathbf{B} = 0$). Then we have, substituting $\mathbf{z} := \mathbf{y} - \mathbf{x}$,

$$N_A(\mathbf{x}) = \int_{z\leqslant 1} |\mathbf{A}(\mathbf{z}+\mathbf{x})|^2\, \mathrm{d}\mathbf{z}$$
$$= \int_{z\leqslant 1} \mathrm{d}\mathbf{z} \int_0^1 t\,\mathrm{d}t \int_0^1 \tau\,\mathrm{d}\tau\, (\mathbf{B}(\mathbf{x}+t\mathbf{z})\wedge \mathbf{z})(\mathbf{B}(\mathbf{x}+\tau\mathbf{z})\wedge\mathbf{z}). \tag{6.4}$$

We estimate $|\mathbf{B}\wedge\mathbf{z}| \leqslant |\mathbf{B}|$ (since $z \leqslant 1$) and factorize the integrand according to $\frac{t^{1+\epsilon}}{\tau^{\epsilon}}|\mathbf{B}(\mathbf{x}+t\mathbf{z})| \cdot \frac{\tau^{1+\epsilon}}{t^{\epsilon}}|\mathbf{B}(\mathbf{x}+\tau\mathbf{z})|$ with, e.g., $\epsilon = \frac{1}{4}$. Applying the Schwarz inequality, we get upon substituting $\boldsymbol{\xi} := t\mathbf{z}$ for $\mathbf{z}$

$$N_A(\mathbf{x}) \leqslant \left(\int_0^1 \frac{\mathrm{d}\tau}{\tau^{2\epsilon}}\right) \int_0^1 t^{2+2\epsilon}\,\mathrm{d}t \int_{z\leqslant 1} \mathrm{d}\mathbf{z}\, |\mathbf{B}(\mathbf{x}+t\mathbf{z})|^2$$
$$= \int_0^1 \frac{\mathrm{d}\tau}{\tau^{\frac{1}{2}}} \int_0^1 \frac{\mathrm{d}t}{t^{\frac{1}{2}}} \int_{\xi\leqslant t} \mathrm{d}\boldsymbol{\xi}\, |\mathbf{B}(\mathbf{x}+\boldsymbol{\xi})|^2 \leqslant 4 \int_{\xi\leqslant 1} \mathrm{d}\boldsymbol{\xi}\, |\mathbf{B}(\mathbf{x}+\xi)|^2 \tag{6.5}$$

which, upon assumption, tends to 0 as $\mathbf{x} \to \infty$.

A further consequence of $N_B(\mathbf{x}) \to 0$ (as $\mathbf{x} \to \infty$) is that $e\boldsymbol{\sigma}\mathbf{B}$ is $\mathbf{p}_A^2$-compact [32, theorem 5.2.2]. Thus, the essential spectrum of the Pauli operator $(\boldsymbol{\sigma}\mathbf{p}_A)^2$ is also given by $[0, \infty)$. Accordingly, $\sigma_{\mathrm{ess}}(D_A^2) = [m^2, \infty)$, and therefore

$$\sigma_{\mathrm{ess}}(E_A) = [m, \infty). \tag{6.6}$$

In fact, let $\lambda^2 \in [m^2, \infty)$ and $\lambda > 0$. Then, there exists a normalized sequence $\varphi_n \in C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^2$ with $\varphi_n \overset{w}{\rightharpoonup} 0$ such that $\|(E_A - \lambda)(E_A + \lambda)\varphi_n\| \to 0$ as $\to \infty$. Let $\phi \in C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^2$ and note that $C_0^\infty \subset H_2 \subset H_1 \subset L_2$. Then, $(E_A+\lambda)\phi \in H_1(\mathbb{R}^3)\otimes\mathbb{C}^2 = \mathcal{D}(E_A)$ and $(\phi, (E_A + \lambda)\varphi_n) = ((E_A + \lambda)\phi, \varphi_n) \to 0$ (since $\varphi_n \overset{w}{\rightharpoonup} 0$) as $\to \infty$. Moreover, $\liminf_{n\to\infty} \|(E_A + \lambda)\varphi_n\| \geqslant \liminf_{n\to\infty} \|(m + \lambda)\varphi_n\| = m + \lambda > 0$, which shows that $\tilde{\varphi} := (E_A + \lambda)\varphi_n \overset{w}{\rightharpoonup} 0$, such that $\lambda \in \sigma_{\mathrm{ess}}(E_A)$ [34, theorem 7.24, p 191].

In order to derive $\sigma_{\mathrm{ess}}(D_A)$ from (6.6), we note that $\sigma_{\mathrm{ess}}(-E_A) = (-\infty, -m]$. Moreover, since a unitary transformation does not change the essential spectrum, we have from (1.7)

$$\sigma_{\mathrm{ess}}(D_A) = \sigma_{\mathrm{ess}}(U_0 D_A U_0^{-1}) = \sigma_{\mathrm{ess}}(\beta E_A)$$
$$= \sigma_{\mathrm{ess}}\begin{pmatrix} E_A & \\ & 0 \end{pmatrix} \cup \sigma_{\mathrm{ess}}\begin{pmatrix} 0 & \\ & -E_A \end{pmatrix} = [m, \infty) \cup (-\infty, -m]. \tag{6.7}$$

It was proven earlier [11, theorem 1.4] that $\sigma_{\mathrm{ess}}(D_A) = (-\infty, -m] \cup [m, \infty)$ under somewhat stronger assumptions (e.g., $\mathbf{B}(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$), the proof being similar to the one given in [5, p 117] for the Schrödinger case.

From the decomposition of D_A into its (disjoint) positive and negative part, $D_A = \Lambda_{A,+} D_A \Lambda_{A,+} + \Lambda_{A,-} D_A \Lambda_{A,-}$, we get $\sigma_{\mathrm{ess}}(\Lambda_{A,+} D_A \Lambda_{A,+}) = \sigma_{\mathrm{ess}}(E_A) = [m, \infty)$.

Together with theorem 2 we have thus proven

Theorem 3. *Let $H^{(2)}$ be the 'magnetic' Jansen–Hess operator, let the vector potential $\mathbf{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^3)$, let the magnetic field obey $N_B(\mathbf{x}) \to 0$ for $\mathbf{x} \to \infty$ with finite field energy E_f. Then for a Coulomb potential with strength $\gamma < \tilde{\gamma}_c$, the essential spectrum is given by*

$$\sigma_{\mathrm{ess}}(H^{(2)}) = [m, \infty) + E_f, \tag{6.8}$$

where $\tilde{\gamma}_c$ is defined in theorem 2.

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