The single-particle pseudorelativistic Jansen–Hess operator with magnetic field

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Abstract
The pseudorelativistic no-pair Jansen–Hess operator is derived for the case where in addition to the Coulomb potential an external magnetic field $B$ is permitted. With some restrictions on the vector potential, it is shown that this operator is positive provided the strength $\gamma$ of the Coulomb potential is below a critical value $(\gamma_c \leq 0.35$, depending on the magnetic field energy $E_f$). Moreover, for $\gamma < 0.32$ and for $B$ tending asymptotically to zero in a weak sense, the essential spectrum is given by $[m, \infty) + E_f$.

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1. Introduction

The spectral properties of the Dirac operator and its nonrelativistic limit, the Pauli operator, describing an atom in an external magnetic field, are a topic of current interest (see the comprehensive review by Erdős [8]). The Dirac operator for an electron in an electric field $V$ and a magnetic field $B = \nabla \times A$, acting in the Hilbert space $L_2(\mathbb{R}^3) \otimes \mathbb{C}^4$, is given by [3, section 1.3] and

$$H = D_A + V + E_f$$

$$D_A := \alpha p_A + \beta m, \quad p_A := p - eA.$$  \hfill (1.1)

$D_A$ is the free Dirac operator with $\alpha$ and $\beta$ Dirac matrices, $m$ is the electron mass, $V = -\gamma/x$ is the Coulomb field generated by a nucleus of charge $Z$ fixed at the origin ($\gamma = Ze^2$ with $e^2 \approx 1/137.04$ the fine structure constant). In (1.1) the (classical) field energy $E_f$ is included:

$$E_f := \frac{1}{8\pi} \int_{\mathbb{R}^3} B^2(x) \, dx = \frac{1}{8\pi} \|B\|^2$$  \hfill (1.2)

where $\|\cdot\|$ denotes the $L_2$-norm, $x$ is the coordinate and $p = -i\nabla$ the momentum of the electron. Relativistic units ($\hbar = c = 1$) are used and $|x| = x$. There is a simple relation to the Pauli operator, $\frac{1}{2m} (\sigma p_A)^2 = \frac{1}{8m} [(p_A)^2 - e\sigma B]$, where $\sigma$ is the vector of Pauli spin matrices.
These conditions imply the commutation relation $[\mathfrak{pA}, \mathfrak{pA}] = \mathfrak{p}[A,\mathfrak{p}]$ [19, p 438] and $A \in L_c(\mathbb{R}^3)$ which results from a Sobolev inequality [9]. The condition $B \in L_2(\mathbb{R}^3)$ renders $E_f$ finite. If, in addition to $\nabla \cdot A = 0$, $A$ is a $C^1$-function, it was shown ([17], based on [13]) that $(\mathfrak{pA})^2$ is essentially self-adjoint on $C^\infty_0(\mathbb{R}^3) \otimes \mathbb{C}^2$. Later, $A \in L_{2,\text{loc}}(\mathbb{R}^3)$ was established as the weakest possible condition for this property to be true [1], [5, p 9]. As a second condition, we require therefore that $A \in L_{2,\text{loc}}(\mathbb{R}^3)$. Let the magnetic field satisfy

$$N_B(x) := \int_{|x-y| \leq 1} |B(y)|^2 dy \leq C$$

with a constant $C \in \mathbb{R}$ independent of $x$ ([1.5] holds for any $B \in L_2(\mathbb{R}^3)$). This guarantees the essential self-adjointness of the Pauli operator. The proof is based on the work of Udim [32, theorem 4.2], showing that a consequence of (1.5) is the $(\mathfrak{pA})^2$-boundedness of $e\sigma B$ with bound zero. This property establishes the required essential self-adjointness according to the Kato–Rellich theorem [28, theorem X.12].

From the symmetry of $\mathfrak{pA}$, we have $(\psi, (\mathfrak{pA})^2 \psi) = \|\mathfrak{pA} \psi\|^2 \geq 0$ for $\psi \in C^\infty_0(\mathbb{R}^3) \otimes \mathbb{C}^2$. Thus, $(\mathfrak{pA})^2$ is a non-negative, self-adjoint operator (by means of closure). It follows [18, theorem 3.35, p 281] that this is also true for

$$E_A := |D_A| = \sqrt{(\mathfrak{pA})^2 + m^2} \geq m$$

which is the kinetic energy term of the pseudorelativistic operator that will be introduced in section 2.

Due to the positron degrees of freedom, the Dirac operator $H$ has a spectrum which is unbounded from below. However, in the spectroscopy of static or slowly moving ions, pair creation plays no role. One of the current techniques, used in the field-free case (A = 0), to construct from $H$ an operator which solely describes the electronic states is the application of a unitary transformation scheme to $H$ (see, e.g., [7, 15, 30]). A perturbative expansion in the central field strength $\gamma$ provides pseudorelativistic operators which are block diagonal in the free (i.e., $Z = 0$) electronic positive and negative spectral subspaces up to a given order in $\gamma$. The zero- plus first-order term in this series, the Brown–Ravenhall operator, has obtained widespread interest because it is simply the restriction of $H$ to the positive spectral subspace. The terms up to second order, comprising the Jansen–Hess operator, provide, however, a much better representation of the bound-state energies [35]. This operator has been proven to be positive with essential spectrum $\sigma_{\text{ess}} = [m, \infty)$ for sufficiently small $\gamma$ [4, 12, 14].

If $A \neq 0$, investigations are scarce. It is known that in the absence of the Coulomb field $V$, the Dirac operator can be block diagonalized by means of a Foldy–Wouthuysen transformation $U_0$ [6, section 3.1],

$$U_0 D_A U_0^{-1} = \beta E_A$$

$$U_0 := \left( m + E_A \right)^{-\frac{1}{2}} + \frac{\beta \mathfrak{pA}}{2E_A(m + E_A)} \mathfrak{p}\mathfrak{p} A \mathfrak{p} A + \beta \mathfrak{p} A \mathfrak{p} A + E_A$$

(1.7)

$U_0^{-1}$ is obtained from $U_0$ by replacing $\beta \mathfrak{p} A \mathfrak{p} A \beta = -\beta \mathfrak{p} A \mathfrak{p} A$. For later use, we note that $E_A$ commutes with $U_0$, $[E_A, U_0] = 0$, because $[\beta \mathfrak{pA}, E_A] = [\beta, E_A] \mathfrak{pA} A + \beta [\mathfrak{pA}, E_A]$ vanishes (the first commutator being zero since $E_A$ is block diagonal). There are also
a few studies of the ‘magnetic’ Brown–Ravenhall operator showing that this operator is either unbounded from below (if $\mathbf{A}$ is disregarded in the projector onto the positive spectral subspace [10]) or that it is positive for $\gamma < \frac{3}{2}$ (if $\mathbf{A}$ is not disregarded) which assures stability of relativistic matter in this model [22, 23].

The aim of the present work is to derive the ‘magnetic’ Jansen–Hess operator $H^{(2)}$ from the corresponding transformation scheme (section 2), to show under which conditions it is positive (theorem 1, section 4) and to provide criteria for $\sigma_{\text{ess}} = [m, \infty \to E_f$ to hold (theorem 3, section 6). An auxiliary step is the invariance of the essential spectrum upon removal of the Jansen–Hess potential (theorem 2, section 5). Consequently, theorem 3 also holds for the ‘magnetic’ Brown–Ravenhall operator (which results from dropping the second-order term in $\gamma$). The basic difference from the $\mathbf{A} = 0$ case in constructing and analysing $H^{(2)}$ is due to the fact that the kinetic energy operator $E_A$ is no longer a multiplicator in momentum space (as is $E_{A=0} := E \rho \sqrt{p^2 + m^2}$). Hence, formal techniques have to replace Fourier analysis (sections 2 and 3). Moreover, in contrast to the ‘magnetic’ Brown–Ravenhall operator, the required bounds on $\gamma$ for self-adjointness and positivity depend nontrivially on the magnetic field. Therefore, these bounds are inferior to the $\mathbf{A} = 0$ case. With $\gamma \to 0$ for $B \to \infty$, our analysis makes the Jansen–Hess operator an unlikely candidate for stability of matter. However, for laboratory magnetic fields up to $10^{12}$ G this operator should be superior to the ‘magnetic’ Brown–Ravenhall operator regarding electron spectroscopy.

2. The transformed Dirac operator

Let us define the projector onto the positive magnetic spectral subspace of the electron (defined by switching off $V$ but fully including $\mathbf{A}$),

$$\Lambda_{A,+} := \frac{1}{2} \left( 1 + \frac{D_A}{|D_A|} \right).$$

(2.1)

For any $\psi \in \mathcal{H}_{+}, \Lambda_{A,+}(H_1(\mathbb{R}^3) \otimes \mathbb{C}^4)$ (where the Sobolev space $H_1(\mathbb{R}^3) \otimes \mathbb{C}^4$ is the domain of $D_A$), we have trivially $\Lambda_{A,+}\psi = \psi$ and $D_A\psi = E_A\psi$, and one easily verifies that with $\psi := (\psi_0^u, u \in H_1(\mathbb{R}^3) \otimes \mathbb{C}^2$, $\psi_0$ can be expressed as

$$\psi_0 = U_0^{-1}\psi$$

(namely using (1.7), $D_A(U_0^{-1}\psi) = U_0^{-1} E_A\psi = U_0^{-1} E_A(\beta\psi) = U_0^{-1} E_A\psi = E_A U_0^{-1}\psi$).

Let $H_V := D_A + V$. We construct a unitary transformation $U$ such that the transformed Dirac operator decouples the magnetic spectral subspaces of the electron,

$$U^{-1}HU = \Lambda_{A,+}(U^{-1}H_VU)\Lambda_{A,+} + \Lambda_{A,-}(U^{-1}H_VU)\Lambda_{A,-} + E_f,$$

(2.2)

with $\Lambda_{A,+}$ from (2.1) and $\Lambda_{A,-} = 1 - \Lambda_{A,+}$. The choice of the projector $\Lambda_{A,+}$ in (2.3) preserves the gauge invariance of the transformed operator [22]. The field energy $E_f$ is a constant which is not affected by $U$. If one defines $P$, as the projector onto the positive spectral subspace of the Dirac operator $H_V$, then (2.3) is equivalent to the condition

$$U^{-1}PU = \Lambda_{A,+}.$$

(2.4)

If, in addition, the Foldy–Wouthuysen transformation $U_0$ is applied, the desired block-diagonal operator is obtained as a consequence of $U_0\Lambda_{A,+} U_0^{-1} = \frac{1}{2}(1 + \beta)$ (see (1.7) and the discussion below):

$$M := \frac{1}{2}(1 + \beta)M \frac{1}{2}(1 + \beta) + \frac{1}{2}(1 - \beta)M \frac{1}{2}(1 - \beta) =: \begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix}$$

(2.5)

where $h, g$ are matrices in $\mathbb{C}^{2 \times 2}$. 

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Rather than solving (2.4) for $U$ (which was recently achieved in the field-free case [29, 30]), we start from (2.3) and apply a technique [15] which is equivalent to the Douglas–Kroll transformation scheme [7, 16]. We formally expand $U = \exp(i \sum_{k=1}^{\infty} B_k)$, where $B_k$ is an operator which contains the potential $V$ to $k$th order, and we are interested in the transformed operator which is block diagonal up to second order in the potential strength $\gamma$. Denoting by $H^{(2)}$ the second-order solution of (2.3) restricted to $\mathcal{H}_{k+1}$ (the ‘magnetic’ Jansen–Hess operator) we have, in analogy to the $\Lambda = 0$ case,

$$H^{(2)} := \Lambda_{A,+} \left\{ D_A + V + \frac{i}{2} [W_1, B_1] + E_f \right\} \Lambda_{A,+}$$

(2.6)

with $W_1 := \Lambda_{A,+} V \Lambda_{A,-} + \Lambda_{A,-} V \Lambda_{A,+}$ being the off-diagonal part of $V$. $B_1$ is determined from the condition

$$W_1 = -i [D_A, B_1].$$

(2.7)

Alternatively, we can obtain $B_1$ from (2.4). Using the integral representation of $P_+$ [18, chapter II.1.4] and expanding $P_+$ in terms of $V$ by means of the second resolvent identity, we have

$$P_+ = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta \frac{1}{D_A + V + i\eta}$$

$$= \Lambda_{A,+} - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta \frac{1}{D_A + i\eta} \frac{1}{V} \frac{1}{D_A + V + i\eta} = \Lambda_{A,+} + F_A + R$$

(2.8)

where $\Lambda_{A,+}$ and $F_A$ are the zero- and first-order terms, respectively, while the remainder $R$ is of higher order in $V$. Defining $\tilde{D}_A := D_A/|D_A|$ and solving (2.4) up to first order in $V$, we get

$$2F_A - iB_1 \tilde{D}_A + i\tilde{D}_A B_1 = 0.$$ 

(2.9)

Multiplication of (2.9) by $\tilde{D}_A$ from the left and, respectively, from the right and addition of the resulting equations provides the useful relation

$$F_A \tilde{D}_A = -\tilde{D}_A F_A.$$ 

(2.10)

Whereas (2.9) is also an only an implicit equation for $B_1$, a trial for $B_1$ can be found from the formal solution $U$ of (2.4) which is completely analogous to the field-free case [29]. $U^{-1} = [1 + (\Lambda_{A,+} - \Lambda_{A,-})(P_+ - \Lambda_{A,+})](1 - (P_+ - \Lambda_{A,+})^2)^{-1}$. An expansion of this formal solution up to first order in $V$ leads to

$$B_1 = i \tilde{D}_A F_A.$$ 

(2.11)

With the help of (2.10), it is easily verified that (2.11) is indeed a solution to (2.9). Insertion into (2.6) finally results in

$$H^{(2)} = \Lambda_{A,+} [D_A + V + B_{2m} + E_f] \Lambda_{A,+}$$

$$B_{2m} := \frac{1}{2} [V F_A \tilde{D}_A + \tilde{D}_A F_A V + \tilde{D}_A V F_A + F_A V \tilde{D}_A].$$ 

(2.12)

3. Relative form boundedness of the Jansen–Hess potential

In order to establish self-adjointness of $H^{(2)}$, the form boundedness of the potential contributions to $H^{(2)}$ (restricted to the ‘positive’ space $\mathcal{H}_{k+1}$) relative to the kinetic energy operator $E_A$ is needed. We have to fix the potential strength $\gamma$ such that this bound becomes smaller than one. We start by showing the relative boundedness of the linear term (in $\gamma$) $V$, then we prove the boundedness of the operator $B_1$ (introduced by the transformation $U$) and subsequently the relative boundedness of the quadratic term. The resulting form boundedness...
of the Jansen–Hess potential relative to $E_A$ is stated in lemma 1, and the condition for $H^{(2)}$ being self-adjoint is part of theorem 1.

3.1. $E_A$-boundedness of $V$ and boundedness of $B_1$

A basic ingredient is the inequality $(\varphi, \exp(-p^2 t)^{\varphi}) \leq (\varphi, \exp(-t^2)^{\varphi})$, valid for $t \geq 0$ and $A \in L_{2, \text{loc}}(\mathbb{R}^3)$ ([1] and references therein). Making use of $(\varphi, p^2 \varphi) = -\lim_{t \to 0} (\varphi, \exp(-p^2 t)^{-1} \varphi)$ [24], one derives

$$<p, (p - eA)^2 \varphi> \geq (p, p^2 \varphi)$$

(3.1)

which is known as diamagnetic inequality (see also earlier work [13] of the related inequality $(|\varphi|, p^2 |\varphi|) \leq (\varphi, (p - eA)^2 \varphi)$). A consequence is

$$|\varphi - eA| \geq p.$$  (3.2)

Further, let $O_\gamma := \{(|O| - \sigma) \geq 0$ be the negative part of an operator $O$ and $\text{tr} O_\gamma$ its trace (i.e., the sum over the absolute values of the negative eigenvalues of $O$ times the spin degrees of freedom). Then by means of (3.1) and the Lieb–Thirring inequality [21, 23] for any $\mu > 0$ and $d > 0$ one has

$$\text{tr} |\mu(p - eA)^2 + e\sigma B| \leq \mu^d \text{tr} \left[ p^2 + \frac{e\sigma B}{\mu} \right] \leq 2\mu^d L_{d, 1.3} \int_{\mathbb{R}^3} \left( \frac{\sigma}{\mu} \right)^{d\mu} dx.$$  (3.3)

with constants $L_{d, 1.3} \leq 0.0604$ and $L_{1.3} \leq 0.0403$.

Then, following [23] we get the form estimate for $\varphi \in H^1(\mathbb{R}^3) \otimes \mathbb{C}^4$, $\|\varphi\| = 1$, using Kato’s inequality $\frac{1}{\mathcal{H}} \leq \frac{\varphi}{\mathcal{H}}$ and (3.2) as well as the trace inequality for non-negative, self-adjoint operators, $\text{tr}(O_1 - O_2)_\gamma \leq \text{tr}(O_1^\gamma - O_2^\gamma)_{1/2}$,

$$(\varphi, E_A \varphi) - (\varphi, \frac{\gamma_0}{\gamma} \varphi) \geq (\varphi, \sqrt{E_A^2 - m^2} - \frac{\gamma_0 \pi}{2} (p - eA) |\varphi|)$$

$$\geq -\text{tr} \left[ \left( E_A^2 - m^2 \right) - \frac{\gamma_0 \pi}{2} (p - eA) \right]_{\gamma}$$

$$\geq -2L_{d, 1.3} \left| 1 - \frac{\gamma_0 \pi}{2} \right|^2 \|\varphi\|^2.$$  (3.4)

for $\gamma_0 < \frac{1}{\gamma}$.

Moreover, using $\text{tr} O_\gamma \leq (\text{tr} O_\gamma)^{1/2}$ [27, p 210] and Hardy’s inequality $\frac{\gamma}{\gamma^2} \leq 4 \rho^2$,

$$\|E_A \varphi\|^2 - \|\frac{\gamma_0}{\gamma} \varphi\|^2 \geq (\varphi, \left(1 - 4\gamma_0^2\right)(p - eA)^2 - e\sigma B) |\varphi|$$

$$\geq -\text{tr} \left[ (1 - 4\gamma_0^2)(p - eA)^2 - e\sigma B \right]_{\gamma}$$

$$\geq -2L_{d, 1.3} \left| 1 - 4\gamma_0^2 \right|^2 \|\varphi\|^2.$$  (3.5)

for $\gamma_0 < \frac{1}{\gamma}$. Thus, we obtain the $E_A$-boundedness of the potential $V$ in the form and in the norm [28, p 162],

$$|(\varphi, V \varphi)| \leq \frac{\gamma}{\gamma_0} (\varphi, E_A \varphi) + \gamma c_B (\varphi, \varphi), \quad c_B := \frac{2}{\gamma_0} L_{d, 1.3} \left| 1 - \frac{\gamma_0 \pi}{2} \right|^2 \|\varphi\|^2.$$  (3.6)
and
\[
\| V \varphi \| \leq \frac{\gamma}{\gamma_1} \| E_A \varphi \| + \gamma d_B \| \varphi \|, \quad d_B := \frac{2}{\gamma_1} L_{1,3} \frac{e^2}{[1 - 4\gamma_1^2]^2} \| B \|^2. \tag{3.7}
\]

The boundedness of \( B_1 \) is a consequence of the boundedness of \( F_A \), since
\[
\| B_1 \| \leq \| \hat{D}_A \| \| F_A \| = \| F_A \|. \tag{3.8}
\]

With (3.6) at hand, the boundedness of \( F_A \) is easy to show. Following the proof of [30, lemma 1], we have for \( \varphi_+, \psi_+ \in \mathcal{H}_{+,1} \) from (2.8)
\[
\| F_A \| = \frac{1}{2\pi} \left\| \int_{-\infty}^{\infty} d\eta \frac{1}{D_A + i\eta} V \frac{1}{D_A + i\eta} \right\| \\
\leq \frac{\gamma}{2\pi} \sup_{\| \varphi_+, \| = 1} \int_{-\infty}^{\infty} d\eta \left\| \left( \varphi_+, \frac{1}{D_A + i\eta} \right) \left( \frac{1}{x^{1/2}} \frac{1}{D_A + i\eta} \right) \psi_+ \right\| \\
\leq \frac{\gamma}{2\pi} \sup_{\| \varphi_+, \| = 1} \int_{-\infty}^{\infty} d\eta \left\| \frac{1}{x^{1/2}} \frac{1}{D_A + i\eta} \right\| \left( \frac{1}{x^{1/2}} \frac{1}{D_A + i\eta} \psi_+ \right) \right\| . \tag{3.9}
\]

An application of the Schwarz inequality leads to
\[
\| F_A \| \leq \frac{\gamma}{2\pi} \sup_{\| \varphi_+, \| = 1} \left( \int_{-\infty}^{\infty} d\eta \left\| \frac{1}{x^{1/2}} \frac{1}{D_A + i\eta} \right\| \left( \frac{1}{x^{1/2}} \frac{1}{D_A + i\eta} \psi_+ \right) \right)^{1/2} \left( \int_{-\infty}^{\infty} d\eta \left\| \frac{1}{x^{1/2}} \frac{1}{D_A + i\eta} \right\| \right)^{1/2}. \tag{3.10}
\]

Setting \( \varphi := \frac{1}{D_A - i\eta} \varphi_+ \) (note that \( D_A^2 > 0 \) for \( m \neq 0 \) such that \((D_A - i\eta)^{-1} \) is bounded for \( \eta \in \mathbb{R} \)), we have from (3.6)
\[
\left\| \frac{1}{x^{1/2}} \frac{1}{D_A - i\eta} \right\|^2 = \left( \varphi, \frac{1}{x} \varphi \right) \leq \frac{1}{\gamma_0} (\varphi, E_A \varphi) + c_B (\varphi, \varphi) \tag{3.11}
\]
and thus we get for the two (equal) integrals in (3.10), using \( D_A \varphi_+ = E_A \varphi_+ \),
\[
\int_{-\infty}^{\infty} d\eta \left\| \frac{1}{x^{1/2}} \frac{1}{D_A - i\eta} \right\|^2 \leq \frac{1}{\gamma_0} (\varphi_+, \int_{-\infty}^{\infty} d\eta \frac{1}{E_A + i\eta} \frac{1}{E_A - i\eta} \psi_+) \]
\[
+ c_B (\varphi_+, \int_{-\infty}^{\infty} d\eta \frac{1}{E_A + i\eta^2} \psi_+) = \frac{1}{\gamma_0} \cdot \pi \| \psi_+ \|^2 + c_B \pi \left( \varphi_+, \frac{1}{E_A} \psi_+ \right). \tag{3.12}
\]

We estimate \( E_A \geq m \) and finally obtain the boundedness of \( \| F_A \| \):
\[
\| F_A \| \leq \frac{\gamma}{2\gamma_0} \left( 1 + c_B \frac{\gamma_0}{m} \right). \tag{3.13}
\]

We note that due to the existence of zero modes [25] the lower bound \( m \) of \( E_A \) is sharp: there is a field \( B_0 = \nabla \times A_0 \in L_2(\mathbb{R}^3) \), satisfying (1.4) and hence (1.5), and a function \( \psi_0 \in H_1(\mathbb{R}^3) \setminus \{0\} \otimes \mathbb{C}^2 \) such that
\[
\sigma (p - e A_0) \psi_0 = 0. \tag{3.14}
\]

From this it follows that the 4-spinor \( \psi_0 \) obeys \( D_{A_0} (\psi_0) = m (\psi_0) \), i.e. it lies in the positive magnetic spectral subspace of the electron, and \( m \) is the lowest positive eigenvalue of \( D_{A_0} \).
3.2. Relative boundedness of the Jansen–Hess potential

From (2.12) we get for \( \psi_+ \in \mathcal{H}_{+,1} \), with \( \| \Lambda_{A,+} \| = 1 \),
\[
\| (4 \Lambda_{A,+} B_{2m} \Lambda_{A,+}) \psi_+ \| \leq \| 4 B_{2m} \psi_+ \|
\leq \| V F_A \hat{D}_A \psi_+ \| + \| \hat{D}_A F_A V \psi_+ \| + \| \hat{D}_A V F_A \psi_+ \| + \| F_A V \hat{D}_A \psi_+ \|.
\tag{3.15}
\]

We shall estimate each of these four terms separately, using the boundedness (3.13) of \( F_A \) and the relative boundedness (3.7) of \( V \). First, we show
\[
[D_A, F_A] = \frac{1}{2}[\hat{D}_A, V].
\tag{3.16}
\]

We multiply (2.7) with \( \hat{D}_A \) and insert \( B_i \) from (2.11). This gives
\[
\hat{D}_A W_i = -i \hat{D}_A (i D_A \hat{D}_A F_A - i \hat{D}_A F_A D_A) = [D_A, F_A].
\tag{3.17}
\]

Inserting for \( W_i \) (below (2.6)) results in (3.16).

Using that \( \| \hat{D}_A \| = 1 \) and \( \hat{D}_A \psi_+ = \psi_+ \), (3.15) gives
\[
\| 4 B_{2m} \psi_+ \| \leq 2 \| V F_A \psi_+ \| + 2 \| F_A \| \| V \psi_+ \|.
\tag{3.18}
\]

With (3.7) and (3.16), defining \( F_A \psi_+ =: \varphi \), we estimate the first term by
\[
\| V F_A \psi_+ \| \leq \frac{\gamma}{\gamma_1} \| D_A \| \| \varphi \| + \gamma d_B \| \varphi \| \leq \frac{\gamma}{\gamma_1} \| D_A F_A \psi_+ \| + \gamma d_B \| F_A \| \| \psi_+ \|
\leq \frac{\gamma}{\gamma_1} \left( \| F_A \| \| D_A \psi_+ \| + \frac{1}{2} \| \hat{D}_A \| \| V \psi_+ \| + \frac{1}{2} \| V \psi_+ \| \right) + \gamma d_B \| F_A \| \| \psi_+ \|. \tag{3.19}
\]

Thus, we get
\[
\| B_{2m} \psi_+ \| \leq \frac{\gamma}{2 \gamma_1} \left( \frac{\gamma}{\gamma_1} + \| F_A \| \right) \| D_A \psi_+ \| + \gamma d_B \left( \frac{\gamma}{2 \gamma_1} + \| F_A \| \right) \| \psi_+ \|. \tag{3.20}
\]

Using (3.13) this results in
\[
\| B_{2m} \psi_+ \| \leq c \| E_A \psi_+ \| + C \| \psi_+ \|
\]
\[
c := \frac{\gamma^2}{2 \gamma_1} \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_0} + \frac{c_B}{m} \right), \quad C := \frac{\gamma^2 d_B}{2} \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_0} + \frac{c_B}{m} \right). \tag{3.21}
\]

Note that both constants, \( c \) and \( C \), depend on the field energy through \( \| B \| = (8\pi E_f)^{1/2} \).

From the \( E_A \)-boundedness of \( B_{2m} \) follows the \( E_A \)-form boundedness of \( B_{2m} \) with the same relative bound \( c \) [28, p 168]. Thus, we have proven

**Lemma 1.** Let \( H^{(2)} = D_A + V + B_{2m} + E_f \) be the ‘magnetic’ Jansen–Hess operator acting on \( \mathcal{H}_{+,1} \). Then \( V + B_{2m} \) is \( E_A \)-form bounded,
\[
\| (\psi_+, (V + B_{2m}) \psi_+) \| \leq \left( \frac{\gamma}{\gamma_0} + c \right) (\psi_+, E_A \psi_+) + \tilde{C} (\psi_+, \psi_+), \tag{3.22}
\]
with \( c = \frac{\gamma^2}{2 \gamma_1} \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_0} + \frac{c_B}{m} \right) \), where \( c_B \) and \( d_B \) are defined in (3.6) and (3.7), and \( \tilde{C} \) is some \( \| B \| \)-dependent constant. (The parameters \( \gamma_0 \leq \frac{1}{2} \) and \( \gamma_1 \leq \frac{1}{2} \) can be chosen arbitrarily.)
4. Positivity of $H^{(2)}$

Let $\delta > 0$ and recall that $E_A \geq m$ is bounded below. If in (3.21), the $\delta E_A$-bound $\frac{\delta}{m}$ of $B_{2m}$ is smaller than unity, then according to [18, theorem 4.11, p 291] $\delta E_A + B_{2m}$ is also bounded below by means of

$$
(\psi_+, (\delta E_A + B_{2m})\psi_+) \geq \left( \delta m - \max \left\{ \frac{C}{1 - c/\delta}, C + cm \right\} \right) (\psi_+, \psi_+),
$$

(4.1)

where the constants $c$ and $C$ are defined in (3.21).

Using the above results, we can estimate

$$
(\psi_+, H^{(2)}\psi_+) = (\psi_+, E_A\psi_+) - |(\psi_+, V\psi_+)| + (\psi_+, B_{2m}\psi_+) + E_f (\psi_+, \psi_+)
\geq (\psi_+, \left( 1 - \frac{\gamma}{\gamma_0} \right) E_A + B_{2m} \psi_+) - \gamma c_B (\psi_+, \psi_+) + E_f (\psi_+, \psi_+)
\geq \left( 1 - \frac{\gamma}{\gamma_0} \right) m - \max \left\{ \frac{C(1 - \gamma/\gamma_0)}{1 - \gamma/\gamma_0}, C + cm \right\} - \gamma c_B + E_f (\psi_+, \psi_+).
$$

(4.2)

This results in

**Theorem 1.** Let $H^{(2)} = D_A + V + B_{2m} + E_f$ be the ‘magnetic’ Jansen–Hess operator acting on $\mathcal{H}_{+, 1}$. If the $E_A$-form bound of $V + B_{2m}$ is smaller than unity,

$$
\frac{\gamma}{\gamma_0} + c < 1,
$$

(3.3)

then $H^{(2)}$ is bounded below and thus extends to a self-adjoint operator on $\Lambda_{A, 4}(L_2(\mathbb{R}^3) \otimes \mathbb{C}^2)$. If in addition

$$
\left( 1 - \frac{\gamma}{\gamma_0} \right) m - \gamma c_B - \max \left\{ \frac{C(1 - \gamma/\gamma_0)}{1 - \gamma/\gamma_0}, C + cm \right\} + E_f > 0,
$$

(3.4)

then $H^{(2)}$ is positive. This restricts the potential strength to $\gamma < \gamma_c$ where $\gamma_c \leq 0.353$ depending on the magnetic field $B$.

In order to derive the conditions on the bound for $\gamma$ which are required for theorem 1, we first consider the case $B = 0$. Then, we can set $\gamma_0 = \frac{1}{2}$ and $\gamma_1 = \frac{1}{2}$, and both inequalities, (3.3) and (3.4), are satisfied for $\gamma < \gamma_c^{(0)}$, where $\gamma_c^{(0)} = 0.353(Z \leq 48)$ is a solution of

$$
\frac{\gamma}{\gamma_0} + c = \frac{\gamma}{2} + \gamma^2 \left( \frac{2}{2} + \frac{\pi}{2} \right) = 1.
$$

(3.5)

This is considerably smaller than the critical $\gamma$ obtained earlier for the field-free case ($\gamma_c^{(0)} = 1.006$ [4]), where one is able to work in momentum space and to use Mellin transform techniques.

When $B$ is turned on, the bound on $\gamma$ from the self-adjointness condition decreases slowly. For example, if $\|B\| = 2.5$, then by optimizing $\gamma_0$ and $\gamma_1$ one gets from (3.3) $\gamma_c = 0.335(\gamma_0 = 0.6, \gamma_1 = 0.498)$, whereas positivity is guaranteed for $\gamma < 0.316(\gamma_0 = 0.6, \gamma_1 = 0.47)$. The relativistic ground-state binding energy of an electron, $|E_g - m| := m|\sqrt{1 - \gamma^2} - 1| = 0.0644$ (in units where $m = 1$, using $\gamma = \gamma_c^{(0)}$), may be used as a reference value with which to compare the field energy $E_f$. Even for quite large fields$^1$, e.g. $\|B\| = 10$ (where $E_f \approx 60|E_g - m|$), the critical potential strength (with

$^1$ In conventional units, $B = 1m^2e^3c/\hbar^3 = 2.35 \times 10^9$ G.
The Jansen–Hess operator with magnetic field

\( \gamma_0 = 0.54, \gamma_1 = 0.499 \) has only slightly decreased, \( \gamma_c = 0.299 \) (\( Z < 41 \)) while \( H^{(2)} > 0 \) for \( \gamma < 0.275 \) (\( \gamma_0 = 0.54, \gamma_1 = 0.45 \)).

However, when \( \|B\| \) becomes extremely large (but still is finite), our estimates (resulting in (4.4)) no longer guarantee positivity because \( C \) is of fourth order in \( \|B\| \) and eventually dominates \( E_J \). In order to remedy this deficiency, different estimates for the \( E_A \)-boundedness of the potential \( V \) are required.

For the magnetic fields which are \( (p_\lambda)^2 \)-bounded with bound \( \kappa \to 0 \) (and hence also \( (p_\lambda)^2 \)-form bounded with the same bound), we have from (1.3)

\[
(\varphi, |B|\varphi) \leq \kappa (\varphi, p_\lambda^2 \varphi) + C_\kappa (\varphi, \varphi)
\leq \kappa (\varphi, E_A^2 \varphi) + \kappa e(\varphi, |B|\varphi) + C_\kappa (\varphi, \varphi)
\]

proving the \( E_A^2 \)-boundedness of \( |B| \) with bound \( \kappa/(1 - \kappa e) \). It can be shown [34, proof of theorem 10.17] that the constant \( C_\kappa \) depends linearly on \( \sup_{x \in R} (N_B(x))^2 \) which in turn can be estimated above by \( \|B\| \). So, we get from Hardy’s inequality and (3.1)

\[
(\varphi, V^2 \varphi) \leq 4\gamma^2 (\varphi, p_\lambda^2 \varphi) \leq 4\gamma^2 (\varphi, E_A^2 \varphi) + 4\gamma^2 e(\varphi, |B|\varphi).
\]

Using (4.6), we eventually obtain the estimate

\[
\|V\varphi\| \leq 2\gamma \left( 1 + \frac{\kappa e}{1 - \kappa e} \right) \frac{1}{\gamma} \|E_A\varphi\| + 2\gamma \left( \frac{e}{1 - \kappa e} \right) C_\kappa \|\varphi\|
\]

in place of (3.7). Note that \( \kappa \) can be taken arbitrarily close to 0 such that the \( E_A \)-bound of \( V \) agrees with the one in (3.7). However, the last term in (4.8) increases only \( \sim \|B\|^{\frac{1}{2}} \). A similar estimate replaces (3.6) for the \( E_A \)-form boundedness of \( V \).

In order to get explicit constants, let us for the moment assume that \( B \) is bounded with \( \|B\|_\infty \leq \|B\| \). Then, the last term in (4.7) is estimated by \( 4\gamma^2 e\|B\|_\infty (\varphi, \varphi) \leq 4\gamma^2 e\|B\|(\varphi, \varphi) \) giving \( \|V\varphi\| \leq 2\gamma \|E_A\varphi\| + 2\gamma e\|B\|\|\varphi\| \). For the form bound, using Kato’s inequality, one gets

\[
| (\varphi, V\varphi) | \leq \gamma \frac{\sqrt{2}}{2} e(\varphi, |B|\varphi) \leq \frac{\sqrt{2}}{2} e(\varphi, E_A^2 \varphi + e|B|\varphi)
\]

\[
\leq \gamma \frac{\sqrt{2}}{2} (\varphi, E_A \varphi) + \gamma \frac{\sqrt{2}}{2} e(\varphi, |B|\varphi).
\]

(4.9)

When (3.6) and (3.7) are replaced by these two inequalities in the subsequent estimates, conditions (4.3) and (4.4) of theorem 1 now read

\[
1 - \gamma \frac{\pi}{2} - c_1 > 0
\]

and

\[
\left( 1 - \gamma \frac{\pi}{2} \right) m - \gamma \frac{\pi}{2} e\|B\|\|\varphi\| - \max \left\{ \frac{C_1 (1 - \gamma \pi/2)}{1 - \gamma \pi/2 - c_1}, C_1 + c_1 m \right\} + E_J > 0
\]

where \( c_1 \) and \( C_1 \) are the changed bounds for \( B_{2m} \), replacing (3.21),

\[
c_1 := \gamma^2 \left( 2 + \frac{\pi}{2} + \frac{e\|B\|\|\varphi\|}{2} \right), \quad C_1 := \gamma^2 \left( \left[ 2 + \frac{\pi}{2} \right] e\|B\|\|\varphi\| + \frac{e\|B\|}{2} \right).
\]

(4.12)

In condition (4.11) for the positivity of \( H^{(2)} \) the leading term in \( \|B\| \) is now \( E_J \), guaranteeing positivity for sufficiently large \( \|B\| \). For example, for \( \|B\| = 10, (4.10) \) and (4.11) hold for \( \gamma < 0.304 \), this limit already exceeding the corresponding one from (4.3).

We close this section by showing that a \( B \)-dependent constant in the form boundedness of \( V \) (which in turn leads to a \( B \)-dependent condition (4.3) for self-adjointness of \( H^{(2)} \)) cannot be avoided [36].
It was proven [2] that for a homogeneous magnetic field $B$, the ground-state energy of the Pauli operator in a central Coulomb field of any given strength $Z_0 e^2$ diverges logarithmically with $B$. This leads to the estimate

$$\frac{1}{2m} \left( \varphi, \left( E_A^2 - m^2 \right) \varphi \right) - \left( \varphi, \frac{Z_0 e^2}{x} \varphi \right) \geq -c_0 \ln B \left( \varphi, \varphi \right),$$

(4.13)

with a suitable ($Z_0$-dependent) constant $c_0$ and sufficiently large $B$. The estimate is sharp since (4.13) turns into an equality if $\varphi$ is the ground-state function. Let $Z_0 = Z/2$. Then, (4.13) is written in the following way:

$$\left( \varphi, \frac{2Z_0 e^2}{x} \varphi \right) = |(\varphi, V \varphi)| \leq c_3 (\varphi, E_A \varphi) + \left( \varphi, E_A \left( \frac{E_A}{m} - c_3 \right) \varphi \right) + [2c_0 \ln B - m] (\varphi, \varphi),$$

(4.14)

where $0 < c_3 < 1$ is an arbitrary real number. Since $E_A \geq m$, the second term in (4.14) is positive and cannot compensate the $B$-dependence of the third term for $B \to \infty$. The fact that a homogeneous $B$-field violates our requirement $\|B\| < \infty$ is no serious problem, since the strong localization of the ground-state function in all three spatial directions [2, 26] allows for the replacement of the homogeneous $B$ by an $L^2$-field (by smoothly cutting off at very large distances) without changing the ground-state energy.

5. Relative compactness of the perturbation

The aim of this section is to prove

**Theorem 2.** Let $H^{(2)} = H_0 + W$ be the ‘magnetic’ Jansen–Hess operator with $H_0 := \Lambda_{A,+}(D + E_f) \Lambda_{A,+}$ and $W := \Lambda_{A,+}(V + B_{2m}) \Lambda_{A,+}$. Then, we have for $\gamma < \gamma_c$

$$\sigma_{ess}(H^{(2)}) = \sigma_{ess}(H_0).$$

(5.1)

The critical potential strength is $\gamma_c \leq \gamma_c^{(0)} = 0.319$ and depends on the magnetic field $B$.

Equivalently [18, problem 5.38, p 244], we have to prove the compactness of the difference $R_\mu$ of the resolvents of $H^{(2)}$ and $H_0$,

$$R_\mu := \frac{1}{H^{(2)} + \mu} - \frac{1}{H_0 + \mu} = -\frac{1}{H_0 + \mu} \Lambda_{A,+}(V + B_{2m}) \Lambda_{A,+} \frac{1}{H^{(2)} + \mu},$$

(5.2)

where the second resolvent identity is used, and $\mu > 0$ has to be chosen suitably. We decompose

$$R_\mu =: R_\mu(V) + R_\mu(B_{2m})$$

$$= -\left\{ \frac{1}{H_0 + \mu} \Lambda_{A,+} V \Lambda_{A,+} + \Lambda_{A,+} B_{2m} \Lambda_{A,+} \right\} \left( \frac{1}{H_0 + \mu} \right)^{1/2} \left( H_0 + \mu \right)^{1/2} \frac{1}{H^{(2)} + \mu}$$

(5.3)

where $\lambda \in \left\{ \frac{1}{2}, 1 \right\}$, and we will show that the two operators in curly brackets are compact while the factor in square brackets is bounded. This will prove the compactness of $R_\mu$.

5.1. Relative compactness of $V^{1/2}$

For the proof of the above assertion we need, with $V = -\gamma/x$, the following lemma.
Lemma 2. Let \( H_0 = \Lambda_{A,+}(D_A + E_f)\Lambda_{A,+} \) with \( D_A \) from (1.1) and \( \Lambda_{A,+} \) from (2.1). Then, the operator

\[
\frac{1}{x^\frac{1}{2}} \Lambda_{A,+} \frac{1}{H_0 + \mu} \tag{5.4}
\]

is compact for \( \mu > 0 \).

According to [18, theorem 4.10, p 159], its adjoint \((H_0 + \mu)^{-1} \Lambda_{A,+} x^{-\frac{1}{2}} \) is then compact too.

**Proof.** We start by showing the boundedness of \( x^{-\frac{1}{2}} (|D_A| + \mu)^{-\frac{1}{2}} \) on \( L_2(\mathbb{R}^3) \otimes \mathbb{C}^4 \). From (3.6), we get

\[
\left\| \frac{1}{x^\frac{1}{2}} \frac{1}{(|D_A| + \mu)^\frac{1}{2}} \psi \right\|^2 \leq \frac{1}{\gamma} \left\| |D_A| \frac{1}{(|D_A| + \mu)^\frac{1}{2}} \psi \right\|^2 + c_B \left\| \frac{1}{(|D_A| + \mu)^\frac{1}{2}} \psi \right\|^2. \tag{5.5}
\]

Since \( (|D_A| + \mu)^{-\frac{1}{2}} \) is bounded for \( \mu > 0 \) and since \( |D_A| (|D_A| + \mu)^{-1} \leq 1 \), the rhs of (5.5) is bounded. This implies the relative boundedness of \( x^{-\frac{1}{2}} \) with respect to \( |D_A| \) with form bound \( a = 0 \). In fact, using [28, p 130], problem 19],

\[
a = \lim_{\mu \to \infty} \left\| \frac{1}{x^\frac{1}{2}} (|D_A| + \mu)^{-\frac{1}{2}} \right\|
\]

we have from (5.5), with \( |D_A| \geq m \),

\[
\left\| x^{-\frac{1}{2}} (|D_A| + \mu)^{-\frac{1}{2}} \psi \right\| \leq \left\| x^{-\frac{1}{2}} (|D_A| + \mu)^{-\frac{1}{2}} \left( \frac{1}{m + \mu} \right)^\frac{1}{2} \right\| \| \psi \|
\]

which proves \( a = 0 \).

Following [31, lemma 11.5], we define a smooth function \( \chi_0 \in C_0^\infty(\mathbb{R}^3) \) mapping to \([0, 1]\) by means of

\[
\chi_0(x) := \begin{cases} 
1, & x < R \\
0, & x \geq R + 1
\end{cases}
\]

with some \( R > 0 \), such that \( \text{supp}(1 - \chi_0) \subset \mathbb{R}^3 \setminus B_R(0) \), where \( B_R(0) \) is a ball of radius \( R \) centred at the origin. Further, let \( (\psi_n)_{n \in \mathbb{N}} \) be a normalized sequence in \( H_1(\mathbb{R}^3) \otimes \mathbb{C}^4 \) weakly converging to zero. We prove the compactness of (5.4) by showing that \( \left\| x^{-\frac{1}{2}} \Lambda_{A,+}(H_0 + \mu)^{-\frac{1}{2}} \psi_n \right\| \to 0 \) for \( n \to \infty \). We decompose

\[
\left\| \frac{1}{x^\frac{1}{2}} \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\| \leq \left\| (1 - \chi_0) \frac{1}{x^\frac{1}{2}} \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\| + \left\| \frac{1}{x^\frac{1}{2}} \chi_0 \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\|. \tag{5.8}
\]

For the first term, we have

\[
\left\| (1 - \chi_0) \frac{1}{x^\frac{1}{2}} \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\| \leq \frac{1}{R^\frac{1}{2}} \left\| (1 - \chi_0) \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\| \leq \frac{c}{R^\frac{1}{2}} \tag{5.9}
\]

with some constant \( c \). Thus, it can be made smaller than \( \epsilon/2 \) if \( R > (2c/\epsilon)^2 \).

For the second term, we define \( \tilde{\psi}_n := \chi_0 \Lambda_{A,+}(H_0 + \mu)^{-\frac{1}{2}} \psi_n \) and use the \( |D_A| \)-boundedness of \( x^{-\frac{1}{2}} \) with bound \( a \to 0 \),

\[
\left\| \frac{1}{x^\frac{1}{2}} \chi_0 \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\| \leq a \| |D_A| \tilde{\psi}_n \| + b \| \tilde{\psi}_n \|, \tag{5.10}
\]

with some constant \( b \). In order to establish that \( \| |D_A| \tilde{\psi}_n \| \) is finite (such that \( a \| |D_A| \tilde{\psi}_n \| \) can be dropped), we consider

\[
\mathcal{O} := [D_A, \chi_0] = \alpha(p \chi_0) \tag{5.11}
\]
which is bounded because \( \chi_0 \) is a \( C_0^\infty \)-function. Thus, we can decompose
\[
\chi_0 |D_A|^2 \chi_0 = \chi_0 D_A \cdot D_A \chi_0 = D_A \chi_0^2 D_A - \mathcal{O} \chi_0 D_A + D_A \chi_0 \mathcal{O} - \mathcal{O}^2 = D_A \chi_0^2 D_A - \mathcal{O} \chi_0 D_A + D_A \chi_0 \mathcal{O} - \mathcal{O}^2
\]
and estimate
\[
||D_A|\tilde{\psi}_n|^2 = \left( \psi_n, \frac{1}{H_0 + \mu} \Lambda_{A,+}(D_A \chi_0^2 D_A - \mathcal{O} \chi_0 D_A + D_A \chi_0 \mathcal{O} - \mathcal{O}^2) \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right) \\
\leq \|\chi_0\|^2_{L^\infty} \left\| D_A \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\|^2 + \left\| \mathcal{O} \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\|^2 \\
\times \|\chi_0\| \left\| D_A \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\| \left( 2 + \left\| \mathcal{O} \Lambda_{A,+} \frac{1}{H_0 + \mu} \psi_n \right\| \right) .
\]
Since \( D_A \Lambda_{A,+}(H_0 + \mu)^{-1} = \Lambda_{A,+} D_A \Lambda_{A,+}(\lambda_{A,+} + \mu, A \lambda_{A,+} + \mu + \mathcal{E}_f \lambda_{A,+} + \mu)^{-1} \leq 1 \), all terms on the rhs of (5.13) are bounded.

Regarding the last term of (5.10), we will establish the compactness of the operator 
\( K := \chi_0 \Lambda_{A,+}(H_0 + \mu)^{-1} \). Then \( \|\tilde{\psi}_n\| \to 0 \) for \( n \to \infty \). Collecting results, this shows that the second term of (5.8) can be made smaller than \( \epsilon/2 \) for \( n \) sufficiently large and thus proves the desired compactness of the operator (5.4).

The strategy to show the compactness of \( K \) is to start with the operator \( K_1 := \chi_0 (p^2 + m^2)^{-1} \) which is compact as a product of bounded functions \( f(x), g(p) \), each of which tending to zero as \( x, \) respectively \( p \), go to infinity (see, e.g., [31, lemma 7.10]). Then, bounded operators \( \mathcal{O}_1, \mathcal{O}_2 \) are constructed such that \( K_1 \cdot \prod \mathcal{O}_i = K \).

Let \( \mathcal{O}_1 := \sqrt{p^2 + m^2} D_A^{-1} \). For showing the boundedness of \( \mathcal{O}_1 \) let \( \psi := D_A^{-1} \psi \). Then from the diamagnetic inequality and (4.6),
\[
\|\mathcal{O}_1 \psi\|^2 = (\psi, (p^2 + m^2) \psi) \leq (\psi, (E^2 + e|B|) \psi) \\
\leq \left( 1 + \frac{\kappa e}{1 - \kappa e} \right) \|\psi\|^2 + \frac{eC_1}{1 - \kappa e} \|D_A^{-2}\| \|\psi\|^2 ,
\]
the rhs being obviously bounded.

With \( \mathcal{O}_2 := D_A \Lambda_{A,+}(H_0 + \mu)^{-1} \leq 1 \) (as shown above), we arrive at \( K_1 \cdot \mathcal{O}_1 \cdot \mathcal{O}_2 = K \).
\( \Box \)

We remark that in the same way the compactness of \( x^{-1}\frac{1}{2}(|D_A| + \mu)^{-1} \) on \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^d \) can be shown. The only additional ingredient is the boundedness of \( D_A(|D_A| + \mu)^{-1} \) in the equation corresponding to (5.13), which follows from \( \|D_A(|D_A| + \mu)^{-1} \psi_n\|^2 = \langle \psi_n, |D_A|^2(|D_A| + \mu)^{-2} \psi_n \rangle \leq \|\psi_n\|^2 \).

5.2. Boundedness of \((H_0 + \mu)^2 (H^{(2)} + \mu)^{-1}\)

Let first \( \lambda = 1 \). From (3.7) and (3.21), we have the relative form boundedness of the potential for \( \psi \in H_1(\mathbb{R}^3) \otimes \mathbb{C}^d \) and \( \psi := \Lambda_{A,+} \psi \),
\[
\|\Lambda_{A,+}(V + B_{2m}) \Lambda_{A,+} \psi\| \leq \|V \psi\| + \|B_{2m} \psi\| \leq a_0 \|D_A \psi\| + b_0 \|\psi\| \leq a_0 \|D_A \psi\| + b_0 \|\psi\|
\]
with
\[
a_0 := \frac{\gamma}{y_1} + \frac{\gamma^2}{2y_1} \left( \frac{1}{y_1} + \frac{1}{y_0} + \frac{c_B}{m} \right), \quad b_0 := \gamma dB + C ,
\]
where \( c_B \) is defined in (3.6). We have to restrict \( \gamma < \gamma_c \) such that \( a_0 < 1 \). \( \gamma_c \) depends on \( B \), its maximum value (for \( B = 0 \)) being \( \gamma_c^{(0)} = 0.319 \) [43], obtained as solution to
\[
2\gamma + \gamma^2 \left( 2 + \frac{4}{\gamma} \right) = 1 .
\]

Let \( \epsilon := 1 - a_0 \) with \( 0 < \epsilon < 1 \). With \( \psi := (H^{(2)} + \mu)^{-1} \psi_+ \), we want to show
\[
\left\| (H_0 + \mu) \frac{1}{H^{(2)} + \mu} \psi_+ \right\|^2 = \| (H_0 + \mu) \psi_+ \|^2 \leq c_1^2 \| (H^{(2)} + \mu) \psi_+ \|^2
\] (5.17)
for a suitable \( c_1 > 0 \). We estimate, using \( \|D_A \psi_+\| = \| \Lambda_A \cdot D_A \Lambda_A \cdot \psi \| \leq \| H_0 \psi \| \) and \( \| \psi_+ \| \leq \| \Lambda_A \cdot \psi \| \),
\[
c_1 \| (H^{(2)} + \mu) \psi_+ \| \geq c_1 \| (H_0 + \mu) \psi_+ \| - c_1 \| \Lambda_A \cdot (V + B_{2m}) \Lambda_A \cdot \psi \|
\geq c_1 \| (H_0 + \mu) \psi_+ \| - c_1 \| a_0 \| H_0 \psi_+ \| + b_0 \| \psi_+ \|
\]
\[
\geq c_1 \| (H_0 + \mu) \psi \| + (1 - c_1)(\| H_0 \psi \| + \| \mu \psi \|) \geq \| (H_0 + \mu) \psi \|.
\] (5.18)
Condition (5.18) is satisfied if \(-c_1 a_0 \geq 1 - c_1\) as well as \(-c_1 b_0 \geq (1 - c_1) \mu \), requiring the choice \( c_1 \geq 1/\epsilon \) and \( \mu \geq c_1 b_0/(c_1 - 1) \).

For \( \lambda = \frac{1}{2} \), the bound on \( \gamma \) can be improved by working with quadratic forms. From (3.22), we have
\[
(\psi, \Lambda_A \cdot (V + B_{2m}) \Lambda_A \cdot \psi) \geq -a_1 (\psi, \Lambda_A \cdot D_A \Lambda_A \cdot \psi) - \tilde{C} (\psi, \psi)
\] (5.19)
with \( a_1 := a_0 - \gamma \left( \frac{1}{\lambda^2} - \frac{1}{\lambda_0^2} \right) \). Trivially, we have \( (H_0 + \mu)^{\frac{1}{2}} (H^{(2)} + \mu)^{\frac{1}{2}} - \gamma \cdot (H^{(2)} + \mu)^{\frac{1}{2}} \cdot (H^{(2)} + \mu)^{\frac{1}{2}} \) where the last factor is bounded. For the boundedness of the other factor, we use the strategy of (5.17) to require \( \| (H_0 + \mu)^{\frac{1}{2}} \psi_+ \|^2 \leq c_2 \| (H^{(2)} + \mu)^{\frac{1}{2}} \psi_+ \|^2 \) which is satisfied if \( c_2 \geq 1/(1 - a_1) \) and \( \mu \geq c_2 \tilde{C}/(c_2 - 1) \). The necessary condition for \( c_2 > 0 \) is \( a_1 < 1 \), i.e. inequality (4.3). The corresponding maximum value for \( \gamma \) is \( \gamma^{(0)} = 0.353 \).

5.3. Compactness of \( R_\mu(V) \)

We take \( \lambda = \frac{1}{2} \) and decompose
\[
\frac{1}{H_0 + \mu} \Lambda_A \cdot V \Lambda_A \cdot \frac{1}{(H_0 + \mu)^{\frac{1}{2}}} = \gamma \left( \frac{1}{H_0 + \mu} \Lambda_A \cdot \frac{1}{x^2} \right) \left[ \frac{1}{x^2} \Lambda_A \cdot \frac{1}{(H_0 + \mu)^{\frac{1}{2}}} \right].
\] (5.20)
The factor in square brackets is bounded according to a (5.5)-type estimate by using that \( \Lambda_A, D_A \Lambda_A, \leq H_0 + \mu \). Together with lemma 2 and the result of section 5.2, this proves the compactness of \( R_\mu(V) \) for \( \gamma < \gamma^{(0)} \) determined from (4.3).

5.4. Compactness of \( R_\mu(B_{2m}) \)

According to the four contributions of \( B_{2m} \) from (2.12), we define
\[
\frac{1}{H_0 + \mu} (\Lambda_A \cdot B_{2m} \Lambda_A + \frac{1}{(H_0 + \mu)^{\frac{1}{2}}} = -\gamma \frac{4}{4} \sum_{i=1}^4 C_1(\lambda).
\] (5.21)
For \( i = 1 \), we take \( \lambda = 1 \) and decompose
\[
C_1(1) = \left( \frac{1}{H_0 + \mu} \Lambda_A \cdot \frac{1}{x^2} \right) \left[ \frac{1}{x^2} F_A D_A \Lambda_A \cdot \frac{1}{H_0 + \mu} \right].
\] (5.22)
In order to show the boundedness of the operator in square brackets, we use a (5.5)-type estimate for \( x^{-2} \) and note that \( F_A \overline{D_A} \Lambda_A \cdot (H_0 + \mu)^{-1} \) is bounded. It then remains to show the boundedness of \( M := \left| D_A \right|^2 F_A \overline{D_A} \Lambda_A \cdot (H_0 + \mu)^{-1} \). We estimate for \( \varphi \in L_2(\mathbb{R}^3) \otimes \mathbb{C}^4 \), noting that \( F_A \overline{D_A} = -\overline{D_A} F_A \),
\[
\left\| M \varphi \right\|^2 \leq \left\| \overline{D_A} F_A \Lambda_A \cdot \frac{1}{H_0 + \mu} \varphi, \right\| \left\| \overline{D_A} \Lambda_A \cdot \frac{1}{H_0 + \mu} \varphi \right\|
\]
\[
\leq \left\| \overline{D_A} F_A \Lambda_A \cdot \frac{1}{H_0 + \mu} \right\| \left\| \varphi \right\| \left\| \overline{D_A} \Lambda_A \cdot \frac{1}{H_0 + \mu} \varphi \right\|
\] (5.23)
We use (3.16) to commute $D_A$ with $F_A$, being left with two terms involving the potential $V$. In turn, these terms can be estimated according to (3.7) by replacing $V$ with $|D_A|$ plus a bounded remainder. For example, we get

\[
\left| \frac{1}{2} \tilde{D}_A V A_{A,+} \frac{1}{H_0 + \mu} \varphi \right| \leq \frac{\gamma}{2} \| \tilde{D}_A \| \left| |D_A| A_{A,+} \frac{1}{H_0 + \mu} \varphi \right| + \frac{\gamma d_B}{2} \| \tilde{D}_A \| \left| A_{A,+} \frac{1}{H_0 + \mu} \varphi \right|
\]

which obviously is bounded.

For $i = 2$, we take $\lambda = \frac{1}{2}$ and decompose

\[
\mathcal{O}_2 \left( \frac{1}{2} \right) = \frac{1}{H_0 + \mu} A_{A,+} \tilde{D}_A F_A (|D_A| + \mu) \cdot \left\{ \frac{1}{|D_A| + \mu + \frac{1}{x^2}} \right\} \cdot \frac{1}{x^2} A_{A,+} \frac{1}{H_0 + \mu}.
\]

Referring to our previous considerations, it remains to show the boundedness of the adjoint of the first term, $|D_A| F_A D_A A_{A,+} (H_0 + \mu)^{-1}$, since $\mu (H_0 + \mu)^{-1} A_{A,+} D_A F_A$ is trivially bounded (and since any bounded operator has a bounded adjoint). With $|D_A| F_A D_A = -D_A F_A$, we arrive at the last term of (5.23), the boundedness of which has just been shown.

For $i = 3$, we take again $\lambda = 1$. Then,

\[
\mathcal{O}_3 (1) = \frac{1}{H_0 + \mu} A_{A,+} \tilde{D}_A \frac{1}{x} F_A A_{A,+} \frac{1}{H_0 + \mu} = \tilde{D}_A \cdot \left( \frac{1}{H_0 + \mu} A_{A,+} \frac{1}{x^2} \right) \cdot \frac{1}{x^2} F_A A_{A,+} \frac{1}{H_0 + \mu},
\]

of which the first factor is compact and the second factor bounded. For the factor in square brackets we estimate according to (5.5), and further

\[
\left| |D_A| F_A A_{A,+} \frac{1}{H_0 + \mu} \varphi \right|^2 \leq \left| F_A A_{A,+} \frac{1}{H_0 + \mu} \varphi \right|^2 \cdot \left| |D_A| F_A A_{A,+} \frac{1}{H_0 + \mu} \varphi \right|. \tag{5.27}
\]

Since $\| |D_A| \varphi \|^2 = (\varphi, D_A^2 \varphi) = \| D_A \varphi \|^2$, the second factor agrees with the one from (5.23).

For $i = 4$ and $\lambda = 1$, we have $\mathcal{O}_4 (1) = (H_0 + \mu)^{-1} A_{A,+} F_A \frac{1}{x^2} \tilde{D}_A A_{A,+} (H_0 + \mu)^{-1} = \mathcal{O}_3 (1)^*$. Together with the result from section 5.2, this proves compactness of $R_B (B_{2n})$ for $\gamma < \tilde{\gamma}_c$ defined below (5.16).

6. The essential spectrum

For the Schrödinger operator with purely magnetic field, $p_A^2$, it was shown, following the work of Jörgens [17], that its essential spectrum is given by

\[
\sigma_{\text{ess}} (p_A^2) = (0, \infty) \tag{6.1}
\]

provided $A \in L_{2,\text{loc}} (\mathbb{R}^3)$ and

\[
N_A (x) = \int_{|x-y| \leq 1} |A(y)|^2 \, dy \to 0 \tag{6.2}
\]

as $x \to \infty$ ([20], see also [33]). In particular, condition (6.2) is satisfied if $B \to 0$ as $x \to \infty$ [20]. It is, however, easy to show that it is sufficient that $N_B (x) \to 0$ as $x \to \infty$ for (6.2) to hold. We use the relation between $A$ and $B$ introduced in [11],

\[
A(y) = \int_0^1 t \, dt \, B (x + t (y - x)) \wedge (y - x) \tag{6.3}
\]
which satisfies $\nabla \times A = B$ (since $\nabla \cdot B = 0$). Then we have, substituting $z := y - x$,

$$N_A(x) = \int_{z \leq 1} |A(z + x)|^2 \, dz = \int_{z \leq 1} dz \int_0^1 t \, dr \int_0^1 \tau \, d\tau \, \rho(B(x + tz) \wedge z)(B(x + rz) \wedge z).$$

(6.4)

We estimate $|B \wedge z| \leq |B|$ (since $z \leq 1$) and factorize the integrand according to $\frac{|B(x + tz)|}{t^2}, |B(x + rz)|$ with, e.g., $\epsilon = \frac{1}{2}$. Applying the Schwarz inequality, we get upon substituting $\xi := tz$ for $z$

$$N_A(x) \leq \left( \int_0^1 \frac{d\xi}{t^2} \right) \int_0^1 t \, dr \int_{z \leq 1} dz |B(x + tz)|^2$$

$$= \int_0^1 \frac{d\xi}{t^2} \int_0^1 \frac{d\xi}{t^2} \int_{\xi \leq 1} d\xi |B(x + \xi)|^2 \leq 4 \int_{\xi \leq 1} d\xi |B(x + \xi)|^2$$

(6.5)

which, upon assumption, tends to 0 as $x \to \infty$.

A further consequence of $N_B(x) \to 0$ (as $x \to \infty$) is that $e\sigma B$ is $p_A$-compact [32, theorem 5.2.2]. Thus, the essential spectrum of the Pauli operator $(\sigma p_A)^2$ is also given by $[0, \infty]$. Accordingly, $\sigma_{\text{ess}}(D_A^2) = [m^2, \infty)$, and therefore

$$\sigma_{\text{ess}}(E_A) = [m, \infty).$$

(6.6)

In fact, let $\lambda^2 \in [m^2, \infty)$ and $\lambda > 0$. Then, there exists a normalized sequence $\psi_n \in C_0^\infty(R^3) \otimes \mathbb{C}^2$ with $\psi_n \rightharpoonup 0$ such that $\|(E_A - \lambda)(E_A + \lambda)\psi_n\| \to 0$ as $x \to \infty$. Let $\phi \in C_0^\infty(R^3) \otimes \mathbb{C}^2$ and note that $C_0^\infty \subset H_2 \subset H_1 \subset L_2$. Then, $\|(E_A + \lambda)\phi \in H_1(R^3) \otimes \mathbb{C}^2 = D(E_A)$

and $(\phi, (E_A + \lambda)\psi_n) = ((E_A + \lambda)\phi, \psi_n) \to 0$ (since $\psi_n \rightharpoonup 0$) as $x \to \infty$. Moreover, $\lim_{x \to \infty} \|(E_A + \lambda)\psi_n\| \geq \lim_{x \to \infty} \|(m + \lambda)\psi_n\| = m + \lambda > 0$, which shows that $\hat{\psi} := (E_A + \lambda)\psi_n \to 0$, such that $\lambda \in \sigma_{\text{ess}}(E_A)$ [34, theorem 7.24, p 191].

In order to derive $\sigma_{\text{ess}}(D_A)$ from (6.6), we note that $\sigma_{\text{ess}}(-E_A) = (-\infty, -m)$. Moreover, since a unitary transformation does not change the essential spectrum, we have from (1.7)

$$\sigma_{\text{ess}}(D_A) = \sigma_{\text{ess}}(U_0 D_A U_0^{-1}) = \sigma_{\text{ess}}(\beta E_A)$$

$$= \sigma_{\text{ess}} \left( \begin{array}{cc} E_A & 0 \\ 0 & -E_A \end{array} \right) \cup \sigma_{\text{ess}} \left( \begin{array}{c} 0 \\ -E_A \end{array} \right) = [m, \infty) \cup (-\infty, -m].$$

(6.7)

It was proven earlier [11, theorem 1.4] that $\sigma_{\text{ess}}(D_A) = (-\infty, -m] \cup [m, \infty)$ under somewhat stronger assumptions (e.g., $B(x) \to 0$ as $x \to \infty$), the proof being similar to the one given in [5, p 117] for the Schrödinger case.

From the decomposition of $D_A$ into its (disjoint) positive and negative part, $D_A = \Lambda_A + D_A \Lambda_A^* + \Lambda_A - D_A \Lambda_A^*$, we get $\sigma_{\text{ess}}(\Lambda_A + D_A \Lambda_A^*) = \sigma_{\text{ess}}(E_A) = [m, \infty)$.

Together with theorem 2 we have thus proven

**Theorem 3.** Let $H^{(2)}$ be the ‘magnetic’ Jansen–Hess operator, let the vector potential $A \in L_{2, \text{loc}}(R^3)$, let the magnetic field obey $N_B(x) \to 0$ for $x \to \infty$ with finite field energy $E_f$. Then for a Coulomb potential with strength $\gamma < \tilde{\gamma}$, the essential spectrum is given by

$$\sigma_{\text{ess}}(H^{(2)}) = [m, \infty) + E_f,$$

(6.8)

where $\tilde{\gamma}$ is defined in theorem 2.
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