THE HVZ THEOREM FOR A PSEUDO-RELATIVISTIC OPERATOR

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Abstract. The localization of the essential spectrum of a relativistic two-electron ion is provided. The analysis is performed with the help of the pseudo-relativistic Brown-Ravenhall operator which is the restriction of the Coulomb-Dirac operator to the electrons' positive spectral subspace.

1. Introduction

We consider two interacting electrons in a central Coulomb field, generated by a point nucleus of charge number Z and infinite mass. In contrast to the case of a single electron in the central field, the additional electron-electron interaction potential does not vanish at infinity if simultaneously, the distance between the two electrons is kept fixed. Therefore, the determination of the essential spectrum with the help of relative compactness arguments is not possible. In order to cope with this difficulty, two-cluster decompositions of the involved particles (including the nucleus) are made and a subordinate partition of unity is defined. This method is described in [4, §3] and [21] for the Schrödinger case and is applied by Lewis, Siedentop and Vugalter [13] to the scalar pseudo-relativistic Hamiltonian which is obtained from the Schrödinger operator by replacing the single-particle kinetic energy operator $p^2/2m$ with $\sqrt{p^2+m^2}-m,$ $\mathbf{p}$ and m being the momentum and mass of the electron, respectively, and which has been analyzed in [25] and [9]. Historically, the location of the essential spectrum of multiparticle Schrödinger operators, called HVZ theorem, was provided by Hunziker (using diagrammatic techniques [10]), van Winter [26] and Zhislin [28] (see e.g. [18, p.120,343]). Alternative methods for the determination of the essential spectrum of generalized Schrödinger operators involve C*-algebra techniques (see e.g. [7] and references therein).

The Brown-Ravenhall operator to be discussed below was introduced [2] as projection of the Coulomb-Dirac operator onto the positive spectral subspace of the free electrons, and was analyzed in a series of papers (e.g. [6, 23, 24, 1]). It also emerges as the first-order term in unitary transformation schemes applied to the Coulomb-Dirac operator [5, 11]. Such transformations allow for the decoupling of the electon and positron spectral subspaces to arbitrary order n in the potential

Date: November 23, 2005.

strength. The convergence of the resulting series of operators was shown only recently [19].

Let (in relativistic units, $\hbar = c = 1$)

$$H \;=\; \sum_{k=1}^{2} (D_0^{(k)} + V^{(k)}) \tag{1.1}$$

be the Dirac operator for two noninteracting electrons, where $D_0^{(k)} := \boldsymbol{\alpha}^{(k)}\mathbf{p}_k + \beta^{(k)}m$ (with $\mathbf{p} := -i\nabla_{\mathbf{x}}$) is the free Dirac operator [22], and $V^{(k)} := -\frac{\gamma}{x_k}$ (with $\gamma = Ze^2$, $e^2 \approx 1/137.04$ the fine-structure constant and $x := |\mathbf{x}|$) is the central Coulomb potential for electron k. H acts in the Hilbert space $\mathcal{A}(L_2(\mathbb{R}^3) \otimes \mathbb{C}^4)^2$ (we shall use this notation as shorthand for $\mathcal{A}(\bigotimes_{k=1}^{2} (L_2(\mathbb{R}^3) \otimes \mathbb{C}^4)^{(k)})$ where $\mathcal{A}$ denotes antisymmetrization of the two-electron function, and the form domain is the subspace $\mathcal{A}(H_{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^4)^2$.

Let $V^{(12)} := \frac{e^2}{|\mathbf{x}_1 - \mathbf{x}_2|}$ be the electron-electron interaction. Then the (two-particle) Brown-Ravenhall operator is defined by

$$H^{BR} \;=\; \Lambda_{+,2}\,(H \;+\; V^{(12)})\,\Lambda_{+,2} \tag{1.2}$$

$$= \frac{1+\beta^{(1)}}{2}\,\frac{1+\beta^{(2)}}{2}\,U_0^{(1)}U_0^{(2)}\,(H + V^{(12)})\,(U_0^{(1)}U_0^{(2)})^{-1}\,\frac{1+\beta^{(1)}}{2}\,\frac{1+\beta^{(2)}}{2}$$

where $\Lambda_{+,2} = \Lambda_+^{(1)}\Lambda_+^{(2)}$ (as short-hand for $\Lambda_+^{(1)} \otimes \Lambda_+^{(2)}$) is the (tensor) product of the single-particle projectors $\Lambda_+^{(k)}$ onto the positive spectral subspace of $D_0^{(k)}$, and the second equality results from the representation of a single-particle function $\varphi_+^{(k)}$ in this subspace in terms of the Foldy-Wouthuysen transformation $U_0^{(k)}$, viz. $\varphi_+^{(k)} = U_0^{(k)-1}\binom{u_+}{0} = U_0^{(k)-1}\frac{1+\beta^{(k)}}{2}\binom{u_+}{u_-}$, with $\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $I \in \mathbb{C}^{2,2}$ the unit matrix, $u_+, u_- \in H_{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^2$, and

$$U_0^{(k)} \;:=\; A(p_k) \;+\; \beta^{(k)}\boldsymbol{\alpha}^{(k)}\mathbf{p}_k\,g(p_k), \tag{1.3}$$

$$A(p) := \left(\frac{E_p + m}{2E_p}\right)^{1/2}, \quad g(p) := \frac{1}{\sqrt{2E_p(E_p + m)}}$$

where $E_p = |D_0| = \sqrt{p^2 + m^2}$. The inverse $U_0^{(k)-1} = A(p_k) + \boldsymbol{\alpha}^{(k)}\mathbf{p}_k\,g(p_k)\beta^{(k)}$. Since H^{BR} is sandwiched between the projectors $\frac{1+\beta^{(k)}}{2}$ it has a block-diagonal form with one nonvanishing entry, h^{BR}, defined by means of [6]

$$(\phi_+, H^{BR}\,\phi_+) \;=\; (u, h^{BR}\,u) \tag{1.4}$$

with $\phi_+ \in \Lambda_{+,2}(\mathcal{A}(H_{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^4)^2)$ and $u \in \mathcal{A}(H_{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^2)^2$, thus reducing the single-particle spinor degrees of freedom from 4 to 2. One obtains (see e.g. [6])

$$h^{BR} \;=\; \sum_{k=1}^{2} (T^{(k)} \;+\; b_{1m}^{(k)}) \;+\; v^{(12)} \tag{1.5}$$

$$T^{(k)} := E_{p_k}, \quad b_{1m}^{(k)} := -\gamma \left(A(p_k)\, \frac{1}{x_k}\, A(p_k) + \boldsymbol{\sigma}^{(k)} \mathbf{p}_k\, g(p_k)\, \frac{1}{x_k}\, g(p_k)\, \boldsymbol{\sigma}^{(k)} \mathbf{p}_k \right)$$

where $\boldsymbol{\sigma}^{(k)}$ is the vector of Pauli matrices, while $v^{(12)}$ results from the electron-electron interaction term and is specified later (section 5).

We note that h^{BR} is a well-defined (in the form sense), positive operator for potential strengths $\gamma \leq \gamma_{BR} = \frac{2}{\pi/2+2/\pi} \approx 0.906$ which relies on the estimates $(u, \sum_{k=1}^{2} (T^{(k)} + b_{1m}^{(k)})\, u) \geq 2m(1-\gamma)(u,u)$ [24] and $V^{(12)} \geq 0$. In particular from the respective property of the single-particle operator [6] and using [3, 11] that $(\phi_+, V^{(12)} \phi_+) \leq \frac{e^2}{2\gamma_{BR}} (\phi_+, \sum_{k=1}^{2} T^{(k)} \phi_+)$, the total potential $V := b_{1m}^{(1)} + b_{1m}^{(2)} + v^{(12)}$ is $(T^{(1)} + T^{(2)})$-form bounded with form bound less than one for $\gamma < \gamma_{BR}$. Thus h^{BR} is a self-adjoint operator by means of the Friedrichs extension of the restriction of h^{BR} to $\mathcal{A}(C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^2)^2$.

2. The HVZ theorem and the strategy of proof

We introduce the three two-cluster decompositions of our operator,

$$h^{BR} = T + a_j + r_j, \qquad j = 0, 1, 2, \tag{2.1}$$

where $T := T^{(1)} + T^{(2)}$ and a_j collects all interactions not involving particle j ($j = 1, 2$ denotes the two electrons and $j = 0$ refers to the nucleus which is fixed at the origin). The remainder r_j is supposed to vanish when particle j is moved to infinity (respectively both electrons are moved to infinity in the case $j = 0$). Define for $j = 0, 1, 2$,

$$\Sigma_0 := \min_j \inf \sigma(T + a_j) = \min \{\inf \sigma(T + b_{1m}^{(1)}), \inf \sigma(T + v^{(12)})\} \tag{2.2}$$

(note that the two electrons move in the same potential, such that $b_{1m}^{(1)}(\mathbf{x}_1) = b_{1m}^{(2)}(\mathbf{x}_2)$). Then we have

Theorem 1 (HVZ theorem).
Let $h^{BR} = T + b_{1m}^{(1)} + b_{1m}^{(2)} + v^{(12)}$ be the two-electron Brown-Ravenhall operator with potential strength $\gamma < \gamma_{BR}$, and let (2.1) be its two-cluster decompositions. Then the essential spectrum of h^{BR} is given by

$$\sigma_{ess}(h^{BR}) = [\Sigma_0, \infty). \tag{2.3}$$

Physically this means that the bottom of the essential spectrum is given by the ground state of the one-electron ion, increased by the rest mass of the second electron.

The strategy of proof is based on Simon [20] (see also [4]) as well as on Lieb and Yau [14] and Lewis et al [13]. We start by characterizing the essential spectrum by means of a Weyl sequence, located outside a ball $B_n(0) \subset \mathbb{R}^6$ of radius n centered at the origin.

Lemma 1. *Let $h^{BR} = T + V$, let V be relatively form bounded with respect to T. Then $\lambda \in \sigma_{ess}(h^{BR})$ iff there exists a sequence of functions $\varphi_n \in \mathcal{A}(C_0^\infty(\mathbb{R}^6 \backslash B_n(0)) \otimes \mathbb{C}^4)$ with $\|\varphi_n\| = 1$ such that*

$$\|(h^{BR} - \lambda)\,\varphi_n\| \;\longrightarrow\; 0 \qquad \text{as } n \to \infty. \tag{2.4}$$

Recall that φ_n is a two-particle function, each particle being described by a two-spinor. For Schrödinger operators, Lemma 1 is proven in [4, Thm 3.11]. We note that Lemma 1 holds also for the single-particle operators, $h := T^{(1)}$ or $h := T^{(1)} + b_{1m}^{(1)}$, with a proof closely following the one given in section 7.

Lemma 2 (Persson's theorem).
Let $h^{BR} = T + V$, let V be relatively form bounded with respect to T, and let $\varphi \in \mathcal{A}(C_0^\infty(\mathbb{R}^6 \backslash B_R(0)) \otimes \mathbb{C}^4)$. Then

$$\inf \sigma_{ess}(h^{BR}) \;=\; \lim_{R\to\infty} \inf_{\|\varphi\|=1} (\varphi, h^{BR}\,\varphi). \tag{2.5}$$

The proof given in [4, Thm 3.12] relies on Lemma 1 and on the min-max principle [18, XIII.1], [12, p.60]. It also holds in our case.

Let us now introduce the Ruelle-Simon partition of unity $(\phi_j)_{j=0,1,2} \in C^\infty(\mathbb{R}^6)$ which is subordinate to the cluster decomposition (2.1), see e.g. [4, p.33], [21]. It is defined on the unit sphere and has the following properties,

$$\sum_{j=0}^{2} \phi_j^2 \;=\; 1, \qquad \phi_j(\lambda \mathbf{x}) \;=\; \phi_j(\mathbf{x}) \quad \text{for } x = 1 \text{ and } \lambda \geq 1,$$

$$\operatorname{supp}\phi_j \,\cap\, \mathbb{R}^6 \backslash B_1(0) \;\subseteq \{\mathbf{x} \in \mathbb{R}^6 \backslash B_1(0) : \, |\mathbf{x}_1 - \mathbf{x}_2| \geq Cx \text{ and } x_j \geq Cx\}, \;\; j = 1,2,$$
$$\operatorname{supp}\phi_0 \,\cap\, \mathbb{R}^6 \backslash B_1(0) \;\subseteq\; \{\mathbf{x} \in \mathbb{R}^6 \backslash B_1(0) : \, x_k \geq Cx \;\; \forall\, k \in \{1,2\}\,\}, \tag{2.6}$$

where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, $x = |\mathbf{x}|$ and C is a positive constant.

According to Lemma 2, we can always assume $\varphi \in \mathcal{A}(C_0^\infty(\mathbb{R}^6 \backslash B_R(0)) \otimes \mathbb{C}^4)$ in the following. For later use we introduce a smooth auxiliary function $\chi \in C^\infty(\mathbb{R}^6)$, $\operatorname{ran}\chi = [0,1]$, which is unity on the support of φ. Then, $\chi\varphi = \varphi$. Having in mind (2.2) we aim at a localization formula for our operator. We write

$$(\varphi, h^{BR}\varphi) \;=\; (\sum_{j=0}^{2} \phi_j\, \chi\phi_j\, \varphi, h^{BR}\varphi)$$

$$=\; \sum_{j=0}^{2} (\phi_j\varphi, h^{BR}\chi\phi_j\,\varphi) \;-\; \sum_{j=0}^{2} (\phi_j\varphi, [h^{BR}, \chi\phi_j]\,\varphi) \tag{2.7}$$

where $[B, A] = BA - AB$ denotes the commutator. One can show that not only the contribution of r_j to the first term of (2.7) vanishes uniformly as $R \to \infty$, but also the second term containing the commutator. More precisely, one has

Lemma 3. *Let $h^{BR} = T + a_j + r_j$, $(\phi_j)_{j=0,1,2}$ the Ruelle-Simon partition of unity and $\varphi \in \mathcal{A}(C_0^\infty(\mathbb{R}^6 \backslash B_R(0)) \otimes \mathbb{C}^4)$ with $R > 1$. Then*

$$|(\phi_j\varphi, r_j\,\phi_j\varphi)| \;\leq\; \frac{c}{R}\,\|\varphi\|^2, \qquad j = 0,1,2, \tag{2.8}$$

where c is some constant.

We call an operator $\mathcal{O}$ $\frac{1}{R}$-bounded if $\mathcal{O}$ is bounded by $\frac{c}{R}$. Thus Lemma 3 states that $\phi_j r_j \phi_j$ is $\frac{1}{R}$-bounded.

Lemma 4. *Assume h^{BR}, ϕ_j and φ as in Lemma 3, $R>2$. Then*

$$\begin{aligned}
&(a) \qquad |\sum_{j=0}^{2}(\phi_j\varphi,[T,\phi_j]\,\varphi)| &&\le \frac{c}{R^2}\,\|\varphi\|^2 \\
&(b) \qquad |(\phi_j\varphi,[b_{1m}^{(k)},\phi_j]\,\varphi)| &&\le \frac{c}{R}\,\|\varphi\|^2 \qquad (2.9)\\
&(c) \qquad |(\phi_j\varphi,[v^{(12)},\phi_j]\,\varphi)| &&\le \frac{c}{R}\,\|\varphi\|^2
\end{aligned}$$

where c is a generic constant.

The proof of (a) in Lemma 4 is provided in [13].

With Lemmata 3 and 4, (2.7) turns into the localization formula

$$(\varphi, h^{BR}\varphi) \;=\; \sum_{j=0}^{2}(\phi_j\varphi,(T+a_j)\,\phi_j\varphi) \;+\; O(\frac{1}{R})\,\|\varphi\|^2 \qquad (2.10)$$

for $R>2$. Using Persson's theorem (Lemma 2), we obtain

$$\inf\sigma_{ess}(h^{BR}) \;=\; \lim_{R\to\infty}\,\inf_{\|\varphi\|=1}\,\sum_{j=0}^{2}(\phi_j\varphi,(T+a_j)\,\phi_j\varphi). \qquad (2.11)$$

Recalling the definition (2.2) of Σ_0 in terms of the smallest infimum of $\sigma(T+a_j)$, we can estimate

$$\inf_{\|\varphi\|=1}\,\sum_{j=0}^{2}(\phi_j\varphi,(T+a_j)\,\phi_j\varphi) \;\ge\; \sum_{j=0}^{2}\Sigma_0\;(\phi_j\varphi,\phi_j\varphi) \;=\; \Sigma_0 \qquad (2.12)$$

since $\sum_{j=0}^{2}\phi_j^2 = 1 = \|\varphi\|$. This proves the inclusion $\sigma_{ess}(h^{BR}) \subset [\Sigma_0,\infty)$.

For the remaining inclusion, $[\Sigma_0,\infty)\subset\sigma_{ess}(h^{BR})$, in the literature called the 'easy part' of the proof of the HVZ theorem, we use the strategy of Weyl sequences [4, 21].

Let $\lambda\in[\Sigma_0,\infty)$. Consider the case that $\Sigma_0=\inf\sigma(T+a_j)$ for $j=1$, and assume that $\lambda\in\sigma(T+a_1)$. (This assumption is proven in (2.16) where it is shown that $\sigma(T+a_1)=\sigma_{ess}(T+a_1)$.) Since $T+a_1 = T^{(1)} + (T^{(2)}+b_{1m}^{(2)})$ describes two independent particles, we can decompose $\lambda=\lambda_1+\lambda_2$ with $\lambda_1\in\sigma_{ess}(T^{(1)})$ and $\lambda_2\in\sigma(T^{(2)}+b_{1m}^{(2)})$.

Let $(\varphi_n^{(1)})_{n\in\mathbb{N}}$ be a Weyl sequence corresponding to λ_1, i.e. $\varphi_n^{(1)}$ is characterized by

$$\|\varphi_n^{(1)}\| = 1, \quad \varphi_n^{(1)} \xrightarrow{w} 0, \qquad \|(T^{(1)}-\lambda_1)\,\varphi_n^{(1)}\| \to 0 \;\text{ as }\; n\to\infty. \qquad (2.13)$$

From Lemma 1 we can require $\varphi_n^{(1)} \in C_0^\infty(\mathbb{R}^3\backslash B_n(0))\otimes\mathbb{C}^2$. Let $(\phi_n^{(2)})_{n\in\mathbb{N}} \in C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^2$ be a defining sequence for λ_2 according to [27, Thm 7.22] with

$\|\phi_n^{(2)}\| = 1$. Since $\|(T^{(2)} + b_{1m}^{(2)} - \lambda_2)\,\phi_n^{(2)}\| \to 0$ as $n \to \infty$, for any given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|(T^{(2)} + b_{1m}^{(2)} - \lambda_2)\,\phi_N^{(2)}\| \;<\; \epsilon. \tag{2.14}$$

We define the sequence $\psi_n := \varphi_n^{(1)} \cdot \phi_N^{(2)}$ for $n \in \mathbb{N}$ and the antisymmetric sequence $\mathcal{A}\,\psi_n := \frac{1}{\sqrt{2}}\,(\varphi_n^{(1)}\phi_N^{(2)} - \varphi_n^{(2)}\phi_N^{(1)})$ and claim that a subsequence of $\mathcal{A}\,\psi_n$ is a Weyl sequence for $\lambda \in \sigma_{ess}(h^{BR})$.

(i) The weak convergence, $\mathcal{A}\,\psi_n \xrightarrow{w} 0$, follows from

$$\sqrt{2}\,|(\mathcal{A}\,\psi_n, f^{(1)}g^{(2)})| \;\le\; |(\varphi_n^{(1)}, f^{(1)})| \cdot |(\phi_N^{(2)}, g^{(2)})| \;+\; |(\phi_N^{(1)}, f^{(1)})| \cdot |(\varphi_n^{(2)}, g^{(2)})|$$

$$\le\; |(\varphi_n^{(1)}, f^{(1)})|\, \|\phi_N^{(2)}\|\, \|g^{(2)}\| \;+ |(\varphi_n^{(2)}, g^{(2)})|\, \|\phi_N^{(1)}\|\, \|f^{(1)}\| \;\to 0 \tag{2.15}$$

for all $f^{(1)}, g^{(2)} \in L_2(\mathbb{R}^3) \otimes \mathbb{C}^2$, since by (2.13) $\varphi_n^{(1)} \xrightarrow{w} 0$.

(ii) $(\psi_n)_{n\in\mathbb{N}}$ obeys the Weyl criterion for $\lambda \in \sigma_{ess}(T + a_1)$ since

$$\|(T^{(1)} - \lambda_1 + T^{(2)} + b_{1m}^{(2)} - \lambda_2)\;\psi_n\| \tag{2.16}$$

$$\le\; \|(T^{(1)} - \lambda_1)\,\varphi_n^{(1)}\|\, \|\phi_N^{(2)}\| \;+\; \|(T^{(2)} + b_{1m}^{(2)} - \lambda_2)\,\phi_N^{(2)}\|\, \|\varphi_n^{(1)}\| \;<\; 2\epsilon$$

for arbitrary ϵ and n sufficiently large because of (2.13) and (2.14).

(iii) Using that h^{BR} is symmetric upon particle exchange, we have

$$\|(h^{BR} - \lambda)\,\mathcal{A}\,\psi_n\| \;\le\; 2\,\|(h^{BR} - \lambda)\,\frac{1}{\sqrt{2}}\,\varphi_n^{(1)}\phi_N^{(2)}\|$$

$$\le\; \sqrt{2}\,\|(T + a_1 - \lambda)\,\psi_n\| \;+\; \sqrt{2}\,\|r_1\psi_n\| \;<\; 2\sqrt{2}\,\epsilon \;+\; \sqrt{2}\,\|b_{1m}^{(1)}\psi_n\| \;+\; \sqrt{2}\,\|v^{(12)}\psi_n\| \tag{2.17}$$

where (2.16) was used. One can show that the two remaining terms also tend to zero as $n \to \infty$. More precisely, one has

Lemma 5. *Let $\varphi \in C_0^\infty(\mathbb{R}^3 \backslash B_R(0)) \otimes \mathbb{C}^2$ and $R > 1$. Then for some constant c,*

$$\|b_{1m}^{(1)}\,\varphi\| \;\le\; \frac{c}{R}\,\|\varphi\|. \tag{2.18}$$

Lemma 6. *Let ψ_n as defined above. Then for all $\varphi \in (C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^2)^2$,*

$$|(\varphi, v^{(12)}\psi_n)| \;\le\; \frac{c}{n}\,\|\varphi\|\,\|\psi_n\| \tag{2.19}$$

with some constant c.

With Lemma 5, one has $\|b_{1m}^{(1)}\psi_n\| \;=\; \|b_{1m}^{(1)}\varphi_n^{(1)}\|\,\|\phi_N^{(2)}\| \;\le\; \frac{c}{n}\,\|\varphi_n^{(1)}\|\,\|\phi_N^{(2)}\|$. Moreover, one has the equivalence for an essentially self-adjoint operator A and $\psi \in \mathcal{D}(A)$ [16, p.260]

$$(i) \qquad \|A\,\psi\| \;\le\; \frac{c}{n}\,\|\psi\| \tag{2.20}$$

$$(ii) \qquad |(\varphi, A\,\psi)| \;\le\; \frac{c}{n}\,\|\varphi\|\,\|\psi\| \quad \forall\,\varphi \text{ in the core of } \mathcal{D}(A).$$

Choosing $A := v^{(12)}$, $\psi := \psi_n$ and $\varphi \in (C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^2)^2$ and using Lemma 6, this proves that the r.h.s. of (2.17) is smaller than 4ϵ for sufficiently large n.

(iv) Concerning the normalizability of $\mathcal{A}\,\psi_n$, we have

$$2\,\|\mathcal{A}\,\psi_n\|^2 \;=\; \|\varphi_n^{(1)}\|\,\|\phi_N^{(2)}\| \;+\; 2\;\mathrm{Re}\;(\varphi_n^{(1)}\phi_N^{(2)},\varphi_n^{(2)}\phi_N^{(1)}) \;+\; \|\varphi_n^{(2)}\|\,\|\phi_N^{(1)}\|. \tag{2.21}$$

Since $\phi_N^{(1)} \in C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^2$ there exists an $R_0 > 0$ such that $x_1 < R_0$ on $\mathrm{supp}\,\phi_N^{(1)}$, and in addition, $x_1 > n$ on $\mathrm{supp}\,\varphi_n^{(1)}$. Hence we have

$$(\varphi_n^{(1)}\phi_N^{(2)},\varphi_n^{(2)}\phi_N^{(1)}) \;=\; \left|\int_{\mathbb{R}^3} d\mathbf{x}_1\;\overline{\varphi_n^{(1)}}\;\phi_N^{(1)}\right|^2 \;=\; 0 \qquad \text{if } n > R_0. \tag{2.22}$$

Thus $\|\mathcal{A}\,\psi_n\| = 1$ for sufficiently large n. Therefore a subsequence of $\mathcal{A}\,\psi_n$ is a Weyl sequence for λ, resulting in $\lambda \in \sigma_{ess}(h^{BR})$.

Due to the symmetry upon particle exchange, this proves the case $j = 2$ as well. Consideration of the case $j = 0$ can be omitted, since $V^{(12)} \geq 0$ and hence $v^{(12)} \geq 0$ (whereas $V^{(1)} \leq 0$ and so $b_{1m}^{(1)} \leq 0$). Thus $\inf\sigma(T + v^{(12)}) \geq \inf\sigma(T + b_{1m}^{(1)})$ such that one has $\Sigma_0 = \inf\sigma(T + b_{1m}^{(1)})$.

3. Ingredients for the proofs of the lemmata

The main difference in the proofs of Lemmata 1, 3, 5 and 6 as contrasted to the proof of Lemma 4 lies in the fact that the momentum representation is used for the former, whereas the proof of Lemma 4 is carried out in coordinate space.

An important estimate which holds in either space is the Lieb and Yau formula (which is related to the Schur test for the boundedness of integral operators and is easily derived from the Schwarz inequality), generalized to the two-particle case [14, 11].

Lemma 7 (Generalized Lieb and Yau formula).
Let A be an essentially self-adjoint integral operator and $k_A(\boldsymbol{\xi},\boldsymbol{\xi}')$ its kernel, $\boldsymbol{\xi} \in \mathbb{R}^{3l}$ with $l \in \{1,2\}$ denoting the number of particles. Then for $\psi,\varphi \in \mathcal{D}(A)$,

$$|(\psi, A\,\varphi)| \;=\; |\int d\boldsymbol{\xi}\;\overline{\psi(\boldsymbol{\xi})}\int d\boldsymbol{\xi}'\;k_A(\boldsymbol{\xi},\boldsymbol{\xi}')\;\varphi(\boldsymbol{\xi}')|$$

$$\leq\;\left(\int d\boldsymbol{\xi}\;|\psi(\boldsymbol{\xi})|^2\;I(\boldsymbol{\xi})\right)^{\frac{1}{2}}\cdot\left(\int d\boldsymbol{\xi}'\;|\varphi(\boldsymbol{\xi}')|^2\;J(\boldsymbol{\xi}')\right)^{\frac{1}{2}} \tag{3.1}$$

with

$$I(\boldsymbol{\xi}) := \int d\boldsymbol{\xi}'\;|k_A(\boldsymbol{\xi},\boldsymbol{\xi}')|\;\frac{f(\boldsymbol{\xi})}{f(\boldsymbol{\xi}')}$$

$$J(\boldsymbol{\xi}') := \int d\boldsymbol{\xi}\;|k_A(\boldsymbol{\xi},\boldsymbol{\xi}')|\;\frac{f(\boldsymbol{\xi}')}{f(\boldsymbol{\xi})}, \tag{3.2}$$

where $f(\boldsymbol{\xi})$ is a nonnegative convergence generating function and all integrals run over $\mathbb{R}^{3l}$.

If $|k_A|$ is symmetric in $\boldsymbol{\xi}, \boldsymbol{\xi}'$ then $J(\boldsymbol{\xi}') = I(\boldsymbol{\xi}')$. Provided we can show that I and J are $\frac{1}{R}$-bounded functions (for all values of ξ and ξ'), then the uniform $\frac{1}{R}$-boundedness of A follows from (3.1) and from the equivalence (2.20),

$$|(\psi, A\,\varphi)| \;\leq\; \left(\int d\boldsymbol{\xi}\; |\psi(\boldsymbol{\xi})|^2\, \frac{c}{R}\right)^{\frac{1}{2}} \left(\int d\boldsymbol{\xi}'\; |\varphi(\boldsymbol{\xi}')|^2\, \frac{c}{R}\right)^{\frac{1}{2}}$$
$$=\; \frac{c}{R}\, \|\psi\|\, \|\varphi\|, \tag{3.3}$$

with c a suitable constant.

Let us now consider the properties of the smooth auxiliary function $\chi \in C^\infty(\mathbb{R}^{3l})$ where $l = 2$ denotes the two-particle and $l = 1$ the one-particle case. For $\varphi \in C_0^\infty(\mathbb{R}^{3l} \backslash B_R(0)) \otimes \mathbb{C}^{2l}$ we define for $\mathbf{x} \in \mathbb{R}^{3l}$

$$\chi(\frac{\mathbf{x}}{R}) := \begin{cases} 0, & x < R/2 \\ 1, & x \geq R \end{cases} \tag{3.4}$$

such that $\chi = 1$ on $\operatorname{supp}\varphi$. Moreover, define

$$\chi_0(\frac{\mathbf{x}}{R}) :=\; 1 \;-\; \chi(\frac{\mathbf{x}}{R}) \tag{3.5}$$

with $\chi_0 \in \mathcal{S}(\mathbb{R}^{3l})$, $\mathcal{S}$ being the Schwartz space.

In our proofs we shall introduce commutators with χ, such that χ can be replaced by χ_0, viz. $[B,\chi] = -[B,\chi_0]$, and for the operator B we shall choose a multiplication operator in momentum space. Then one can readily work in Fourier space since the Fourier transform of the Schwartz function χ_0 is again a Schwartz function, making the resulting integrals converge. Marking the Fourier transform with a hat, we have for $\mathbf{p} \in \mathbb{R}^{3l}$,

$$\left(\widehat{\chi_0(\frac{\cdot}{R})}\right)(\mathbf{p}) \;=\; \frac{1}{(2\pi)^{3l/2}} \int_{\mathbb{R}^{3l}} d\mathbf{x}\; e^{-i\mathbf{p}\mathbf{x}}\; \chi_0(\frac{\mathbf{x}}{R})$$
$$=\; \frac{1}{(2\pi)^{3l/2}}\; R^{3l} \int_{\mathbb{R}^{3l}} d\mathbf{z}\; e^{-i\boldsymbol{p}R\boldsymbol{z}}\; \chi_0(\mathbf{z}) \;=\; R^{3l}\; \hat{\chi}_0(\mathbf{p}R). \tag{3.6}$$

In Lemma 4 the commutator with the partition of unity, ϕ_j, is needed. As neither ϕ_j nor $1-\phi_j$ is a Schwartz (or even an L_2) function, its Fourier transform is not well defined. Therefore we work in coordinate space instead. The strategy we apply is to construct commutators $[B,\phi_j]$ where B is again a multiplication operator in momentum space. Then its kernel k_B factorizes, $k_B(\mathbf{p},\mathbf{p}') = B(\mathbf{p})\,\delta(\mathbf{p}-\mathbf{p}')$. Our aim is to estimate this kernel in coordinate space and then apply the Lieb and Yau formula.

Consider the one-particle case, $\mathbf{p} \in \mathbb{R}^3$, and let $B(\mathbf{p}) := (\boldsymbol{\sigma}\mathbf{p})^l g(p)$ where g is spherically symmetric and $l \in \{0,1\}$. The Fourier transformed kernel is defined by

$$\check{k}_B(\mathbf{x}_1,\mathbf{x}_1') \;=\; \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d\mathbf{p}\; e^{i\boldsymbol{p}\mathbf{x}_1} \int_{\mathbb{R}^3} d\mathbf{p}'\; e^{-i\boldsymbol{p}'\boldsymbol{x}_1'}\; (\boldsymbol{\sigma}\mathbf{p})^l\; g(p)\; \delta(\mathbf{p}-\mathbf{p}')$$

$$= (-i\boldsymbol{\sigma}\nabla_{\mathbf{x}_1})^l \, \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d\mathbf{p} \; e^{i\boldsymbol{p}(\boldsymbol{x}_1 - \boldsymbol{x}_1')} \; g(p) \tag{3.7}$$

$$= (-i\boldsymbol{\sigma}\nabla_{\tilde{\mathbf{x}}})^l \, \frac{4\pi}{(2\pi)^3} \, \frac{1}{\tilde{x}} \int_0^\infty p\,dp \; \sin p\tilde{x} \; g(p)$$

where $\tilde{\mathbf{x}} := \mathbf{x}_1 - \mathbf{x}_1'$ is introduced. This integral can be estimated with the help of complex and asymptotic analysis [15].

In the following we prove successively Lemma 5 (section 4), Lemma 6 (section 5), Lemma 3 (section 6), Lemma 1 (section 7) and Lemma 4 (section 8).

4. Proof of Lemma 5

According to the equivalence (2.20) we prove $|(\phi, b_{1m}^{(1)}\varphi)| \leq \frac{c}{R}\,\|\phi\|\,\|\varphi\|$ for all $\phi \in C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^2$.

The operator $b_{1m}^{(1)}$, defined in (1.5) with (1.3), is a sum of terms each of which has the structure $B(\mathbf{p})\frac{1}{x}B(\mathbf{p})$ where the indices on $\mathbf{p}_1$ and $\mathbf{x}_1$ are suppressed and B is an analytic (for $m \neq 0$), bounded multiplication operator in momentum space. Hence one can apply the mean value theorem to find

$$B(\mathbf{p}) \;=\; B(\mathbf{p}') \;+\; (\mathbf{p}-\mathbf{p}')\;\nabla_{\mathbf{p}}\,B(\boldsymbol{\xi}_0), \qquad \boldsymbol{\xi}_0 \;:=\; \lambda\mathbf{p} \;+\; (1-\lambda)\,\mathbf{p}' \tag{4.1}$$

for a suitable $\lambda \in [0,1]$. Since B is bounded for all $\mathbf{p}$, its derivative can be estimated by

$$|\nabla_{\mathbf{p}}\,B(\mathbf{p})| \;\leq\; \frac{c_0}{1+p} \tag{4.2}$$

with some constant c_0. We introduce the auxiliary function χ from (3.4) (with $l = 1$) and estimate (by means of the triangle inequality) each term of $b_{1m}^{(1)}$ separately,

$$|(\phi, B\,\frac{1}{x}\,B\,\varphi)| \;=\; |(B\phi, \frac{1}{x}\,B\chi\,\varphi)| \;\leq\; |(\tilde{\phi}, \frac{1}{x}\,\chi\,\tilde{\varphi})| \;+\; |(\tilde{\phi}, \frac{1}{x}\,[B,\chi]\,\varphi)| \tag{4.3}$$

where we have abbreviated $\tilde{\phi} := B\phi$ and $\tilde{\varphi} := B\varphi$. Recalling that $\chi(\frac{\mathbf{x}}{R})$ is nonvanishing only if $x \geq R/2$ and $\operatorname{ran}\chi \in [0,1]$, the first contribution in (4.3) is easily estimated by

$$|(\tilde{\phi}, \frac{1}{x}\,\chi\,\tilde{\varphi})| \;\leq\; \int_{\mathbb{R}^3} d\mathbf{x}\;|\tilde{\phi}(\mathbf{x})|\;\frac{1}{x}\;\chi(\frac{\mathbf{x}}{R})\;|\tilde{\varphi}(\mathbf{x})|$$

$$\leq\; \frac{2}{R}\;\|\tilde{\phi}\|\;\|\tilde{\varphi}\| \;\leq\; \frac{c}{R}\;\|\phi\|\;\|\varphi\| \tag{4.4}$$

where $\|\tilde{\phi}\| \;\leq\; \|B\|\,\|\phi\| \;\leq c_1\|\phi\|$ was used $(c := 2c_1^2)$. In the second term we set $\chi = 1 - \chi_0$ and note that $\frac{1}{x}\,[B,\chi] \;=\; -\frac{1}{px}\cdot p\,[B,\chi_0]$ with $\frac{1}{px}$ a bounded operator [12, p.307]. Transforming into Fourier space, we get with (3.6)

$$\left(p\,\widehat{[B,\chi_0]\,\varphi}\right)(\mathbf{p}) \;=\; \frac{1}{(2\pi)^{\frac{3}{2}}}\;p\int_{\mathbb{R}^3} d\mathbf{p}'\;R^3\;\hat{\chi}_0((\mathbf{p}-\mathbf{p}')R)\;(B(\mathbf{p})-B(\mathbf{p}'))\;\hat{\varphi}(\mathbf{p}')$$

$$=: \int_{\mathbb{R}^3} d\mathbf{p}'\;k_{p[B,\chi_0]}(\mathbf{p},\mathbf{p}')\;\hat{\varphi}(\mathbf{p}'). \tag{4.5}$$

We define $\psi := \frac{1}{px}\tilde{\phi}$ and apply the Lieb and Yau formula (3.1) with $l = 1$, $\boldsymbol{\xi} := \mathbf{p}$ and $f = 1$. Using our kernel from (4.5) we can estimate with (4.1) and (4.2)

$$I(\mathbf{p}) := \int_{\mathbb{R}^3} d\mathbf{p}' \, \frac{R^3}{(2\pi)^{\frac{3}{2}}} \, p \, |\hat{\chi}_0((\mathbf{p}-\mathbf{p}')R| \, |\mathbf{p}-\mathbf{p}'| \, |\nabla_{\mathbf{p}} B(\boldsymbol{\xi}_0)|$$

$$\leq \frac{c_0}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} d\mathbf{y} \, |\hat{\chi}_0(\mathbf{y})| \, y \, \frac{1}{R} \cdot \frac{p}{1+|\mathbf{p}-(1-\lambda)\mathbf{y}/R|} \, . \tag{4.6}$$

The last factor is bounded for all $p \geq 0$, and the integral is finite because $\hat{\chi}_0 \in \mathcal{S}(\mathbb{R}^3)$. Thus $I(\mathbf{p})$ is $1/R$-bounded, i.e. $I(\mathbf{p}) \leq \frac{c}{R}$.

The second integral $J(\mathbf{p}')$ in the Lieb and Yau formula can be estimated in the same way. There, the last factor in (4.6) is replaced by $|\mathbf{p}'+\mathbf{y}/R|/(1+|\mathbf{p}'+\lambda\mathbf{y}/R|)$ which is also bounded for all $p' \geq 0$. (In the limiting case $m = 0$, one has $B(\mathbf{p}) := A(p) = \frac{1}{\sqrt{2}}$ which commutes with χ, and for $B(\mathbf{p}) := \boldsymbol{\sigma}\mathbf{p}/p$, one should use the explicit result $B(\mathbf{p}) - B(\mathbf{p}') = \frac{\mathbf{p}-\mathbf{p}'}{p}\,(\boldsymbol{\sigma} - (\boldsymbol{\sigma}\mathbf{p}'/p')(\mathbf{p}+\mathbf{p}')/(p+p'))$ such that the last factor in (4.6) is not present.)

Together with (3.3), this proves

$$|(\tilde{\phi}, \frac{1}{x}\,[B,\chi]\,\varphi)| \;\leq\; \frac{c}{R}\,\|\frac{1}{px}\,B\,\phi\|\,\|\varphi\| \;\leq\; \frac{\tilde{c}}{R}\,\|\phi\|\,\|\varphi\|, \tag{4.7}$$

and thus the assertion of Lemma 5.

5. Proof of Lemma 6

We have to show that $|(\varphi, v^{(12)}\psi_n)| \leq \frac{c}{n}\,\|\varphi\|\,\|\psi_n\|$ for $\psi_n = \varphi_n^{(1)}\phi_N^{(2)}$ with $\varphi_n^{(1)} \in C_0^\infty(\mathbb{R}^3\backslash B_n(0)) \otimes \mathbb{C}^2$ and $\phi_N^{(2)} \in C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^2$.

For the definition of the auxiliary function χ we note that $\exists\, R_0 : \operatorname{supp}\psi_n \subset \mathbb{R}^3\backslash B_n(0) \cap B_{R_0}(0)$. Choose n so large that $R_0 < n/2$. Then, on $\operatorname{supp}\psi_n$: $|\mathbf{x}_1 - \mathbf{x}_2| \geq x_1 - x_2 > n - \frac{n}{2} = n/2$. Define

$$\chi_{12}(\frac{\mathbf{x}_1-\mathbf{x}_2}{n}) := \begin{cases} 0, & |x_1 - x_2| < n/4 \\ 1, & |x_1 - x_2| \geq n/2 \end{cases}, \tag{5.1}$$

a smooth function mapping to $[0,1]$ with the property $\psi_n\chi_{12} = \psi_n$.

The operator $v^{(12)}$, defined in (1.2) - (1.5), reads explicitly (note that only terms even in $\boldsymbol{\alpha}^{(k)}$ survive the projection by $(1+\beta^{(k)})/2$)

$$v^{(12)} \;=\; A(p_1)A(p_2)\,\frac{e^2}{|\mathbf{x}_1-\mathbf{x}_2|}\,A(p_1)A(p_2) \tag{5.2}$$

$$+\,A(p_1)g(p_2)\,\boldsymbol{\sigma}^{(2)}\mathbf{p}_2\,\frac{e^2}{|\mathbf{x}_1-\mathbf{x}_2|}\,A(p_1)g(p_2)\,\boldsymbol{\sigma}^{(2)}\mathbf{p}_2$$

$$+\,A(p_2)g(p_1)\,\boldsymbol{\sigma}^{(1)}\mathbf{p}_1\,\frac{e^2}{|\mathbf{x}_1-\mathbf{x}_2|}\,g(p_1)\,\boldsymbol{\sigma}^{(1)}\mathbf{p}_1\,A(p_2)$$

$$+\,g(p_1)\,\boldsymbol{\sigma}^{(1)}\mathbf{p}_1\,g(p_2)\,\boldsymbol{\sigma}^{(2)}\mathbf{p}_2\,\frac{e^2}{|\mathbf{x}_1-\mathbf{x}_2|}\,g(p_1)\,\boldsymbol{\sigma}^{(1)}\mathbf{p}_1\,g(p_2)\,\boldsymbol{\sigma}^{(2)}\mathbf{p}_2$$

with A and g as in (1.3). For the present proof, we again need only the structure of each term in $v^{(12)}$, $B(\mathbf{p}_1,\mathbf{p}_2)\,\frac{e^2}{|\mathbf{x}_1-\mathbf{x}_2|}\,B(\mathbf{p}_1,\mathbf{p}_2)$ with $B(\mathbf{p}_1,\mathbf{p}_2)\,=B_1(\mathbf{p}_1)\cdot B_2(\mathbf{p}_2)$ an analytic (for $m\neq 0$), bounded multiplication operator in momentum space. As in the previous proof (cf. (4.3)) we decompose

$$|(\varphi,\,B\,\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}\,B\,\psi_n)| \tag{5.3}$$

$$\leq\;|(B\varphi,\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}\,\chi_{12}\,B\,\psi_n)|\;+\;|(B\varphi,\,\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|p_k}\cdot p_k\,[B,\chi_{12}]\,\psi_n)|$$

with $k\in\{1,2\}$. We have $|\mathbf{x}_1-\mathbf{x}_2|^{-1}\leq 4/n$ on $\operatorname{supp}\chi_{12}$ such that the first summand in (5.3) can be estimated by $\frac{4}{n}\,\|B\varphi\|\,\|B\psi_n\|\;\leq\;\frac{c}{n}\,\|\varphi\|\,\|\psi_n\|$.

Consider now the second summand. The boundedness of $(|\mathbf{x}_1-\mathbf{x}_2|p_k)^{-1}$ is readily seen by considering e.g. $k=1$ and defining $\varphi_{x_2}(\mathbf{y}_1):=\psi(\mathbf{y}_1+\mathbf{x}_2,\mathbf{x}_2)$ for $\psi\in\mathcal{A}(L_2(\mathbb{R}^3)\otimes\mathbb{C}^2)^2$. Keeping $\mathbf{x}_2$ fixed, we have with $p_1=\sqrt{-\nabla^2_{\mathbf{x}_1}}$ and the substitution $\mathbf{y}_1:=\mathbf{x}_1-\mathbf{x}_2$,

$$\left|\int_{\mathbb{R}^3}d\mathbf{x}_1\,\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}\,\overline{\psi(\mathbf{x}_1,\mathbf{x}_2)}\,\frac{1}{p_1}\,\psi(\mathbf{x}_1,\mathbf{x}_2)\right|\;=\;\left|\int_{\mathbb{R}^3}d\mathbf{y}_1\,\frac{1}{y_1}\,\overline{\varphi_{x_2}(\mathbf{y}_1)}\,\frac{1}{\sqrt{-\nabla^2_{\mathbf{y}_1}}}\,\varphi_{x_2}(\mathbf{y}_1)\right|$$

$$\leq\;c\int_{\mathbb{R}^3}d\mathbf{y}_1\;|\varphi_{x_2}(\mathbf{y}_1)|^2 \tag{5.4}$$

according to Kato's inequality. Hence, $|(\psi,\,\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|p_1}\,\psi)|\;\leq c\int_{\mathbb{R}^6}d\mathbf{x}_2d\mathbf{y}_1\,|\varphi_{x_2}(\mathbf{y}_1)|^2$ $=c\,\|\psi\|^2$.

Furtheron, we aim at a reduction of the commutator to the one-particle case such that the proof of Lemma 5 can be mimicked.

First we note that χ_{12} depends only on the difference of variables such that, defining $\chi_{12,0}:=1-\chi_{12}$, $\hat{\chi}_{12,0}$ splits off a Dirac δ-function. Using $l=2$ we get upon substitution of $\mathbf{z}_1':=\mathbf{z}_1-\mathbf{z}_2$ for $\mathbf{z}_1$ from the second line of (3.6)

$$\left(\widehat{\chi_{12,0}(\frac{\circ-*}{n})}\right)(\mathbf{p}_1,\mathbf{p}_2)\;=\;\frac{1}{(2\pi)^3}\,n^6\int_{\mathbb{R}^6}d\mathbf{z}_1d\mathbf{z}_2\;e^{-i\mathbf{p}_1n\mathbf{z}_1}\,e^{-i\mathbf{p}_2n\mathbf{z}_2}\;\chi_{12,0}(\mathbf{z}_1-\mathbf{z}_2) \tag{5.5}$$

$$(a)\quad=n^6\int_{\mathbb{R}^3}d\mathbf{z}_1'e^{-i\mathbf{p}_1n\mathbf{z}_1'}\chi_{12,0}(\mathbf{z}_1')\delta((\mathbf{p}_1+\mathbf{p}_2)n)=n^3(2\pi)^{\frac{3}{2}}\hat{\chi}_{12,0}(\mathbf{p}_1n)\delta(\mathbf{p}_1+\mathbf{p}_2)$$

$$(b)\qquad\qquad=\;n^3(2\pi)^{\frac{3}{2}}\;\hat{\chi}_{12,0}(-\mathbf{p}_2n)\;\delta(\mathbf{p}_1+\mathbf{p}_2)$$

where line (b) is obtained from $\mathbf{p}_1=-\mathbf{p}_2$.

Now we make use of the factorization of $B(\mathbf{p}_1,\mathbf{p}_2)$ to write

$$p_k\,[B_1(\mathbf{p}_1)B_2(\mathbf{p}_2),\chi_{12}]\;=\;-p_k\,[B_1(\mathbf{p}_1),\chi_{12,0}]\,B_2\;-\;B_1\,p_k\,[B_2(\mathbf{p}_2),\chi_{12,0}] \tag{5.6}$$

such that the second summand in (5.3) can be split (via the triangle inequality) into two parts for each of which p_k is chosen independently ($k=1$ for the first and

$k = 2$ for the second term in (5.6)). Consider the kernel of the first part (cf. (4.5))

$$k_{p_1[B_1,\chi_{12,0}]}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2') \tag{5.7}$$

$$= p_1 \frac{n^3}{(2\pi)^{\frac{3}{2}}} \hat{\chi}_{12,0}((\mathbf{p}_1 - \mathbf{p}_1')n) \, \delta(\mathbf{p}_1 - \mathbf{p}_1' + \mathbf{p}_2 - \mathbf{p}_2') \, (B_1(\mathbf{p}_1) - B_1(\mathbf{p}_1'))$$

where (a) is used for the Fourier transform of $\chi_{12,0}$. Insertion into the Lieb and Yau formula (for $l = 2$) with $f = 1$ gives

$$I(\mathbf{p}_1, \mathbf{p}_2) := \int_{\mathbb{R}^6} d\mathbf{p}_1' d\mathbf{p}_2' \, |k_{p_1[B_1,\chi_{12,0}]}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2')|$$

$$= p_1 \frac{n^3}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} d\mathbf{p}_1' \, |\hat{\chi}_{12,0}((\mathbf{p}_1 - \mathbf{p}_1')n)| \, |B_1(\mathbf{p}_1) - B_1(\mathbf{p}_1')| \tag{5.8}$$

which is independent of $\mathbf{p}_2$ and has the identical form of (4.6) (note that the operators B_1 and B_2 are the same as occurred in $b_{1m}^{(1)}$, and $\hat{\chi}_{12,0} \in \mathcal{S}(\mathbb{R}^3)$). Therefore, I (and also J) is $\frac{1}{n}$-bounded by the proof of Lemma 5. The same holds true for the second part of the second summand in (5.3) which corresponds to the second term in (5.6). There, expression (b) in (5.5) has to be used. Thus, collecting results,

$$|(B\,\varphi, \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} [B, \chi_{12}] \, \psi_n)| \;\le\; |(\frac{1}{p_1|\mathbf{x}_1 - \mathbf{x}_2|} B\,\varphi, p_1 \, [B_1, \chi_{12,0}] \, B_2 \, \psi_n)|$$

$$+ \, |(\frac{1}{p_2|\mathbf{x}_1 - \mathbf{x}_2|} B\,\varphi, B_1 \, p_2 \, [B_2, \chi_{12,0}] \, \psi_n)| \tag{5.9}$$

$$\le \frac{c}{n} \, \|\frac{1}{p_1|\mathbf{x}_1 - \mathbf{x}_2|} B\varphi\| \, \|B_2\psi_n\| \;+\; \frac{c}{n} \, \|B_1 \frac{1}{p_2|\mathbf{x}_1 - \mathbf{x}_2|} B\varphi\| \, \|\psi_n\| \;\le\; \frac{\tilde{c}}{n} \, \|\varphi\| \, \|\psi_n\|$$

which completes the proof of Lemma 6.

6. Proof of Lemma 3

For $j = 0, 1$ and $\varphi \in \mathcal{A}(C_0^\infty(\mathbb{R}^6 \backslash B_R(0)) \otimes \mathbb{C}^4)$ we have to show $|(\phi_j\varphi, r_j \, \phi_j\varphi)| \le \frac{c}{R} \, \|\varphi\|^2$. (The proof for $j = 2$ follows from the symmetry upon electron exchange.) The same strategy is used as in the previous proofs.

a) $j = 1: \quad r_1 \,=\, b_{1m}^{(1)} \,+\, v^{(12)}$

For the definition of the auxiliary function χ we recall that $\operatorname{supp} \phi_1\varphi \subset \mathbb{R}^6 \backslash B_R(0) \cap \{\mathbf{x} \in \mathbb{R}^6 : x_1 \ge Cx\}$. Thus $x = \sqrt{x_1^2 + x_2^2} \ge R$ and $x_1 \ge CR$. For the estimate of $b_{1m}^{(1)}$, we take

$$\chi_1(\frac{\mathbf{x}_1}{R}) := \begin{cases} 0, & x_1 < CR/2 \\ 1, & x_1 \ge CR \end{cases} \tag{6.1}$$

such that $\phi_1\varphi\chi_1 = \phi_1\varphi$, and we introduce $\chi_{1,0} := 1 - \chi_1 \in \mathcal{S}(\mathbb{R}^3)$ as before. Although we are dealing here with two-particle functions, all operators ($b_{1m}^{(1)}$ and

χ_1) act only on particle 1. This reduces the Lieb and Yau formula to the single-particle case,

$$|(\psi, A\,\varphi)| \;=\; |\int_{\mathbb{R}^6} d\mathbf{p}_1 d\mathbf{p}_2\, \overline{\hat{\psi}(\mathbf{p}_1,\mathbf{p}_2)} \int_{\mathbb{R}^3} d\mathbf{p}_1'\; k_A(\mathbf{p}_1,\mathbf{p}_1')\,\hat{\varphi}(\mathbf{p}_1',\mathbf{p}_2)|$$

$$\leq\; \left(\int_{\mathbb{R}^6} d\mathbf{p}_1 d\mathbf{p}_2\, |\hat{\psi}(\mathbf{p}_1,\mathbf{p}_2)|^2\, I(\mathbf{p}_1)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^6} d\mathbf{p}_1' d\mathbf{p}_2\, |\hat{\varphi}(\mathbf{p}_1',\mathbf{p}_2)|^2\, J(\mathbf{p}_1')\right)^{\frac{1}{2}} \tag{6.2}$$

where I and J are given in (3.2) with $\boldsymbol{\xi} := \mathbf{p}_1$. Therefore the proof of Lemma 5 can be copied to obtain

$$|(\phi_1\,\varphi, b_{1m}^{(1)}\,\phi_1\,\varphi)| \;\leq\; \frac{\tilde{c}}{R}\,\|\phi_1\,\varphi\|^2 \;\leq\; \frac{\tilde{c}}{R}\,\|\varphi\|^2. \tag{6.3}$$

For the estimate of $v^{(12)}$, we define χ in analogy to (5.1) by noting that additionally, $\operatorname{supp}\phi_1\varphi \subset \mathbb{R}^6\backslash B_R(0) \cap \{\mathbf{x}\in\mathbb{R}^6 : |\mathbf{x}_1-\mathbf{x}_2| \geq Cx\}$. Therefore,

$$\chi_{12}(\frac{\mathbf{x}_1-\mathbf{x}_2}{R}) := \begin{cases} 0, & |\mathbf{x}_1-\mathbf{x}_2| < CR/2 \\ 1, & |\mathbf{x}_1-\mathbf{x}_2| \geq CR \end{cases} \tag{6.4}$$

such that again, $\phi_1\varphi\chi_{12} = \phi_1\varphi$ and $\chi_{12,0} := 1-\chi_{12} \in \mathcal{S}(\mathbb{R}^3)$. This enables us to adopt the proof of Lemma 6 (with n replaced by R) to obtain

$$|(\phi_1\,\varphi, v^{(12)}\,\phi_1\,\varphi)| \;\leq\; \frac{c}{R}\,\|\phi_1\,\varphi\|^2 \;\leq\; \frac{c}{R}\,\|\varphi\|^2. \tag{6.5}$$

b) $j=0$*:* $\quad r_0 \;=\; b_{1m}^{(1)} \,+\, b_{1m}^{(2)}$

In this case, the support of $\phi_j\varphi$ obeys $\operatorname{supp}\phi_0\varphi \subset \mathbb{R}^6\backslash B_R(0) \cap \{\mathbf{x}\in\mathbb{R}^6 : x_1 \geq Cx$ and $x_2 \geq Cx\}$. For the discussion of $b_{1m}^{(1)}$, we define $\chi_1(\mathbf{x}_1/R)$ as in (6.1) and copy the corresponding proof from *a*). For $b_{1m}^{(2)}$, we choose $\chi_2(\mathbf{x}_2/R)$ according to (6.1) with x_1 replaced by x_2. The proof is done along the same lines as for $b_{1m}^{(1)}$. Hence we obtain

$$\begin{aligned} |(\phi_0\,\varphi, r_0\,\phi_0\,\varphi)| \;&\leq\; |(\phi_0\,\varphi, b_{1m}^{(1)}\,\chi_1\,\phi_0\,\varphi)| \,+\, |(\phi_0\,\varphi, b_{1m}^{(2)}\,\chi_2\,\phi_0\,\varphi)| \\ &\leq\; \frac{2c_1}{R}\,\|\phi_0\,\varphi\|^2 \;\leq\; \frac{2c_1}{R}\,\|\varphi\|^2. \end{aligned} \tag{6.6}$$

7. Proof of Lemma 1

Assume we have a normalized sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ localized outside $B_n(0)$ with the property (2.4) for h^{BR} and the $\lambda\in\mathbb{R}$ under consideration. Since the normalization constant of φ_n tends to zero as $n\to\infty$, for any $\phi\in C_0^\infty(\mathbb{R}^6)$ we have $|(\phi,\varphi_n)| \to 0$ as $n\to\infty$, i.e. $\varphi_n \stackrel{w}{\to} 0$. By the Weyl criterion it follows that $\lambda\in\sigma_{ess}(h^{BR})$.

Conversely, let $\lambda\in\sigma_{ess}(h^{BR})$. Then there exists a Weyl sequence $\psi_n \in \mathcal{A}(C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^2)^2$, $\|\psi_n\| = 1$, with $\psi_n \stackrel{w}{\to} 0$ and $\|(h^{BR}-\lambda)\,\psi_n\| \to 0$ as $n\to\infty$.

Define a smooth symmetric (with respect to interchange of $\mathbf{x}_1$ and $\mathbf{x}_2$) function $\chi_0 \in C_0^\infty(\mathbb{R}^6)$ mapping to $[0,1]$ by means of

$$\chi_0(\frac{\mathbf{x}}{n}) \;=\; \begin{cases} 1, & x \leq n \\ 0, & x > 2n \end{cases} \tag{7.1}$$

and let $\chi_n(\mathbf{x}) := 1 - \chi_0(\frac{\mathbf{x}}{n}), \quad \chi_n \in C^\infty(\mathbb{R}^6 \backslash B_n(0))$. Then $\varphi_n := \psi_n \chi_n \in \mathcal{A}(C_0^\infty(\mathbb{R}^6 \backslash B_n(0)) \otimes \mathbb{C}^4)$ and we claim that a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ satisfies the requirements of Lemma 1.

a) $\|(h^{BR} - \lambda)\, \varphi_n\| \to 0$ as $n \to \infty$:

We decompose

$$\|(h^{BR} - \lambda)\, \chi_n \psi_n\| \;\leq\; \|\chi_n\, (h^{BR} - \lambda)\, \psi_n\| \;+\; \|\, [h^{BR}, \chi_0]\, \psi_n\|, \tag{7.2}$$

and use the equivalence (2.20) again. Concerning the first term in (7.2), we have for any $\phi \in C_0^\infty(\mathbb{R}^6) \otimes \mathbb{C}^4$,

$$|(\phi, \chi_n\, (h^{BR} - \lambda)\, \psi_n)| \;\leq\; \|\chi_n\, \phi\|\, \|(h^{BR} - \lambda)\, \psi_n\| \;\longrightarrow\; 0 \qquad \text{as } n \to \infty \tag{7.3}$$

by assumption, since $\|\chi_n \phi\| \;\leq \|\phi\| < \infty$.

In order to treat the single-particle contribution to the second term in (7.2), $T^{(k)}$ and $b_{1m}^{(k)}$, we change again to Fourier space and introduce the 6-dimensional Fourier transform of χ_0 according to (3.6) with $l = 2$. Then the kernel of $[T^{(1)}, \chi_0]$ in momentum space reads

$$k_{[T^{(1)}, \chi_0]}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2') \;=\; \frac{n^6}{(2\pi)^3}\, \hat{\chi}_0((\mathbf{p}_1 - \mathbf{p}_1')n, (\mathbf{p}_2 - \mathbf{p}_2')n)\; (E_{p_1} - E_{p_1'}) \tag{7.4}$$

and by the mean value theorem (4.1), using $\nabla_{\mathbf{p}} E_p \;=\; \nabla_{\mathbf{p}} \sqrt{p^2 + m^2} \;=\; \mathbf{p}/E_p$,

$$|E_{p_1} - E_{p_1'}| \;=\; |\mathbf{p}_1 - \mathbf{p}_1'| \left| \frac{\boldsymbol{\xi}}{E_\xi} \right| \;\leq\; |\mathbf{p}_1 - \mathbf{p}_1'| \tag{7.5}$$

for all $\boldsymbol{\xi} := \lambda \mathbf{p}_1 + (1 - \lambda)\mathbf{p}_1'$ with $\lambda \in [0,1]$. For the integral I appearing in the Lieb and Yau formula (3.1) we have with $f = 1$ and the substitution $\mathbf{y}_k := (\mathbf{p}_k - \mathbf{p}_k')n, \;\; k = 1, 2$,

$$\begin{aligned} I(\mathbf{p}_1, \mathbf{p}_2) := & \int_{\mathbb{R}^6} d\mathbf{p}_1'\, d\mathbf{p}_2'\; \frac{n^6}{(2\pi)^3}\; |\hat{\chi}_0((\mathbf{p}_1 - \mathbf{p}_1')n, (\mathbf{p}_2 - \mathbf{p}_2')n)|\; |E_{p_1} - E_{p_1'}| \\ \leq & \;\frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} d\mathbf{y}_1\, d\mathbf{y}_2\; |\hat{\chi}_0(\mathbf{y}_1, \mathbf{y}_2)|\; y_1 \cdot \frac{1}{n} \;\leq\; \frac{c}{n} \end{aligned} \tag{7.6}$$

since $\hat{\chi}_0 \in \mathcal{S}(\mathbb{R}^6)$. In a similar way, $J(\mathbf{p}_1', \mathbf{p}_2') \leq c/n$, and hence

$$|(\phi, [T^{(1)}, \chi_0]\, \psi_n)| \;\leq\; \frac{c}{n}\; \|\phi\|\; \|\psi_n\| \tag{7.7}$$

for all $\phi \in C_0^\infty(\mathbb{R}^6) \otimes \mathbb{C}^4$. For the operator $b_{1m}^{(1)}$ we can proceed as in the proof of Lemma 5, because (according to (7.6)) the two-particle nature of χ_0 does not affect the convergence of the single-particle integrals.

For the estimate of the remaining commutator, $[v^{(12)}, \chi_0]$, we follow section 5 to split it into commutators of χ_0 with single-particle (bounded) operators $B_1(\mathbf{p}_1)$, $B_2(\mathbf{p}_2)$. The only difference as compared to the proof of Lemma 6 lies in the two-particle nature of χ_0 (cf. (7.4) in place of (5.7)), but again this does not affect the convergence of the integrals. Thus we get

$$|(\phi, [h^{BR}, \chi_0]\, \psi_n)| \;\le\; \sum_{k=1}^{2} \Big(|(\phi, [T^{(k)}, \chi_0]\, \psi_n)| \;+\; |(\phi, [b_{1m}^{(k)}, \chi_0]\, \psi_n)| \Big)$$

$$+\, |(\phi, [v^{(12)}, \chi_0]\, \psi_n)| \;\le\; \frac{c}{n}\, \|\phi\|\, \|\psi_n\| \tag{7.8}$$

with the generic constant c.

b) $\quad \|\varphi_n\| \neq 0$:

We show that for any $\epsilon > 0$ there is an $N_0 \in \mathbb{N}$ such that $\|\chi_0 \psi_n\| < \epsilon$ for all $n \ge N_0$. Then $\|(1-\chi_n)\, \psi_n\| = \|\psi_n - \varphi_n\| < \epsilon$. As a consequence, $\|\varphi_n\| \neq 0$ for sufficiently large $n > n_0 \ge N_0$ since $\|\psi_n\| = 1$ (i.e. one can choose a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ with normalizable elements).

Since h^{BR} is a self-adjoint positive operator, $h^{BR} + 1 > 0$ with a bounded inverse. Then following [4] we estimate

$$\|\chi_0\, \psi_n\| \;=\; \|\chi_0\, (h^{BR}+1)^{-1}\, [(h^{BR} - \lambda) \;+\; (1+\lambda)]\, \psi_n\|$$

$$\le\; \|\chi_0\, (h^{BR}+1)^{-1}\|\; \|(h^{BR}-\lambda)\, \psi_n\| \;+\; |1+\lambda|\; \|\chi_0\, (h^{BR}+1)^{-1}\, \psi_n\|. \tag{7.9}$$

Since $\chi_0 \in C_0^\infty(\mathbb{R}^6)$ and by assumption $\exists\, N_1 :\; \|(h^{BR}-\lambda)\, \psi_n\| < \tilde{\epsilon}$ for $n \ge N_1$, the first term is bounded by, say, $c_1 \tilde{\epsilon}$ for $n \ge N_1$.

For the second term we show that the bounded function χ_0 is relatively compact with respect to h^{BR}. We write

$$\chi_0\, (h^{BR}+1)^{-1} \;=\; \chi_0\; (T+1)^{-\frac{1}{2}}\; (T+1)^{\frac{1}{2}}\; (h^{BR}+1)^{-1} \tag{7.10}$$

and note that $(T+1)^{\frac{1}{2}}\, (h^{BR}+1)^{-\frac{1}{2}} \cdot (h^{BR}+1)^{-\frac{1}{2}}$ is a product of bounded operators since V is T-from bounded (with form bound < 1 for $\gamma < \gamma_{BR}$).

For the compactness of $\chi_0\, (T+1)^{-\frac{1}{2}}$ we apply a theorem ([22, p.115], [21, Lemma 6.9] stating that for bounded functions $f(x)$ and $g(p)$ with $f, g : [0,\infty) \to \mathbb{C}$ and $\lim\limits_{x\to\infty} f(x) = 0 = \lim\limits_{p\to\infty} g(p)$ the product $K := gf$ is compact.

Clearly, both χ_0 and $(T+1)^{-\frac{1}{2}} = (\sqrt{p_1^2+m^2} + \sqrt{p_2^2+m^2} + 1)^{-\frac{1}{2}}$ are bounded functions, going to zero as $x \to \infty$ and $p := \sqrt{p_1^2+p_2^2} \to \infty$, respectively. Therefore, $\chi_0\, (h^{BR}+1)^{-1}$ is compact and maps the weakly convergent sequence $(\psi_n)_{n\in\mathbb{N}}$ into a strongly convergent sequence. So the second term in (7.9) can be estimated by $c_2 \tilde{\epsilon}$ for, say, $n \ge N_2$. This proves the assertion $\|\chi_0 \psi_n\| < \epsilon := (c_1+c_2)\tilde{\epsilon}$ for $n \ge N_0 := \max\{N_1, N_2\}$.

8. Proof of Lemma 4

We have to show that for $\varphi \in \mathcal{A}(C_0^\infty(\mathbb{R}^6 \backslash B_R(0)) \otimes \mathbb{C}^4)$ and $\chi(\frac{\mathbf{x}}{R})$ the auxiliary function from (3.4) with $l=2$, $(\phi_j\varphi, [b_{1m}^{(1)}, \phi_j\chi]\,\varphi)$ as well as $(\phi_j\varphi, [v^{(12)}, \phi_j\chi]\,\varphi)$ are uniformly $1/R$-bounded.

Let us denote $\phi_j\chi =: \psi_j$. We have $\operatorname{supp}\phi_j\chi \subseteq \operatorname{supp}\chi$ which is located outside $B_{R/2}(0)$ such that the scaling holds,

$$\psi_j(\mathbf{x}) \;=\; \phi_j(\mathbf{x})\,\chi(\frac{\mathbf{x}}{R}) \;=\; \phi_j(\frac{\mathbf{x}}{R/2})\,\chi(\frac{\mathbf{x}}{R}) \tag{8.1}$$

for $R \geq 2$. Since ϕ_j and χ are analytic functions in $\operatorname{supp}\chi$ we can apply the mean value theorem, with $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$,

$$|\psi_j(\mathbf{x}_1,\mathbf{x}_2) \,-\, \psi_j(\mathbf{x}_1', \mathbf{x}_2)| \;=\; |\mathbf{x}_1 - \mathbf{x}_1'|\; |(\nabla_{\mathbf{x}_1}\psi_j)(\boldsymbol{\xi}, \mathbf{x}_2)| \tag{8.2}$$

with $\boldsymbol{\xi}$ some value on the line between $\mathbf{x}_1$ and $\mathbf{x}_1'$. Since from (3.5), $\chi' = -\chi_0' \in \mathcal{S}(\mathbb{R}^6)$ is a bounded function, as is ϕ_j' (because $\phi_j \in C^\infty(\mathbb{R}^6)$ is defined on the compact unit sphere and is homogeneous of degree zero outside the unit ball), we can estimate

$$|\left(\nabla_{\mathbf{x}_1}\,\phi_j(\frac{\mathbf{x}}{R/2})\,\chi(\frac{\mathbf{x}}{R})\right)(\boldsymbol{\xi},\mathbf{x}_2)| \;=\; |\phi_j|\,\frac{1}{R}\,|\chi'(\frac{\boldsymbol{\xi}}{R}, \frac{\mathbf{x}_2}{R})| \,+\, |\chi|\,\frac{1}{R/2}\,|\phi_j'(\boldsymbol{\xi}, \mathbf{x}_2)| \;\leq\; \frac{c_0}{R} \tag{8.3}$$

(the prime refering to the derivative with respect to the first entry). c_0 is a suitable constant which can be chosen independently of j.

a) Using the explicit form (1.5) of $b_{1m}^{(1)}$, we decompose the commutator in the following way,

$$[b_{1m}^{(1)}, \psi_j] \;=\; -\gamma\left\{[A,\psi_j]\,\frac{1}{x_1}\,A \;+\; [\boldsymbol{\sigma}^{(1)}\mathbf{p}_1\,g, \psi_j]\,\frac{1}{x_1}\,g\,\boldsymbol{\sigma}^{(1)}\mathbf{p}_1 \right.$$

$$\left. +\;\; A\,\frac{1}{x_1}\,[A,\psi_j] \;+\; \boldsymbol{\sigma}^{(1)}\mathbf{p}_1\,g\,\frac{1}{x_1}\,[g\,\sigma^{(1)}\mathbf{p}_1, \psi_j]\right\}. \tag{8.4}$$

Since the two terms in the second line of (8.4) are (up to a sign) the hermitean conjugate of the first line, and the interchange of $\phi_j\varphi$ with φ in the quadratic form plays no role, we need not discuss these terms separately.

The terms in the first line of (8.4) have the structure $[\mathcal{O},\psi_j]\,\frac{1}{x_1}\,B$ with $\mathcal{O}$ and B multiplication operators in momentum space and B bounded. Provided we can show

$$|(\varphi, [\mathcal{O},\psi_j]\,\frac{1}{x_1}\,\psi)| \;\leq\; \frac{c}{R}\,\|\varphi\|\,\|\psi\|, \tag{8.5}$$

we can estimate $|(\phi_j\varphi, [\mathcal{O},\psi_j]\,\frac{1}{x_1}\,B\,\varphi)| \;\leq\; \frac{c}{R}\,\|\phi_j\varphi\|\,\|B\varphi\| \;\leq\; \frac{c_1}{R}\,\|\varphi\|^2$ and we are done. The proof of (8.5) is based on the following estimate of the kernel $\check{k}_{\mathcal{O}}$ of $\mathcal{O}$ in coordinate space,

$$|\check{k}_{\mathcal{O}}(\mathbf{x}_1, \mathbf{x}_1')| \;\leq\; \frac{c}{|\mathbf{x}_1 - \mathbf{x}_1'|^3}, \tag{8.6}$$

for our two operators of interest, A and $\boldsymbol{\sigma}^{(1)}\mathbf{p}_1 g$. Assuming (8.6) to hold, we apply the coordinate-space version of the Lieb and Yau formula (3.1) to the l.h.s. of (8.5), identifying $\boldsymbol{\xi}$ with $(\mathbf{x}_1, \mathbf{x}_2)$. Then we have to estimate the two integrals from (3.2), I and J. For I we have

$$I(\mathbf{x}_1,\mathbf{x}_2) := \int_{\mathbb{R}^6} d\mathbf{x}_1' d\mathbf{x}_2' \; |\check{k}_{[\mathcal{O},\psi_j]\frac{1}{x_1}}(\mathbf{x}_1,\mathbf{x}_1')| \; \delta(\mathbf{x}_2-\mathbf{x}_2') \; \frac{f(\mathbf{x}_1,\mathbf{x}_2)}{f(\mathbf{x}_1',\mathbf{x}_2')} \tag{8.7}$$

with the convergence generating function chosen to be $f(\mathbf{x}_1,\mathbf{x}_2) = x_1^{3/2}$. The delta function appears because the momentum operator $\mathcal{O}$ does not affect particle 2. With the help of (8.6) and the mean value theorem (8.2) and (8.3) we obtain

$$\begin{aligned} I(\mathbf{x}_1,\mathbf{x}_2) \; &= \; \int_{\mathbb{R}^3} d\mathbf{x}_1' \; |\check{k}_{\mathcal{O}}(\mathbf{x}_1,\mathbf{x}_1')| \; |\psi_j(\mathbf{x}_1,\mathbf{x}_2) - \psi_j(\mathbf{x}_1',\mathbf{x}_2)| \; \frac{1}{x_1'} \cdot \frac{x_1^{3/2}}{x_1'^{3/2}} \\ &\leq \; \frac{c_0 c}{R} \int_{\mathbb{R}^3} d\mathbf{x}_1' \; \frac{1}{|\mathbf{x}_1-\mathbf{x}_1'|^3} \; |\mathbf{x}_1-\mathbf{x}_1'| \; \frac{x_1^{3/2}}{x_1'^{5/2}}. \end{aligned} \tag{8.8}$$

Using spherical coordinates, the angular integration is performed by means of $\int_{S^2} d\omega' \, \frac{1}{|\mathbf{x}_1-\mathbf{x}_1'|^2} \; = \; \frac{2\pi}{x_1 x_1'} \ln\left|\frac{x_1+x_1'}{x_1-x_1'}\right|$ [8]. With the substitution $x_1' =: x_1 z'$ we get for the r.h.s. of (8.8),

$$I(\mathbf{x}_1,\mathbf{x}_2) \; \leq \; \frac{2\pi c_0 c}{R} \int_0^\infty \frac{dz'}{z'^{3/2}} \; \ln\left|\frac{1+z'}{1-z'}\right| \; \leq \; \frac{C}{R} \tag{8.9}$$

since the integral is convergent. For the integral J we use the identical estimates. Then with $x_1 := x_1' z$,

$$\begin{aligned} J(\mathbf{x}_1',\mathbf{x}_2') &:= \int_{\mathbb{R}^6} d\mathbf{x}_1 d\mathbf{x}_2 \; |\check{k}_{[\mathcal{O},\psi_j]\frac{1}{x_1}}(\mathbf{x}_1,\mathbf{x}_1')| \; \delta(\mathbf{x}_2-\mathbf{x}_2') \; \frac{x_1'^{3/2}}{x_1^{3/2}} \\ &\leq \; \frac{2\pi c_0 c}{R} \int_0^\infty \frac{dz}{z^{1/2}} \; \ln\left|\frac{1+z}{1-z}\right| \; \leq \; \frac{\tilde{C}}{R}. \end{aligned} \tag{8.10}$$

From (8.9) and (8.10), the Lieb and Yau formula together with (3.3) provides the desired result (8.5).

It remains to show the estimate (8.6) for the kernel of our operators. Let us first consider the limiting case $m = 0$ as this is very simple.

For $m = 0$, $\boldsymbol{\sigma}^{(1)}\mathbf{p}_1 g = \frac{1}{\sqrt{2}} \boldsymbol{\sigma}^{(1)} \frac{\mathbf{p}_1}{p_1}$, while A is a constant which need not be considered. From (3.7), using the Fourier transform of the Coulomb potential,

$$\begin{aligned} \check{k}_{\boldsymbol{\sigma}^{(1)}\mathbf{p}_1 g}(\mathbf{x}_1,\mathbf{x}_1') \; &= \; -i\boldsymbol{\sigma}^{(1)} \nabla_{\mathbf{x}_1} \; \frac{1}{\sqrt{2}} \, \frac{4\pi}{(2\pi)^3} \; \frac{1}{|\mathbf{x}_1-\mathbf{x}_1'|^2} \\ &= \; \frac{i\boldsymbol{\sigma}^{(1)}}{\sqrt{2}\pi^2} \; \frac{\mathbf{x}_1-\mathbf{x}_1'}{|\mathbf{x}_1-\mathbf{x}_1'|^4}, \end{aligned} \tag{8.11}$$

such that the desired estimate (8.6) follows immediately.

For $m \neq 0$, we have to consider the two functions $g(p_1)$ and $A(p_1)$ defined in (1.3). Without loss of generality we can set $m = 1$ (otherwise, due to scaling, one would have to consider $m\tilde{x}$ in place of $\tilde{x}$ in the integrand of (3.7)). Then

$$g(p_1) \;=\; \frac{1}{\sqrt{2}} \left(\frac{1}{p_1^2 + 1 + \sqrt{p_1^2+1}} \right)^{\frac{1}{2}}$$

$$A(p_1) \;=\; \frac{1}{\sqrt{2}} \left(1 \,+\, \frac{1}{\sqrt{p_1^2+1}} \right)^{\frac{1}{2}} \;=\; \frac{1}{\sqrt{2}} \,+\, \tilde{g}(p_1), \tag{8.12}$$

$$\tilde{g}(p_1) := \; \frac{1}{\sqrt{2}} \, \frac{1}{\sqrt{p_1^2+1} + \sqrt{p_1^2+1+\sqrt{p_1^2+1}}}$$

where $\tilde{g}(p_1) \sim 2^{-\frac{3}{2}}/p_1$ for $p_1 \to \infty$.

Both functions, if extended to the complex plane, have branch points at $p_1 = \pm i$ and are analytic in the strip $\{z \in \mathbb{C} : \, |\mathrm{Im}\; z| \, < 1\}$ if the cuts are chosen from i to ∞ and from $-\infty$ to $-i$, respectively. According to a corollary of the Paley-Wiener theorem [17, Thm IX.14], for $\tilde{x} := |\mathbf{x}_1' - \mathbf{x}_1| \, > 0$, it follows that $\check{k}_{\tilde{g}}$ and $\check{k}_g$ (if defined with convergence generating factors) are bounded continuous functions decaying exponentially at $\tilde{x} \to \infty$, i.e. for any $b < 1$ and any $\delta > 0$, there exists a constant $c_1 > 0$:

$$|\check{k}_{\tilde{g}}(\tilde{x})| \, \le c_1 e^{-b\tilde{x}} \qquad \text{for } \tilde{x} \ge \delta, \tag{8.13}$$

and similarly for $\check{k}_g$. Here and in the following we have to introduce the convergence generating factors by means of replacing for $\epsilon > 0$ the integral in (3.7) by

$$\lim_{\epsilon \to 0} \; \frac{1}{\tilde{x}} \int_0^\infty p_1 \, dp_1 \;\; \sin p_1 \tilde{x} \;\, \tilde{g}(p_1) \; e^{-\epsilon p_1}. \tag{8.14}$$

In determining the behaviour of $\check{k}_{\tilde{g}}$ near $\tilde{x} = 0$, we apply two partial integrations to (8.14) and obtain

$$\frac{1}{\tilde{x}} \left| \int_0^\infty dp_1 \;\, \sin p_1 \tilde{x} \;\, \left(p_1 \tilde{g}\, e^{-\epsilon p_1} \right) \right| \;=\; \frac{1}{\tilde{x}^3} \left| \int_0^\infty dp_1 \;\, \sin p_1 \tilde{x} \;\, \left\{ \left(\frac{d^2}{dp_1^2} \, p_1 \tilde{g} \right) \; e^{-\epsilon p_1} \right.\right.$$

$$\left.\left. -\, 2 \; \frac{1}{p_1} \; \left(\frac{d}{dp_1} \, p_1 \tilde{g} \right) \cdot [\epsilon p_1 \, e^{-\epsilon p_1}] \;+\; \frac{\tilde{g}}{p_1} \; [(\epsilon p_1)^2 \; e^{-\epsilon p_1}] \right\} \right| \;\le\; \frac{c}{\tilde{x}^3} \tag{8.15}$$

because the boundary terms vanish, $|\sin p_1 \tilde{x}| \le 1$ and the term in curly brackets is bounded by $\frac{c}{(p_1+1)^2}$ for all ϵ, which is an integrable function.

The continuity of $\check{k}_{\tilde{g}}$ at $\tilde{x} > 0$ together with (8.13) thus ensures the estimate $|\check{k}_{\tilde{g}}(\tilde{x})| \, \le \, \frac{c}{\tilde{x}^3}$.

For the second kernel, $\check{k}_{\boldsymbol{\sigma}^{(1)}\mathbf{p}_1 g}(\tilde{x})$, we need more careful estimates due to the presence of the derivative $\boldsymbol{\sigma}^{(1)} \nabla_{\tilde{\mathbf{x}}}$ in (3.7). First we prove analyticity of the kernel

$\check{k}_g$. We use (8.14) with $\tilde{g}$ replaced by g. Writing $\sin p_1\tilde{x} = \frac{1}{2i}(e^{ip_1\tilde{x}} - e^{-ip_1\tilde{x}})$, substituting $p_1 = iy$ and using symmetry, (8.14) turns into

$$\lim_{\epsilon\to 0} \frac{1}{2i\tilde{x}} \left(\int_{-i\infty}^{0} \tilde{f}(y)\, e^{-i\epsilon y}\, dy \;+\; \int_{0}^{i\infty} \tilde{f}(y)\, e^{i\epsilon y}\, dy \right)$$

$$\tilde{f}(y) := \; y\, e^{-\tilde{x}y}\, \frac{1}{\sqrt{2}} \left(\frac{1}{1 - y^2 + \sqrt{1-y^2}} \right)^{\frac{1}{2}} \tag{8.16}$$

Using analyticity of the integrands outside the cuts as well as their exponential decay as $\mathrm{Re}(y) \to \infty$, we can use Cauchy's integral theorem to deform the integration paths to the real axis. The resulting integrals are finite for all $\tilde{x} > 0$ and $\epsilon = 0$ since the singularity at $y = 1$ is integrable. Thus (8.16) gives

$$\frac{1}{\tilde{x}\sqrt{2}} \int_1^\infty ydy\; e^{-\tilde{x}y}\; \mathrm{Re} \left(\frac{1}{y^2 - 1 + i\sqrt{y^2-1}} \right)^{\frac{1}{2}} \tag{8.17}$$

with positive choice of the real part. Its derivative with respect to $\tilde{x}$ also converges absolutely and hence (8.17) represents an analytic function for $\tilde{x} > 0$.

For the large-$\tilde{x}$ behaviour we note that an expansion around the branch point $p_1 = i$ provides, as outlined by Murray [15],

$$\frac{1}{\tilde{x}} \int_0^\infty p_1 dp_1\; \sin p_1\tilde{x}\; g(p_1)\, e^{-\epsilon p_1^4} \;\sim\; z_0\, \frac{e^{-\tilde{x}}}{\tilde{x}^{7/4}}\, e^{-\epsilon} \qquad \text{as } \tilde{x} \to \infty \tag{8.18}$$

with a constant z_0.

For the estimate of $\check{k}_g$ near $\tilde{x} = 0$, we separate the 'Coulombic' tail from g, $g(p) = \frac{1}{\sqrt{2}p} + (g(p) - \frac{1}{\sqrt{2}p})$ and obtain with (8.11) and (3.7)

$$\check{k}_g(\tilde{x}) \;=\; \frac{1}{2\sqrt{2}\pi^2} \left(\frac{1}{\tilde{x}^2} \;-\; \lim_{\epsilon\to 0} I_\epsilon(\tilde{x}) \right) \tag{8.19}$$

$$I_\epsilon(\tilde{x}) := \frac{1}{\tilde{x}} \int_0^\infty dp_1\; \sin p_1\tilde{x}\; \frac{(1 + \sqrt{p_1^2+1})^{\frac{1}{2}}}{(p_1^2+1)^{\frac{1}{4}}\,[p_1 + (p_1^2 + 1 + \sqrt{p_1^2+1}\,)^{\frac{1}{2}}]}\; e^{-\epsilon p_1}.$$

The fraction multiplying $\sin p_1\tilde{x}$ decreases according to $1/p_1$ for $p_1 \to \infty$. With one partial integration, we get

$$I_\epsilon(\tilde{x}) \;=\; \frac{1}{\tilde{x}^2} + \frac{1}{\tilde{x}^2} \int_0^\infty dp_1\; \cos p_1\tilde{x} \left\{ (\frac{d}{dp_1}[\cdots])\, e^{-\epsilon p_1} \;-\; \frac{[\cdots]}{p_1}\, (\epsilon p_1\, e^{-\epsilon p_1}) \right\} \tag{8.20}$$

where the first term comes from the boundary at $p_1 = 0$ and $[\cdots]$ denotes the fraction in (8.19). The integral is finite (and independent of $\tilde{x}$ for $\tilde{x} \to 0$) because the curly bracket can be estimated by $\frac{c_0}{(1+p_1)^2}$ (independent of ϵ) which is integrable. Hence, $\check{k}_g(\tilde{x}) \sim \frac{\tilde{c}}{\tilde{x}^2}$ for $\tilde{x} \to 0$ with some constant $\tilde{c}$. Performing the derivative inherent in $\boldsymbol{\sigma}^{(1)}\mathbf{p}_1$, we find $\check{k}_{\boldsymbol{\sigma}^{(1)}\mathbf{p}_1 g}(\tilde{x}) \sim 2i\tilde{c}\,\frac{\boldsymbol{\sigma}^{(1)}\tilde{\mathbf{x}}}{\tilde{x}^4}$ $(\tilde{x} \to 0)$ and

$\sim iz_0 \frac{\boldsymbol{\sigma}^{(1)}\tilde{\mathbf{x}}}{\tilde{x}^4} (\tilde{x}^{5/4} + \frac{7}{4}\tilde{x}^{1/4}) e^{-\tilde{x}}$ for $\tilde{x} \to \infty$. From the analyticity of $\check{k}_g$ we finally obtain

$$|\check{k}_{\boldsymbol{\sigma}^{(1)}\mathbf{p}_1 g}(\tilde{x})| \leq \frac{c}{\tilde{x}^3} \quad \text{for all } \tilde{x} > 0. \tag{8.21}$$

b) Let us now turn to the commutator with the two-body interaction $v^{(12)}$. According to the explicit expression (5.2) for $v^{(12)}$, we split $[v^{(12)}, \psi_j] =: e^2 \sum_{k=1}^4 M_k$ into four contributions and write them in the following way (with the short-hand notation $A_k := A(p_k), \;\; g_k := g(p_k), \;\; k = 1, 2$).

$$\begin{aligned}
M_1 :=\; & [A_1, \psi_j] \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} \cdot |\mathbf{x}_1 - \mathbf{x}_2| \, A_2 \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} A_1 A_2 \\
& + A_1 \, [A_2, \psi_j] \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} A_1 A_2 \; - \; h.c. \\
M_2 :=\; & [A_1, \psi_j] \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} \cdot |\mathbf{x}_1 - \mathbf{x}_2| \, g_2 \boldsymbol{\sigma}^{(2)}\mathbf{p}_2 \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} A_1 g_2 \boldsymbol{\sigma}^{(2)}\mathbf{p}_2 \\
& + A_1 \, [g_2 \boldsymbol{\sigma}^{(2)}\mathbf{p}_2, \psi_j] \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} A_1 g_2 \boldsymbol{\sigma}^{(2)}\mathbf{p}_2 \; - \; h.c. \\
M_4 :=\; & [g_1 \boldsymbol{\sigma}^{(1)}\mathbf{p}_1, \psi_j] \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} \cdot |\mathbf{x}_1 - \mathbf{x}_2| \, g_2 \boldsymbol{\sigma}^{(2)}\mathbf{p}_2 \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} g_1 \boldsymbol{\sigma}^{(1)}\mathbf{p}_1 g_2 \boldsymbol{\sigma}^{(2)}\mathbf{p}_2 \\
& + g_1 \boldsymbol{\sigma}^{(1)}\mathbf{p}_1 \, [g_2 \boldsymbol{\sigma}^{(2)}\mathbf{p}_2, \psi_j] \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} g_1 \boldsymbol{\sigma}^{(1)}\mathbf{p}_1 g_2 \boldsymbol{\sigma}^{(2)}\mathbf{p}_2 \; - \; h.c.
\end{aligned} \tag{8.22}$$

where $h.c.$ denotes the hermitean conjugate, and M_3 results from M_2 upon interchanging $\mathbf{p}_1, \boldsymbol{\sigma}^{(1)}$ with $\mathbf{p}_2, \boldsymbol{\sigma}^{(2)}$ and therefore need not be considered separately.

In addition to estimate the commutators $[\mathcal{O}, \psi_j] \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}$ (with the same operators $\mathcal{O}$ as in a) except for a possible particle exchange), we have to prove boundedness of $|\mathbf{x}_1 - \mathbf{x}_2| \, \mathcal{O} \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}$ which appears as a factor in the first contribution to $M_k, \;\; k = 1, ..., 4$.

For the estimate of the commutators we proceed as in a) except for the choice $f(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_1 - \mathbf{x}_2|^{3/2}$. With the inequality (8.6) for the kernel of $\mathcal{O}$ and the mean value theorem (8.2) and (8.3) we obtain for I, defined as the integral over the kernel of $[\mathcal{O}, \psi_j] \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}$ where $\mathcal{O}$ depends e.g. on $\mathbf{p}_2$,

$$\begin{aligned}
I(\mathbf{x}_1, \mathbf{x}_2) :=\; & \int_{\mathbb{R}^3} d\mathbf{x}_2' \, |\check{k}_{\mathcal{O}}(\mathbf{x}_2, \mathbf{x}_2')| \, |\psi_j(\mathbf{x}_1, \mathbf{x}_2) - \psi_j(\mathbf{x}_1, \mathbf{x}_2')| \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2'|} \frac{f(\mathbf{x}_1, \mathbf{x}_2)}{f(\mathbf{x}_1, \mathbf{x}_2')} \\
\leq\; & \frac{c_0 c}{R} \int_{\mathbb{R}^3} d\mathbf{x}_2' \frac{1}{|\mathbf{x}_2 - \mathbf{x}_2'|^3} |\mathbf{x}_2 - \mathbf{x}_2'| \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2'|} \frac{|\mathbf{x}_1 - \mathbf{x}_2|^{3/2}}{|\mathbf{x}_1 - \mathbf{x}_2'|^{3/2}}.
\end{aligned} \tag{8.23}$$

We abbreviate $\mathbf{x}_0 := \mathbf{x}_2 - \mathbf{x}_1$ and substitute $\mathbf{x}_2' - \mathbf{x}_1 = x_0 \mathbf{y}$. Then the second line in (8.23) is written as ($\mathbf{e}_{x_0} := \mathbf{x}_0 / x_0$)

$$\begin{aligned}
I(\mathbf{x}_1, \mathbf{x}_2) \;\leq\; & \frac{c_0 c}{R} \int_{\mathbb{R}^3} d\mathbf{y} \frac{1}{|\mathbf{e}_{x_0} - \mathbf{y}|^2} \frac{1}{y^{5/2}} \\
=\; & \frac{2\pi c_0 c}{R} \int_0^\infty \frac{dy}{y^{3/2}} \ln \left| \frac{1+y}{1-y} \right| \;\leq\; \frac{C}{R}
\end{aligned} \tag{8.24}$$

with the same estimate as in (8.9) above. Likewise, the estimate (8.10) holds for the second integral J.

Finally we have to show the boundedness of $|\mathbf{x}_1 - \mathbf{x}_2|\,\mathcal{O}\,\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}$. Since commutators help to regularize integrals, we decompose

$$|\mathbf{x}_1 - \mathbf{x}_2|\,\mathcal{O}\,\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|} \;=\; \mathcal{O} \;+\; [\,|\mathbf{x}_1-\mathbf{x}_2|, \mathcal{O}]\,\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}. \tag{8.25}$$

$\mathcal{O}$ is bounded so we can concentrate on the second term. The difference to the commutators occurring in (8.22) is the replacement of ψ_j by $|\mathbf{x}_1 - \mathbf{x}_2|$. From the mean value theorem we get (for $\mathcal{O}$ depending on $\mathbf{p}_2$)

$$|\,|\mathbf{x}_1-\mathbf{x}_2| \;-\; |\mathbf{x}_1-\mathbf{x}_2'|\,| \;=\; \left|(\mathbf{x}_2-\mathbf{x}_2')\,\frac{\boldsymbol{\xi}-\mathbf{x}_1}{|\boldsymbol{\xi}-\mathbf{x}_1|}\right| \;\leq\; |\mathbf{x}_2-\mathbf{x}_2'| \tag{8.26}$$

with $\boldsymbol{\xi}$ some point between $\mathbf{x}_2$ and $\mathbf{x}_2'$. This means that the earlier estimates (8.24) and (8.10) hold except for the factor c_0/R from (8.3) which is not present now. This proves the desired boundedness.

The operators in M_k, $\;k = 1,..,4\;$ from (8.22) which are not yet considered, are all bounded. Applying the Lieb and Yau formula, this finally shows that $|(\phi_j\varphi, [v^{(12)}, \psi_j]\,\varphi)| \;\leq \frac{c}{R}\,\|\varphi\|^2$ and completes the proof of the HVZ theorem.

Acknowledgment

I should like to thank H.Siedentop, H.Kalf, R.Lasser, L.Erdös, S.Vugalter and S.Morozov for stimulating discussions and critical comments. It is a pleasure to thank A.Sobolev for his hint to work in momentum space and to use the auxiliary function in the proofs, as well as for his hospitality during my stay at the University of Sussex in 2005. Support by the EU Network Analysis and Quantum (contract HPRN-CT-2002-00277) is gratefully acknowledged.

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