96e:57019 57R40 57R55 57R57 57R95 58D27

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Embedded surfaces and the structure of Donaldson's polynomial invariants.

J. Differential Geom. **41** (1995), *no.* 3, 573–734.

FEATURED REVIEW.

Since 1982, most of the progress in four-dimensional differential topology has arisen from the applications of gauge theory pioneered by S. K. Donaldson. In particular, Donaldson's polynomial invariants have been used to prove a variety of results about the topology and geometry of four-manifolds. The paper under review is one of the pinnacles of this development. It gives a conceptual framework and an organizing principle for some of the disparate observations and calculations of Donaldson invariants that had been made earlier, it reveals a deep structure encoded in the Donaldson invariants which is related to embedded surfaces in four-manifolds, and it has been the point of departure and the motivating example for important further developments, most spectacularly for Witten's introduction of the so-called Seiberg-Witten invariants.

Let X be a smooth closed oriented four-manifold with $b_1(X)=0$. Recall that Donaldson's polynomial invariants are defined by evaluating certain tautological cohomology classes on the compactifications of moduli spaces of (smooth) anti-self-dual connections on SU(2)- or SO(3)-bundles over X. If $b_2^+(X)>1$, the invariants are differentiable invariants of X; they do not depend on the choice of Riemannian metric used to write down the anti-self-duality equation. It is convenient to think of the invariants as an infinite sequence of graded homogeneous polynomials on $H_0(X,\mathbf{R})\oplus H_2(X,\mathbf{R})$, where H_i is assigned degree $4-i$. This is a slight generalisation of Donaldson's original definition, which used only $H_2(X,\mathbf{R})$. This generalisation is one of the keys to the work under review.

Although these invariants are hard to calculate, they were used successfully from the very beginning to distinguish 4-manifolds which are homeomorphic but not diffeomorphic. Usually this required the calculation of one or two coefficients of a particular polynomial, and more complete calculations were never done—and indeed they could not be done. Nevertheless, workers in the field observed that the invariants did not appear to be completely independent of each other. There were restrictions on the parity of the values, and more subtle relations between them were observed in specific examples, like the $K3$ surfaces.

Another important problem about 4-manifolds, besides their diffeomorphism classification, is to determine the minimal genus of a smoothly embedded surface in a given class in $H_2(X, \mathbf{Z})$. (It is clear that the genus of such a surface is unbounded above because one can always add homologically trivial handles.) When X is a complex algebraic surface, and Σ is homologous to a smooth reduced complex curve, then the so-called Thom conjecture asserts that the genus of Σ is at least that of the complex curve, which is given by the adjunction formula to be $1 + \frac{1}{2}(K_X(\Sigma) + \Sigma^2)$, where K_X is the canonical class.

The initial attempts to apply Donaldson's polynomials to the Thom conjecture produced disappointingly weak results [see, e.g., S. K. Donaldson, Inst. Hautes Etudes Sci. Publ. Math. No. 68 (1988), 91–97 (1989); MR 90k:57041]. It seemed that the invariants did not capture the information about the minimal genus, but might instead be related to knotting phenomena. In particular, the fact that homologous embeddings of Σ are not necessarily isotopic appeared as an obstacle in some of the proposed arguments. In his lecture at the 1990 ICM, Kronheimer [in Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 529–539, Math. Soc. Japan, Tokyo, 1991; MR 93d:57066] suggested an approach, using connections singular along Σ, to proving that if X has a nontrivial Donaldson invariant, then the minimal genus satisfies $g(\Sigma) \geq 1 + \frac{1}{2}\Sigma^2$, unless Σ is an inessential sphere or a sphere of self-intersection $= -1$. He carried out this approach in joint work with Mrowka [Topology 32 (1993), no. 4, 773–826; MR 94k:57048; Topology 34 (1995), no. 1, 37–97; MR 96b:57038], thus proving the Thom conjecture for $K3$ surfaces (for which $K_X = 0$). Although this represented tremendous progress, the result was still far from optimal for other manifolds, and the method of proof still seemed ill-adapted to the problem. Subsequent work of Kronheimer [Bull. Amer. Math. Soc. (N.S.) 29 (1993), no. 1, 63–69; MR 94b:57037] improved the above bound, again through the use of singular connections, and this proved some more instances of the Thom conjecture.

At a technical level, the paper under review develops further the same ideas about singular connections that were used in the earlier work of the authors [op. cit.]. In addition, they use a complete, cylindrical end metric on the complement of Σ in X to define relative Donaldson invariants, which, under suitable hypotheses, take values in the homology of the moduli space of flat connections on the circle bundle over Σ which is the cross-section of the end. (This part uses earlier work of C. H. Taubes and of J. W. Morgan, T. S. Mrowka and D. Ruberman.) The authors show that under favourable circumstances the Donaldson invariants of X can be expressed in terms of the invariants derived from the singular connections by using the relative invariants as an intermediary. An inherent symmetry in the invariants derived from singular connections, already exploited in the earlier work, then gives universal relations between the Donaldson invariants of X. These relations depend on X and Σ only through the genus of Σ and its homology class. In particular, there is no annoying dependence on the isotopy class of the embedding $\Sigma \hookrightarrow X$. These universal relations hold for all smooth 4-manifolds X with $b_2^+(X) > 1$. Even if one started out looking only at the Donaldson invariants evaluated on classes in $H_2(X, \mathbf{R})$, the relations involve $H_0(X, \mathbf{Z})$.

At this point two crucial new ideas intervene. Firstly, the authors package all the Donaldson invariants arising from SO(3)-bundles with a fixed second Stiefel-Whitney class into a single formal power series. Secondly, they observe that if $x \in H_0(X, \mathbf{Z})$ is the generator, then, for a large class of manifolds, this power series is invariant under the substitution $x^2 = 4$. Manifolds satisfying this condition are said to be of simple type. The remarkable fact is that for manifolds of simple type the universal relations discussed above simplify dramatically and become computable. This leads to a closed formula for the formal power series encoding the Donaldson invariants. In a suitable normalisation, this formula reads

$$D^w = \exp(Q/2) \sum_{r=1}^{s} (-1)^{(w^2+K_r w)/2} \beta_r \exp(K_r), \tag{1}$$

where Q is the intersection form of X thought of as an element of $\mathrm{Sym}^2(H_2(X, \mathbf{R}))$, w is the Stiefel-Whitney class of the SO(3)-bundles under consideration, the $K_r \in H^2(X, \mathbf{Z})$ are characteristic elements for Q and the β_r are rational numbers. The K_r have the further property that for every homologically non-trivial embedded surface $\Sigma \subset X$ with non-negative self-intersection number, the genus satisfies $2g(\Sigma) - 2 \geq \Sigma^2 + K_r(\Sigma)$. The authors also prove that if X is of simple type then so is $X \# \overline{\mathbf{C}\mathrm{P}^2}$ and their Donaldson series are related by a simple formula involving trigonometric functions.

These remarkable results and their proofs clarify many observed properties and special calculations of Donaldson invariants. They show that, at least for manifolds of simple type, the infinite sequence of polynomial invariants is in fact determined by finitely many rational numbers and finitely many cohomology classes, and that to a large degree the invariants determine, and are determined by, the smooth embeddings of surfaces in X.

The above results are more general than those the authors announced earlier [Bull. Amer. Math. Soc. (N.S.) 30 (1994), no. 2, 215–221; MR 94k:57046], and the proofs are more self-contained.

Combining the Donaldson invariants into a formal power series, the Donaldson series, may not seem a very deep idea, but it is crucial in uncovering the structure in the invariants, as is the involvement of $x \in H_0(X, \mathbf{Z})$ and the simple type condition. The authors show that many smooth 4-manifolds, in particular many algebraic surfaces, are of simple type, and they give some sufficient conditions for an algebraic surface of simple type to have the canonical class K_X as one of the basic classes K_r. When this is the case, the above results prove the Thom conjecture for embedded surfaces of non-negative self-intersection number.

The structure theorem for the Donaldson invariants of manifolds of simple type discussed above has led to the rapid consolidation of much work in gauge theory. For simply connected 4-manifolds, R. Fintushel and R. J. Stern [J. Differential Geom. 42 (1995), no. 3, 577–633] have given an alternative proof of the structure theorem which is technically simpler than the argument in the paper under review. Instead of connections singular in codimension 2, Fintushel and Stern use only smooth connections. They find universal relations among the Donaldson invariants not by considering embedded surfaces of large genus, but by considering immersed spheres. This means that instead of estimating the minimal genus of an embedded surface, they can only estimate the minimal number of positive double points of an immersed sphere in terms of its homology class. More importantly, Fintushel and Stern ["The blowup formula for Donaldson invariants", Ann. of Math., to appear] have generalised the Kronheimer-Mrowka formula relating the invariants of $X\#\overline{\mathbf{CP}^2}$ to those of X. The Fintushel-Stern formula, valid for every X, not necessarily of simple type, involves elliptic functions which, when X is of simple type, specialise to the trigonometric functions found in the paper under review.

Given the blowup formula for arbitrary X, one can try to pin down the universal relations discussed above. Unpublished work of Kronheimer and Mrowka and of Fintushel and Stern goes a long way in this direction. They have derived a closed formula for the Donaldson series of any X, not necessarily of simple type, involving versions of the elliptic functions appearing in the blowup formula. However, once again all the information reduces to finitely many cohomology classes and finitely many rational numbers. This work, the details of which have not been written down, suggests that all smooth 4-manifolds with $b_2^+(X) > 1$ and $b_1(X) = 0$ are of finite type, meaning that their Donaldson series vanish when evaluated on $(x^2 - 4)^r$ for some sufficiently large r depending on X.

The work of Fintushel and Stern inspired by the paper under review has produced further sufficient conditions for manifolds to be of simple type. In particular, they have described very general surgical procedures, called rational blowdowns, which construct new smooth 4-manifolds from old ones preserving the simple type condition. Applications of rational blowdowns have led to the statement that "all known 4-manifolds" with $b_2^+ > 1$ are of simple type. On the other hand, the reviewer and P. Lisca [Math. Ann. 303 (1995), no. 2, 345–371] have shown that $\mathbf{C}\mathrm{P}^2$ is not of simple type, and L. Gottsche has given a conjectural generalization to show that $\mathbf{C}\mathrm{P}^2$ is not of finite type. (In this case the Donaldson invariants are well defined, but the criteria for simple type of Kronheimer and Mrowka and of Fintushel and Stern are not applicable because $b_2^+(\mathbf{C}\mathrm{P}^2) = 1$.)

Perhaps the most profound and exciting development arising out of the work under review is the abelian Seiberg-Witten gauge theory. While trying to understand formula (1) from the point of view of quantum field theory, and using ideas from his joint work with Seiberg on $N = 2$ supersymmetric Yang-Mills theory, Witten found what are now called the Seiberg-Witten equations. In fact, Witten says the equations are implicit in his papers with Seiberg, though they did not appear in print until Witten's paper [E. Witten, Math. Res. Lett. 1 (1994), no. 6, 769–796; MR 96d:57035]. The Seiberg-Witten equations are part of a field theory that is in some sense dual to Yang-Mills theory; under this duality they correspond to the anti-self-dual Yang-Mills equations used in Donaldson theory. Much like the Yang-Mills equations, the Seiberg-Witten equations have moduli spaces of solutions which lead to differentiable invariants of 4-manifolds. In fact, one can imitate Donaldson's construction of invariants, except that it is simpler to carry through in the context of Seiberg and Witten.

The Seiberg-Witten invariants of a smooth 4-manifold are indexed by the Spin^c structures of the manifold. The information contained in them is a priori finite, in that each smooth 4-manifold has at most finitely many Spin^c structures with non-trivial invariants. Witten conjectured that the first Chern classes of these Spin^c structures are precisely the basic classes K_r appearing in (1), and that the numerical invariants attached to them coincide with the numbers β_r, suitably normalized. Witten's reason for making this conjecture is a physical argument related to the duality between the Donaldson and Seiberg-Witten theories. His argument involves elliptic functions which may well be related to those appearing in the blowup formula for Donaldson invariants. Moreover, there is an interpretation of the simple type condition in this setting, and again it leads to the reduction of the general elliptic functions to elementary trigonometric functions. The fact that each manifold has only finitely many non-trivial Seiberg-Witten invariants is a finite type condition which may be equivalent to the finite type condition in Donaldson theory.

Witten's conjecture concerning equation (1) has been verified by direct calculations for large classes of manifolds, for example for all elliptic surfaces and their blowups, and many manifolds obtained from these by rational blowdowns [R. Fintushel and R. J. Stern, "Rational blowdowns of smooth 4-manifolds", Preprint, 1995; per revr.]. There are also some very promising attempts to prove Witten's conjecture in general through embedding the moduli spaces of both the Donaldson and Seiberg-Witten theories into a larger moduli space of solutions to suitable coupled equations. This should also shed light on cases not directly covered by Witten's conjecture, such as the case of manifolds with $b_2^+ = 1$. For example, the basic case of $\mathbf{C}\mathrm{P}^2$ is still not understood. All the Seiberg-Witten invariants of $\mathbf{C}\mathrm{P}^2$ are trivial, because $\mathbf{C}\mathrm{P}^2$ admits metrics of positive scalar curvature; on the other hand the structure of its Donaldson invariants is extremely complex. Witten has suggested that in order to understand the relationship between the two theories for manifolds with $b_2^+ = 1$, one has to come to grips with the elliptic functions occurring in the physical duality arguments. No reduction to elementary functions should be possible.

The Seiberg-Witten invariants have been used to prove the Thom conjecture for all embedded surfaces of non-negative self-intersection. In spite of what was said above about the triviality of these invariants for $\mathbf{C}\mathrm{P}^2$, the Thom conjecture for $\mathbf{C}\mathrm{P}^2$ can be deduced from fairly formal properties of the invariants on a blowup of $\mathbf{C}\mathrm{P}^2$ [P. B. Kronheimer and T. S. Mrowka, Math. Res. Lett. 1 (1994), no. 6, 797–808; MR 96a:57073].

While the paper under review was a major conceptual advance in our understanding of Donaldson theory, these advances are not being pursued much, except where they relate to Seiberg-Witten theory. There is a lot more to be explored here.

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