ON IRREDUCIBLE FOUR–MANIFOLDS

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1. Introduction

For many years, four–manifold folklore suggested that all simply connected smooth four–manifolds should be connected sums of complex algebraic surfaces, with both their complex and non–complex orientations allowed[1]. The first counterexamples were constructed in 1990 by Gompf and Mrowka [5], and many others followed. Then, Gompf [4] showed that many, and possibly all, these counterexamples arise from symplectic four–manifolds. Having no indication to the contrary, many people have put forward the following:

Conjecture 1. *Every smooth, closed, oriented and simply connected 4–manifold is the connected sum of symplectic manifolds, with both the symplectic and the opposite orientations allowed.*

This conjecture is ambitious; it would imply the smooth Poincaré conjecture.

Note that such a connected sum can have summands with definite intersection forms, for example copies of $\mathbb{C}P^2$. In fact, any definite summand has a diagonalizable intersection form by Donaldson's theorem [1], and is therefore homeomorphic to $n\mathbb{C}P^2$ by Freedman's classification [3].

When the manifolds under consideration are not simply connected, the situation is more complicated. Then there are obvious counterexamples to Conjecture 1, e.g. rational homology spheres which are not homotopy spheres. Thus, one has to allow definite summands which are more general than $n\mathbb{C}P^2$ or $n\overline{\mathbb{C}P^2}$. The natural conjecture is:

Conjecture 2. *Every smooth, closed and oriented 4–manifold is the connected sum of symplectic manifolds, with both the symplectic and the opposite orientations allowed, and of some manifolds with definite intersection forms.*

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[1]The 4–sphere is the empty connected sum.

This has occurred to several people, especially in the light of the examples constructed in [7], and has been dubbed the current "minimal conjecture" by Taubes. In this note we show that it is false[2]. Conjecture 1 remains open.

Recall the following definition:

Definition 1. A smooth closed 4–manifold X is *irreducible* if for every smooth connected sum decomposition $X \cong X_1 \# X_2$ one of the summands X_i must be a homotopy sphere.

We will show:

Theorem 1. *There exist oriented irreducible* 4*–manifolds* X *with indefinite intersection forms and with* $\pi_1(X) = \mathbb{Z}_2$ *and* $b_2^+(X) \equiv b_2^-(X) \equiv 0 \pmod 2$.

This follows from Proposition 1 in the next section.

Corollary 1. *There exist orientable irreducible* 4*–manifolds* X *with indefinite intersection forms, which are not almost complex (and therefore not complex and not symplectic) with respect to either orientation, and for which the Donaldson and Seiberg–Witten invariants are not defined (or must vanish by definition).*

If one drops the requirement that X have indefinite intersection form, rational homology spheres give obvious examples.

Recall that a symplectic 4–manifold is called minimal if it contains no symplectically embedded 2–sphere of selfintersection -1. Conjectures 1 and 2 are complementary to Gompf's conjecture [4] that minimal symplectic 4–manifolds are irreducible. In section 3 we deduce from the recent work of Taubes [11] on the Gromov and Seiberg–Witten invariants that Gompf's conjecture is true in many cases, including all simply connected manifolds with $b_2^+ > 1$. We shall prove:

Theorem 2. *Let* X *be a minimal symplectic* 4*–manifold with* $b_2^+(X) > 1$. *If* $X \cong X_1 \# X_2$ *is a smooth connected sum decomposition of* X*, then one of the* X_i *is an integral homology sphere whose fundamental group has no non–trivial finite quotient.*

This strengthening of Proposition 1 in [7] goes a long way towards confirming a conjecture made there.

Corollary 2. *Minimal symplectic* 4*–manifolds with* $b_2^+ > 1$ *and with residually finite fundamental groups are irreducible.*

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[2]We do not have any proposal for a new "minimal conjecture" in the non–simply connected case.

See section 3 for a result in and comments on the case $b_2^+ = 1$.

For Kähler surfaces, Theorem 2 and Corollary 2 could be deduced easily from the reduction of the Seiberg–Witten equation to the Kähler vortex equation and the study of effective divisors on complex surfaces, due to Kronheimer–Mrowka and Witten [13].

2. Irreducibility of quotient manifolds

Theorem 1 will follow from the following application of the covering trick introduced in [6]:

Proposition 1. *Let X be a smooth, closed, simply connected and oriented spin 4–manifold. If $b_2^+(X) > 1$, assume that X has a non–trivial Donaldson or Seiberg–Witten invariant. Suppose a non–trivial finite group G acts freely by orientation–preserving diffeomorphisms of X. Then the quotient $Y = X/G$ is an orientable irreducible 4–manifold.*

Proof. Let $Y \cong M\#N$ be a smooth connected sum decomposition. As $\pi_1(Y) \cong G$ is finite, it cannot be a non–trivial free product and we may assume $\pi_1(M) \cong G$ and $\pi_1(N) \cong \{1\}$.

Let d be the order of G. The connected sum decomposition of Y induces a connected sum decomposition $X \cong \overline{M}\#dN$, where $\overline{M}$ is the universal covering of M. As either $b_2^+(X) \leq 1$ or X is assumed to have a non–trivial Donaldson or Seiberg–Witten invariant, it follows that the intersection form of N is negative definite. By Donaldson's theorem [1] it is diagonalizable over $\mathbb{Z}$, and therefore either trivial or odd.

On the other hand, the intersection form of N must be even, because it is a direct summand of the intersection form of X, which is even because X is spin. We conclude $b_2(Y) = 0$. As N is simply connected, it is a homotopy sphere and Y is irreducible. $\square$

To obtain examples as in the statement of Theorem 1, take for X the Fermat surface of degree $d \equiv 2 \pmod 4$ in $\mathbb{C}P^3$ with $d \geq 6$. This is the surface defined in homogeneous coordinates by

$$x^d + y^d + z^d + t^d = 0 \ . \tag{1}$$

It is simply connected, and spin because its canonical class is the restriction of $(d-4)H$ to X, and therefore 2–divisible. Being an algebraic surface with $b_2^+ > 1$, X has non–trivial Donaldson and Seiberg–Witten invariants. Furthermore, the equation (1) is invariant under complex conjugation on $\mathbb{C}P^3$, and has no non–trivial real solutions (because d is even). Thus, complex conjugation acts freely on X and the Proposition shows that $Y = X/\mathbb{Z}_2$ is irreducible. We have $\pi_1(Y) = \mathbb{Z}_2$. Using the multiplicativity of the Euler characteristic and the signature under finite unramified coverings, one can calculate $b_2^{\pm}(Y) = \frac{1}{2}(b_2^{\pm}(X) - 1)$ which are both positive (because $d \geq 6$) and even (because $d \equiv 2 \bmod 4$).

This completes the proof of Theorem 1.

Remark 1. Wang [12] has shown that the quotients of simply connected minimal algebraic surfaces of general type by free anti–holomorphic involutions have trivial Seiberg–Witten invariants, even when the invariants do not vanish by definition as in the above examples.

The assumptions in Proposition 1 are such that X has to be irreducible, and then Y turns out irreducible as well. Here is another result in the same spirit, but which does not require a spin condition and uses instead Corollary 2. We could apply this to the examples discussed above, but Proposition 1 is much more elementary.

Proposition 2. *Let X be a closed, simply connected minimal symplectic 4–manifold with $b_2^+(X) > 1$. Suppose a non–trivial finite group G acts freely by orientation–preserving diffeomorphisms of X. Then the quotient $Y = X/G$ is an orientable irreducible 4–manifold.*

Proof. If $Y \cong M \# N$ with $\pi_1(M) \cong G$ and $\pi_1(N) \cong \{1\}$, then $X \cong \overline{M} \# dN$. Thus Corollary 2 implies that N is a homotopy sphere. Hence Y is irreducible. $\square$

3. Irreducibility of symplectic manifolds

In this section we give the proof of Theorem 2. This requires some familiarity with Seiberg–Witten invariants, particularly the work of Taubes [9, 10, 11]. See also [7, 8, 13].

Let X be a closed symplectic 4–manifold with $b_2^+(X) > 1$. If X splits as a connected sum $X \cong M \# N$, then by Proposition 1 of [7] we may assume that N has a negative definite intersection form and that its fundamental group has no non–trivial finite quotient. In particular $H_1(N, \mathbb{Z}) = 0$. This implies that the homology and cohomology of N are torsion–free.

Donaldson's theorem about (non–simply connected) definite manifolds [2] implies that the intersection form of N is diagonalizable over $\mathbb{Z}$. If N is not an integral homology sphere, let $e_1, \ldots, e_n \in H^2(N, \mathbb{Z})$ be a basis with respect to which the cup product form is the standard diagonal form. This basis is unique up to permutations and sign changes.

It is a theorem of Taubes [9] that the Seiberg–Witten invariants of X are non–trivial for the canonical $Spin^c$–structures with auxiliary line bundles $\pm K_X$. Note that we can write

$$K_X = K_M + \sum_{i=1}^{n} a_i e_i \ ,$$

where $K_M \in H^2(M, \mathbb{Z})$ and the a_i are odd integers because $a_i^2 = -1$ and K_X is characteristic. Considering $-K_X$ and using a family of Riemannian metrics which stretches the neck connecting M and N, we conclude that M has a non–trivial Seiberg–Witten invariant for a $Spin^c$–structure with auxiliary line bundle $-K_M$.

Now we can reverse the process and glue together solutions to the Seiberg–Witten equation for $-K_M$ on M and reducible solutions on N for the unique $Spin^c$–structure with auxiliary line bundle $e_1 - \sum_{i \neq 1} e_i$, as in the proof of Proposition 2 in [7]. This gives a Seiberg–Witten invariant of X which is equal (up to sign) to the Seiberg–Witten invariant of M for $-K_M$, which is non–zero.

This implies that $L = -K_M + e_1 - \sum_{i \neq 1} e_i$ has selfintersection number $= K_X^2$ because for symplectic manifolds all the non–trivial Seiberg–Witten invariants come from zero–dimensional moduli spaces, see [11]. Thus, $a_i = \pm 1$ for all $i \in \{1, \ldots, n\}$. Without loss of generality we may assume $a_i = 1$ for all i.

The line bundle L is obtained from $-K_X$ by twisting with e_1. Thus, by Taubes's main theorem in [11], the non–triviality of the Seiberg–Witten invariant of X with respect to L implies that e_1 can be represented by a symplectically embedded 2–sphere in X. This contradicts the minimality of X.

We conclude that N must be an integral homology sphere. This completes the proof of Theorem 2.

Remark 2. Gompf [4] has shown that all finitely presentable groups occur as fundamental groups of minimal symplectic 4–manifolds, and conjecturally all these manifolds are irreducible. As was the case in [6, 7], our arguments do not give an optimal result because we cannot deal with fundamental groups without non–trivial finite quotients. With regard to Theorem 2, note that there are such groups which occur as fundamental groups of integral homology 4–spheres. Let G be the Higman 4–group, an infinite group without non–trivial finite quotients, which has a presentation with 4 generators and 4 relations. Doing surgery on $4(S^1 \times S^3)$ according to the relations produces an integral homology sphere with fundamental group G.

Remark 3. In another direction, the assumption $b_2^+(X) > 1$ can probably be removed from Theorem 2 and Corollary 2. To do this one needs to understand how the neck–stretching in the proof of Theorem 2 and the perturbations in Taubes's arguments [9, 11] interact with the chamber structure of the Seiberg–Witten invariants for manifolds with $b_2^+ = 1$. We will return to this question in the future.

However, some results about the case when $b_2^+ = 1$ can be deduced from Theorem 2. For example, all manifolds with non–tivial finite fundamental groups are dealt with by the following:

Corollary 3. *Let X be a minimal symplectic 4–manifold with $b_2^+(X) = 1$ and $b_1(X) \leq 1$. If $\pi_1(X)$ is a non–trivial residually finite group, then X is irreducible.*

Proof. Suppose $X \cong M\#N$. We may assume that N has negative definite intersection form and its fundamental group has no non–trivial finite quotient. Residual finiteness then implies that N is simply connected, and $\pi_1(M) \cong \pi_1(X)$. By assumption, X has a finite cover $\overline{X}$ of degree $d > 1$ which is diffeomorphic to $\overline{M}\#dN$, where $\overline{M}$ is a d–fold cover of M. But $\overline{X}$ is minimal symplectic because X is, and so Corollary 2 implies that N is a homotopy sphere whenever $b_2^+(\overline{X}) > 1$.

If $b_1(X) = 0$, the multiplicativity of the Euler characteristic and of the signature imply $b_2^+(\overline{X}) \geq 3$. If $b_1(X) = 1$, we can take $d \geq 3$ to obtain $b_2^+(\overline{X}) \geq 2$. □

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