Γ-convergence: An Application

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Γ-convergence

X topological space satisfying 1st axiom of countability, $F_n, F : X \to \overline{\mathbb{R}}$.
We say $F = \Gamma\text{-}\lim\limits_{n\to\infty} F_n$ iff

1. $\forall (x_n)_n \subset X$ with $x_n \xrightarrow{n\to\infty} x \in X$ it is
$$F(x) \leq \liminf_{n\to\infty} F_n(x_n).$$
2. $\forall x \in X$ there is $(x_n)_n \subset X$ with $x_n \xrightarrow{n\to\infty} x \in X$ s.t.
$$F(x) = \lim_{n\to\infty} F_n(x_n).$$

The sequence in 2. is called *recovering sequence*.

Remark: There is a more general definition for arbitrary topol. spaces X coinciding with this definition.

Γ-limits – an easy example

Set $X := \mathbb{R}$ and $F_n(x) := \sin(nx)$. Then,

1. $-1 \leq \liminf_{n\to\infty} \sin(nx_n)$ and
2. For $x \in \mathbb{R}$ define x_n as the nearest point with $\sin(nx_n) = -1$. $\sin(n\,\cdot\,)$ is $\frac{2\pi}{n}$-periodic $\Longrightarrow |x_n - x| \leq \frac{\pi}{n} \Longrightarrow x_n \to x$. Furthermore, $-1 = \lim_{n\to\infty} \sin(nx_n)$.

$$\Longrightarrow \Gamma\text{-}\lim_{n\to\infty} \sin(n\,\cdot\,) = -1.$$

We see:

1. Need additional assumptions for comparison with pointwise convergence.
2. This example points towards minimization problems!

Coercivity, Γ-limits and minimization

X reflexive Banach space:

$F : X \to \overline{\mathbb{R}}$ coercive w.r.t. weak topology $\iff F(x) \xrightarrow{\|x\|\to\infty} \infty$.

X topological space:

$(F_n)_n$ equi-coercive $\iff \exists$ l.sc. and coercive $\Psi : X \to \overline{\mathbb{R}}$ s.t. $F_n \geq \Psi$.

Theorem

$(F_n)_n$ equi-coercive, $F = \Gamma\text{-}\lim_{n\to\infty} F_n$, $\exists! x_0 \in X$ minimizing F. Choose

$$x_n \in \operatorname*{argmin}_{x \in X} F_n(x).$$

Then, $x_n \xrightarrow{n\to\infty} x_0$ and $(F_n(x_n))_n \xrightarrow{n\to\infty} F(x_0)$.

p-Poisson equation

Strong formulation: u satisfies

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \text{ on } \Omega$$
$$u|_{\partial\Omega} = 0$$

Weak formulation: $u \in W_0^{1,p}(\Omega)$ and for all $\xi \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\xi \, dx = f(\xi).$$

u strong solution $\Longrightarrow$ u weak solution $\Longleftrightarrow$ u is the unique minimizer of

$$\mathcal{J}(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - f(u).$$

Splitted energy

Consider $1 < p < 2$ and minimize

$$\mathcal{J}^s(u,a) := \int_\Omega \tfrac{1}{2}|a|^{p-2}|\nabla u|^2 + (\tfrac{1}{p} - \tfrac{1}{2})|a|^p \, dx - f(u)$$

Lemma

Let (u,a) be a local minimizer of $\mathcal{J}^s$.
Then, u is a weak solution of p-Poisson equation and

$$|a| = |\nabla u|.$$

Sketch of proof: Calculate first variation and use "right" test functions.

Constrained splitted energy

For stability add constraint for a:

$$\mathcal{J}^s_{(\varepsilon)}(u,a) := \mathcal{J}^s(u, \varepsilon \vee |a| \wedge \tfrac{1}{\varepsilon})$$

Lemma

For fixed $u \in W^{1,2}_0(\Omega)$, the function $a = \varepsilon \vee |\nabla u| \wedge \frac{1}{\varepsilon}$ is the unique minimum of $\mathcal{J}^s_{(\varepsilon)}(u, \cdot\,)$ satisfying $a = \varepsilon \vee a \wedge \frac{1}{\varepsilon}$.

So we want to use

$$u_{k+1} := \operatorname{argmin} \mathcal{J}^s_{(\varepsilon)}(\,\cdot\,, a_k) \quad \text{and} \quad a_{k+1} := \varepsilon \vee |\nabla u_{k+1}| \wedge \tfrac{1}{\varepsilon}.$$

Constrained energy

How does ε affect the algorithm?

Define

$$\mathcal{J}_{(\varepsilon)}(u) := \min_{a=\varepsilon\vee|a|\wedge\frac{1}{\varepsilon}} \mathcal{J}^s_{(\varepsilon)}(u,a) = \mathcal{J}^s(u, \varepsilon \vee |\nabla u| \wedge \tfrac{1}{\varepsilon})$$
$$= \int_\Omega \kappa_{(\varepsilon)}(|\nabla u|)\, dx - f(u)$$

Theorem

Let u be minimizer of $\mathcal{J}$ and u_ε be minimizer of $\mathcal{J}_{(\varepsilon)}$.
Then, $u_\varepsilon \rightharpoonup u$ in $W^{1,p}_0(\Omega)$ as $\varepsilon \searrow 0$.

Outline of the proof

We need to check:

1. 1^{st} axiom of countability,
2. equi-coercivity of $\mathcal{J}_{(\varepsilon)}$ and
3. $\mathcal{J} = \Gamma\text{-}\lim_{\varepsilon \searrow 0} \mathcal{J}_{(\varepsilon)}$.
4. Finally, apply theorem from slide 4

Arising problems:

- no coercivity w.r.t. norm-topology
  $\Longrightarrow$ use weak topology
- no 1^{st} axiom of countability on $W_0^{1,p}(\Omega)$ w.r.t. weak topology

Apropriate space

Choose the topological space

$$X := \{w \in W_0^{1,p}(\Omega) \,:\, \mathcal{J}_{(0.5)}(w) \leq 0\} \cup \{w \in W_0^{1,\infty}(\Omega) \,:\, \|\nabla w\|_{L^p(\Omega)} \leq \gamma\}.$$

A priori estimates for u_ε, monotonicity of $\mathcal{J}_{(\varepsilon)}$ in ε

$\Longrightarrow X$ is norm-bounded:

$$0 \geq \mathcal{J}_{(0.5)}(v) \geq \mathcal{J}(v) \geq \frac{1}{p}\|\nabla v\|^p_{L^p(\Omega)} - c\|f\|_{W_0^{-1,p}(\Omega)} \cdot \|\nabla v\|_{L^p(\Omega)}$$

$\Longrightarrow \mathcal{T}_{\text{weak}}$ is induced by a metric (since $\left(W_0^{1,p}(\Omega)\right)^*$ is separable)

$\Longrightarrow B_{\frac{1}{n}}(x_0)$ are countable basis of neighbourhoods of x_0

i.e. 1^{st} axiom of countability!

Equi-coercivity

Need to find l.sc. and coercive $\Psi : X \to \overline{\mathbb{R}}$ such that $\mathcal{J}_{(\varepsilon)} \geq \Psi$.

$\mathcal{J}$ does the job:

1. $\mathcal{J}$ is lower semicontinuous
2. Use

$$\mathcal{J}(v) \geq \frac{1}{p}\|\nabla v\|^p_{L^p(\Omega)} - c\|f\|_{W_0^{-1,p}(\Omega)} \cdot \|\nabla v\|_{L^p(\Omega)}$$

   to see $\mathcal{J}(v) \xrightarrow{\|\nabla v\|_{L^p(\Omega)}} +\infty$
   $\implies \mathcal{J}$ is coercive w.r.t. $\mathcal{T}_{\text{weak}}$
3. $\mathcal{J}_{(\varepsilon)}$ monotonic increasing in ε and $\mathcal{J} = \mathcal{J}_{(0)}$
   $\implies \mathcal{J}_{(\varepsilon)} \geq \mathcal{J}$

Overall: $\mathcal{J}_{(\varepsilon)}$ is equi-coercive!

lim inf-condition

Let $v \in X$ and $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Then,

$$\liminf_{n\to\infty} \mathcal{J}_{(\varepsilon_n)}(v_n) \geq \liminf_{n\to\infty} \mathcal{J}(v_n) \geq \mathcal{J}(v).$$

$\Longrightarrow$ lim inf*-condition is satisfied.*

It remains to find *recovering sequences*.

For $v \in \{w \in W_0^{1,\infty}(\Omega) : \|\nabla w\|_{L^p(\Omega)} \leq \gamma\}$ take $v_n \equiv v$:

$$\lim_{n\to\infty} \mathcal{J}_{(\varepsilon_n)}(v_n) = \lim_{n\to\infty} \mathcal{J}_{(\varepsilon_n)}(v) = \lim_{n\to\infty} \int_\Omega \kappa_{(\varepsilon_n)}(|\nabla v|)\,dx - f(v)$$

$$= \int_\Omega |\nabla v|^p\,dx - f(v) = \mathcal{J}(v)$$

More definitions

For $v \in L^1(\mathbb{R}^d)$ define the *Hardy-Littlewood maximal function*

$$M(v)(x) := \sup_{r>0} \fint_{B_r(x)} |v(y)|\, dy.$$

M continuous for $p > 1$ $\qquad \|Mv\|_{L^p(\mathbb{R}^d)} \leq c\|v\|_{L^p(\mathbb{R}^d)}$

and of weak type (1,1). $\qquad \sup_{\lambda>0} \lambda|\{M(v) > \lambda\}| \leq c\|v\|_{L^1(\mathbb{R}^d)}$

We call

$$\mathcal{O}_\lambda(v) := \{x \in \mathbb{R}^d \, : \, M(|\nabla v|)(x) > \lambda\}$$

the *"bad set"*. Due to lower semicontinuity of $M(|\nabla v|)$, it is open.

Lipschitz truncation

Theorem (Diening, Kreuzer, Süli)

Let $\lambda > 0$ and $v \in W_0^{1,p}(\Omega)$. Then, there is $v_\lambda \in W_0^{1,\infty}(\Omega)$ with the following properties:

(a) $\{v \neq v_\lambda\} \subset \mathcal{O}_\lambda(v) \cap \Omega$,

(b) $\|v_\lambda\|_{L^p(\Omega)} \leq c_1 \|v\|_{L^p(\Omega)}$,

(c) $\|\nabla v_\lambda\|_{L^p(\Omega)} \leq c_2 \|\nabla v\|_{L^p(\Omega)}$ *and*

(d) $|\nabla v_\lambda| \leq c\lambda \chi_{\mathcal{O}_\lambda(v) \cap \Omega} + |\nabla v| \chi_{\mathcal{O}_\lambda(v)^\complement \cap \Omega} \leq c_3 \lambda$ *almost everywhere.*

For $\varepsilon_n \xrightarrow{n\to\infty} 0$ define $\lambda_n := \frac{1}{c_3 \varepsilon_n} \quad \Longrightarrow \quad |\nabla v_{\lambda_n}| \leq \frac{1}{\varepsilon_n}$ a.e.

Recovering sequence

For $v \in \{w \in W_0^{1,p}(\Omega) : \mathcal{J}_{(0.5)}(w) \leq 0\}$ the Lipschitz truncations

$$v_n := v_{\lambda_n}$$

give an admissable recovering sequence. First of all,

$$\|\nabla v_n\|_{L^p(\Omega)} \leq c_2 \|\nabla v\|_{L^p(\Omega)} \leq c \left(\|f\|_{W_0^{-1,p}(\Omega)}\right)^{\frac{1}{p-1}} =: \gamma$$

and $(v_n)_n \subset W_0^{1,\infty}(\Omega)$. Hence, $(v_n)_n \subset X$. It remains to show

$$\mathcal{J}_{(\varepsilon_n)}(v_n) - \mathcal{J}(v) = \int_\Omega \kappa_{(\varepsilon_n)}(|\nabla v_n|) - |\nabla v|^p \, dx - \langle f, v_n - v \rangle \xrightarrow{n \to \infty} 0.$$

Therefore, split Ω into $\Omega_1 := \mathcal{O}_{\lambda_n}(v)^\complement \cap \Omega$ and $\Omega_2 := \mathcal{O}_{\lambda_n}(v) \cap \Omega$.

Convergence on good set

On Ω_1 we have $v \equiv v_n$.

Furthermore $|\nabla v_n| \le c_3 \lambda_n = \frac{1}{\varepsilon_n}$ and additionaly $\kappa_{(\varepsilon_n)}(t) = t^p$ for $t \in [\varepsilon_n, \frac{1}{\varepsilon_n}]$.

$$\begin{aligned}
\Big| \int_{\Omega_1} \kappa_{(\varepsilon_n)}(|\nabla v_n|) - \varphi(|\nabla v|)\, dx \Big| &= \Big| \int_{\Omega_1} \kappa_{(\varepsilon_n)}(|\nabla v_n|) - |\nabla v_n|^p \, dx \Big| \\
&= \Big| \int_{\Omega_1 \cap \{|\nabla v_n| < \varepsilon_n\}} \kappa_{(\varepsilon_n)}(|\nabla v_n|) - |\nabla v_n|^p \, dx \Big| \\
&\le 2|\Omega| \varepsilon_n^p \\
&\xrightarrow{n \to \infty} 0.
\end{aligned}$$

Convergence on bad set

Weak type (1,1) implies $|\Omega_2| \xrightarrow{n\to\infty} 0$, so

$$
\begin{aligned}
\left| \int\limits_{\Omega_2} \kappa_{(\varepsilon_n)}(|\nabla v_n|) - |\nabla v|^p \, dx \right| &\leq \int\limits_{\Omega_2} \kappa_{(\varepsilon_n)}(|\nabla v_n|) + |\nabla v|^p \, dx \\
&\leq \int\limits_{\Omega_2} \varphi(c_3\lambda) + |\nabla v|^p \, dx \\
&\leq \int\limits_{\Omega_2} \underbrace{c_3^p M(|\nabla v|)^p + |\nabla v|^p}_{\in L^1(\Omega)} \, dx \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}
$$

Alternative proof via relaxation

1. Define

$$\tilde{\mathcal{J}}_{(\varepsilon)} : W_0^{1,p}(\Omega) \to \overline{\mathbb{R}}$$
$$u \mapsto \begin{cases} \mathcal{J}_{(\varepsilon)}(u) & \text{for } u \in W_0^{1,2}(\Omega) \\ +\infty & \text{for } u \in W_0^{1,p}(\Omega) \setminus W_0^{1,2}(\Omega). \end{cases}$$

   and $\tilde{\mathcal{J}} := \tilde{\mathcal{J}}_{(0)}$.
2. $\tilde{\mathcal{J}}_{(\varepsilon)}$ converges pointwise and decreasing to $\tilde{\mathcal{J}}$
   $\implies \mathrm{sc}^- \tilde{\mathcal{J}} = \Gamma\text{-}\lim_{\varepsilon \searrow 0} \tilde{\mathcal{J}}_{(\varepsilon)}$
3. $\mathrm{sc}^- \tilde{\mathcal{J}} = \mathcal{J}$
4. Use theorem from slide 4

Thank you for your attention.

Literature:

- Dal Maso: “An Introduction to Γ-convergence”
- Diening, Kreuzer, Süli: “Finite element approximation of steady flows of incompressible fluids with implicit power-law-like rheology”