Introduction to Semigroup Theory

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The prototype of a parabolic PDE is given by the heat equation, this is

\[(\partial_t - \Delta)u = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty)\]

\[u = f \quad \text{on } \mathbb{R}^d \times \{t = 0\}\]

Solutions are given by

\[S(t)f(x) \equiv \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp \left( -\frac{|x - \zeta|^2}{4t} \right) f(\zeta) \, d\zeta \quad (1)\]

This is strongly dependent on \(f \in X\) - what is \(X\)?

\[\rightarrow S(\mathbb{R}^d) \subset \bigcap_{1 \leq p \leq \infty} L^p(\mathbb{R}^d)\] and thus is well-defined for any \(f \in L^q(\mathbb{R}^d), \, 1 \leq q \leq \infty\).

\[\rightarrow\] Let us investigate \(S(t)\)!
Properties of $S(t)$

Recall the Gauss-Weierstraß kernel

$$W_t(x) \equiv \frac{1}{(4\pi t)^{d/2}} \exp\left(\frac{-|x|^2}{4t}\right)$$

for $t > 0$.

Write $S(t)f = W_t * f$ for $t > 0$ and additionally $S(0) = I$.

The following is obvious:

- $S(t)$ is a well-defined, bounded linear operator on $L^p(\mathbb{R}^d)$ for all $p < \infty$
- $\|S(t)\|_{L^p \rightarrow L^p} \leq 1$

Moreover: $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$
**Key topic: Semigroups**

**Definition**

Let $X$ be Banach and $\mathcal{A} \equiv (S(t))_{t \geq 0} \subset \mathcal{L}(X)$ a family of linear, bounded operators $S(t): X \rightarrow X$ for all $t \geq 0$. $\mathcal{A}$ is called a *semigroup* if and only if

$$S(0) = I \quad \text{and} \quad S(s + t) = S(s)S(t) \quad \forall t, s \geq 0$$
Example: Exponential Series

Let $A \in \mathcal{L}(X)$, $X$ Banach. Then set $S(t) = \exp(tA)$, i.e.

$$\exp(tA) \equiv I + \sum_{n \in \mathbb{N}} \frac{t^n A^n}{k!}$$

Check by

$$|| \exp(tA)|| \leq \exp(t||A||) < \infty$$

and $A^n A^l = A^l A^n$ that this indeed defines a semigroup. Even more is valid:

For any $x \in X$,

$$||S(t)x - x|| \leq \sum_{k \in \mathbb{N}} \frac{t^k}{k!} ||A||^k ||x|| = ||x||(\exp(t||A||) - 1) \to 0, \ t \to 0+$$
**UC and \( C_0 \)-Semigroups**

Due to our last example the following notion makes sense:

**Definition**

A semigroup \( \mathcal{A} \equiv \{ S(t) \}_{t \geq 0} \subset \mathcal{L}(X) \) on \( X \) Banach is called a \( C_0 \)- or strongly continuous semigroup if and only if

\[
\|S(t)x - x\| \to 0, \quad t \to 0^+ \quad \forall x \in X
\]

Moreover, it is called a uniformly continuous or UC semigroup if and only if

\[
\|S(t) - I\| \to 0, \quad t \to 0^+
\]

Clearly, UC \( \implies \) \( C_0 \). The converse is not true!
**Infinitesimal Generators**

**Definition**

Let $\mathcal{A} \equiv (S(t))_{t \geq 0} \subset \mathcal{L}(X)$ be a semigroup on $X$ Banach. Set

\[
D(\mathcal{A}) \equiv \left\{ x \in X : \exists \lim_{t \to 0^+} \frac{S(t)x - x}{t} \right\}
\]

\[
\mathcal{A}x \equiv \lim_{t \to 0^+} \frac{S(t)x - x}{t} \quad \text{for} \quad x \in D(\mathcal{A})
\]

$\mathcal{A}$ is called the *(infinitesimal)* generator of $\mathcal{A}$. We write

\[
\langle A \rangle_{gen} = \mathcal{A}
\]
Intermezzo: Example - The Heat Semigroup

Recall our basic example

\[ S(t)f(x) \equiv \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp \left( -\frac{|x - \zeta|^2}{4t} \right) f(\zeta) \, d\zeta, \quad t > 0 \]

\[ S(0) = I \]

We refer to this semigroup as the heat semigroup and write \( \mathcal{A}_{\text{heat}} \)

Our goal:

We want to show \( \langle \Delta \rangle_{\text{gen}} = \mathcal{A}_{\text{gen}} \).
Recall:

- \( \ast : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d) \)
- \( \mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d) \) bijectively
- \( \mathcal{F} : L^2(\mathbb{R}^d) \xrightarrow{\sim} L^2(\mathbb{R}^d) \)
- Alternative characterization of \( W^{m,2} \)-Sobolev functions:
  \[
  W^m(\mathbb{R}^d) = W^{m,2}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : (1 + |\zeta|^2)^{m/2} \mathcal{F} f \in L^2(\mathbb{R}^d) \right\}
  \]
- \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( L^2(\mathbb{R}^d) \)

Density \( \rightarrow \) it suffices to show the claim for the Schwartz class!

Claim:

\[
\lim_{t \rightarrow 0^+} \frac{W_t \ast f - f}{t} = \Delta f \quad \text{for} \quad f \in \mathcal{S}(\mathbb{R}^d)
\]

\((L^2 \text{ convergence})\).
\[
\lim_{t \to 0} \frac{(2\pi)^{d/2} (\mathcal{F} W_t) \cdot (\mathcal{F} f) - (\mathcal{F} f)}{t} = \mathcal{F}(\Delta f)
\]

By Cauchy’s integral formula we easily check that \( \zeta \mapsto \exp(-\zeta^2) \) is a fixed point of \( \mathcal{F} \), whereby a change of variables implies

\[
\mathcal{F} W_t(\zeta) = \frac{1}{(2\pi)^{d/2}} \exp(-t\zeta^2), \quad \mathcal{F}(\Delta f)(\zeta) = -\zeta^2 \mathcal{F} f(\zeta)
\]

\[
\lim_{t \to 0} \frac{\exp(-t\zeta^2)g - g}{t} = -\zeta^2 g \quad \text{for all } g \in \mathcal{S}(\mathbb{R}^d), \zeta \in \mathbb{R}^d
\]

or

\[
\lim_{t \to 0^+} \frac{\exp(tv)g - g}{t} = vg \quad \text{for all } g \in \mathcal{S}(\mathbb{R}^d), \zeta \in \mathbb{R}^d
\]

with \( v(\zeta) = -\zeta^2 \).
Set

\[ \Phi(z) = \frac{\exp(z) - 1}{z} \equiv \sum_{n \geq 2} \frac{z^{n-1}}{n!} \], \quad -1 \leq \Phi(z) \leq 0 \text{ for } z \leq 0

Thus

\[ \left\| \exp(tv)g - g - \frac{vg}{t} \right\|^2 = \int_{\mathbb{R}^d} |\Phi(-t\zeta^2)||\zeta^2g(\zeta)|^2 \, d\zeta \to 0 \]

This estimate is valid for any \( v \) such that \( v \mathcal{F}f \in L^2 \to W^{2,2}(\mathbb{R}^d) \).

Conclusion: \( \langle \Delta \rangle_{\text{gen}} = \mathcal{A} \) and \( D(A) = W^{2,2}(\mathbb{R}^d) \)
Differential properties of Generators

Theorem

Assume $u \in D(A)$ and $\sup_{t \geq 0} \|S(t)\| < \infty$. Then

(i) $S(t)u \in D(A)$ for all $t \geq 0$
(ii) $AS(t)u = S(t)Au$ for all $t \geq 0$
(iii) $t \mapsto S(t)u$ is differentiable for each $t > 0$ and $\frac{d}{dt} S(t)u = AS(t)u$ for $t > 0$

Proof of (iii) & (iv): For $u \in D(A)$, $h > 0$ and $t > 0$. Consider

$$\lim_{h \to 0^+} \left( \frac{S(t)u - S(t-h)u}{h} - S(t)Au \right)$$
**Topological Properties of Generators**

**Note:** Under the assumptions of the last theorem, continuity of $t \mapsto AS(t)u = S(t)Au$ implies that $t \mapsto S(t)u$ is of class $C^1((0, \infty), X)$ for $u \in D(A)$.

**Recall:** An operator $T : X \rightarrow X$ is called **closed** iff its graph is closed with respect to the product topology on $X \times X$.

**Theorem**

*The generator of a $C_0$-semigroup is densely defined and closed.*
EXPO NENTIAL BOUNDS FOR $C_0$-SEMIGROUPS

Theorem

Let $(S(t))_{t \geq 0}$ be a $C_0$-semigroup on $X$ Banach. Then there exist $\omega \in \mathbb{R}$, $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t} \quad \forall t > 0$$

Proof. At first, there exists $\tau > 0$ such that

$$M \equiv \sup_{0 \leq t \leq \tau} \|S(t)\| < \infty$$

Now, rescale: For $t \geq 0$ write $t = n\tau + \theta$ with $n \in \mathbb{N}$ and $0 \leq \theta < \tau$. 
The goal of the game: Cauchy Problems

Theorem

Let $A$ be the infinitesimal generator of some $C_0$-semigroup and let $u \in D(A) \subset X$. Then the mapping $u: [0, \infty) \ni t \mapsto S(t)u \in X$ is $C^1$, $D(A)$-valued and a solution to

$$y' = Ay \quad \& \quad y(0) = u$$

Proof. Differentiability: Compute, compute and estimate. Then:

$$\frac{d}{dt} S(s - t)v(t) = 0 \text{ for } s \leq t. \implies \Phi: [0, s] \ni t \mapsto S(s - t)v(t) \in X$$

satisfies

$$\frac{d}{dt} (\ell \circ \Phi) = \ell \circ \frac{d}{dt} \Phi(t) = 0 \quad \text{Hahn-Banach} \implies \text{Uniqueness.}$$
**Peripety: \( C_0 \)-contraction semigroups**

We have: \( A \) generates a UC-semigroup \( \iff A \in \mathcal{L}(X) \)

**Question:** \( A \) generates a \( C_0 \)-semigroup \( \iff \) ???

We know: If \( A \) is a generator of a \( C_0 \)-semigroup, then \( A \) is densely defined and closed

**We need:** Criterion to decide whether a densely defined, closed operator generates a \( C_0 \)-semigroup \( \rightarrow \) Available for ’Contraction Semigroups’

Recall: Any \( C_0 \)-semigroup satisfies \( \|S(t)\| \leq Me^{\omega t} \) for all \( t \geq 0 \).

**Definition**

If one can choose \( \omega = 0 \) and \( M = 1 \) in the last theorem, then the \( C_0 \)-semigroup is called **contractive** or contraction semigroup.
**On the Road to Hille-Yosida - Spectral Theory**

Spectral Theory in Linear Algebra: $\sigma(A) =$ Eigenvalues

Spectral Theory in Functional Analysis: For $A: D(A) \to X$ (un)bounded and closed operator, define the resolvent set

$$\lambda \in \rho(A) \subset \mathbb{C} \iff \lambda - A: D(A) \to X \text{ bijective}$$

and the resolvent operator

$$R_\lambda: X \ni u \mapsto (\lambda - A)^{-1} u$$

Then the spectrum is the complement of the resolvent set:

$$\sigma(A) = \rho(A)^c = \mathbb{C} \setminus \rho(A)$$

**Standard Assumption from now on:** $A$ is a closed linear operator on $X$ Banach.
The Hilla - Yosida Theorem

Theorem

An operator $A$ is the infinitesimal generator of a $C_0$-contraction semigroup if and only if $A$ is densely defined and closed, $(0, \infty) \subset \rho(A)$ and $\|R_\lambda\| \leq \lambda^{-1} \forall \lambda > 0$.

Proof. IDEA: For $\lambda > 0$ define the bounded Yosida approximations

$$A_\lambda = \lambda A(\lambda - A)^{-1} = \lambda^2 (\lambda - A)^{-1} - \lambda \in \mathcal{L}(X)$$

and define semigroups $(\exp(tA_\lambda))_{t \geq 0}$. Show: $\exists S(t)x \equiv \lim_{\lambda \to \infty} e^{tA_\lambda}x$ and this defines semigroup with operator $A$. 
The Homogeneous Heat Equation I

\[(\partial_t - \Delta)u = 0 \quad \text{in } \Omega \times (0, \infty)\]

\[u = f \quad \text{on } \partial\Omega \times (0, \infty)\]

\[u(\cdot, 0) = u_0 \quad \text{in } \Omega\]

where \(\Omega \subset \mathbb{R}^d\) bounded, open, \(\partial\Omega \in C^{0,1}\).

CLAIM:

Let \(u_0 \in H^2(\Omega) \cap H^1_0(\Omega)\). Then there exists exactly one solution

\[u \in C^1([0, \infty) ; L^2(\Omega)) \cap C^0([0, \infty) ; H^2(\Omega) \cap H^1_0(\Omega))\]

of the above problem such that \(\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} \quad \forall t \geq 0\).
**The Homogeneous Heat Equation II**

*Proof.* Set

\[ X = L^2(\Omega), \quad A = \Delta, \quad D(A) = H^2(\Omega) \cap H^1_0(\Omega) \]

**Densely defined:** Obvious.

\((0, \infty) \subset \rho(A):\) Show \(\forall \lambda \in \mathbb{C}, \Re(\lambda) > 0: \quad \mathcal{R}(\lambda - A) = X.\)

**Equivalently:** For \(\lambda \in \mathbb{C}, \Re(\lambda) > 0\) the problem

\[-\Delta u + \lambda u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega\]

has a unique solution for every \(f \in L^2(\Omega).\)

**Apply** Lax-Milgram!
The Homogeneous Heat Equation III

Set

\[ B[u, v] \equiv \int_{\Omega} \nabla u \cdot \nabla v + \lambda u v \, dx, \quad u, v \in H^1_0(\Omega) \]

\[ F(v) \equiv \int_{\Omega} f v \, dx \quad u \in H^1_0(\Omega) \]

\[ \Rightarrow \quad B \text{ bounded and coercive BLF, } F \text{ bounded linear functional} \]

\[ \Rightarrow \quad \exists! u \in H^{1,2}_0(\Omega) \forall v \in H^1_0(\Omega) \]

USE: \( u \in H^2(\Omega) \longrightarrow \text{Regularity Theory.} \)

\[ \Rightarrow \quad u \in D(A). \]
The Homogeneous Heat Equation IV

∀λ > 0: ||R_λ|| ≤ λ⁻¹: Let λ > 0. Then

\[ \int_{\Omega} |\nabla u|^2 + \lambda |u|^2 \, dx = \int_{\Omega} f \bar{u} \, dx \leq \frac{1}{2\lambda} \int_{\Omega} |f|^2 \, dx + \frac{\lambda}{2} \int_{\Omega} |u|^2 \, dx \]

\[ \Rightarrow \frac{\lambda}{2} \int_{\Omega} |u|^2 \, dx \leq \frac{1}{2\lambda} \int_{\Omega} |f|^2 \, dx \Rightarrow \|u\|_{L^2} \leq \lambda^{-1} \|f\|_{L^2} \]

For \( u_1, u_2 \in D(A) \) solutions to the PDE

\[ -\Delta u + \lambda u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \]

we have \( u_1 = u_2 \).

\[ \Rightarrow R_\lambda \text{ exists, is continuous and } \|R_\lambda f\| \leq \lambda^{-1} \|f\| \]
The Homogeneous Heat Equation V

A is closed: Let \( x_n \in D(A) \), \( x_n \rightarrow x \) in \( D(A) \) and \( Ax_n \rightarrow y \) in \( X \), \( n \rightarrow \infty \).

Show: \( x \in D(A) \) and \( Ax = y \).

We know: For \( x + \lambda y \), there exists \( z \in D(A) \) such that \((\lambda I - A)z = \lambda x - y\)

\[
\|x_n - z\| \leq \lambda^{-1}\|(\lambda - A)(x_n - z)\| = \lambda^{-1}\|\lambda x_n - Ax_n - (\lambda x - y)\| \leq \|x_n - x\| + \lambda^{-1}\|Ax_n - y\| \rightarrow 0 \iff x = z, \ Ax = y
\]

Hille–Yosida \( \rightarrow \) A generates contraction semigroup \((S(t))_{t \geq 0}\).

\( t \mapsto S(t)u_0 \in D(A) \) continuous

\( t \mapsto u'(t) = AS(t)u_0 = S(t)Au_0 \in X \) continuous \( \rightarrow \) claim.
Second-order parabolic PDE

Recall that an elliptic differential operator of 2nd order in divergence form is given by

\[
Lu \equiv - \sum_{1 \leq i, j \leq d} \partial x_j(a^{ij}(x)\partial x_i u) + \sum_{1 \leq i \leq d} b^i(x)\partial x_i u + c(x)u
\]

where \( \zeta \cdot A(x)\zeta \geq \theta|\zeta|^2 \).

Assume: \( a^{ij}, b^i, c \in L^\infty(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \) for all \( i, j \leq d \) and they are not time-dependent. Moreover: \( \partial U \in C^\infty \). Consider the general parabolic equation

\[
(\partial_t + L)u = 0 \quad \text{in } U_T
\]
\[
u = 0 \quad \text{on } \partial U \times [0, T]
\]
\[
u = g \quad \text{on } U \times \{t = 0\}
\]
Then: $A$ defined by $Au \equiv -Lu$ for $u \in D(A) = H^{1,2}_0(U) \cap H^2(U)$ generates a $\gamma$-contraction semigroup! The problem is SOLVED.