Existence and stability of fully localised three-dimensional gravity-capillary solitary water waves

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In this paper we show that the hydrodynamic problem for three-dimensional water waves with strong surface-tension effects admits a *fully localised solitary wave* which decays to the undisturbed state of the water in every horizontal direction. The proof is based upon the classical variational principle that a solitary wave of this type is a critical point of the energy, which is given in dimensionless coordinates by

$$E(\eta,\phi)=\int_{\mathbb{R}^2}\left\{\frac{1}{2}\int_0^{1+\eta}(\phi_x^2+\phi_y^2+\phi_z^2)\,dy+\frac{1}{2}\eta^2+\beta[\sqrt{1+\eta_x^2+\eta_z^2}-1]\right\}\,dx\,dz,$$

subject to the constraint that the momentum

$$I(\eta,\phi)=\int_{\mathbb{R}^2}\eta_x\phi|_{y=1+\eta}\,dx\,dz$$

is fixed; here $\{(x,y,z):x,z\in\mathbb{R},\ y\in(0,1+\eta(x,z))\}$ is the fluid domain, ϕ is the velocity potential and $\beta>1/3$ is the Bond number. These functionals are studied locally for η in a neighbourhood of the origin in $H^3(\mathbb{R}^2)$.

We prove the existence of a minimiser of E subject to the constraint $I=2\mu$, where $0<\mu\ll 1$. The existence of a small-amplitude solitary wave is thus assured, and since E and I are both conserved quantities a standard argument may be used to establish the stability of the set D_μ of minimisers as a whole. ‘Stability’ is however understood in a qualified sense due to the lack of a global well-posedness theory for three-dimensional water waves. We show that solutions to the evolutionary problem starting near D_μ remain close to D_μ in a suitably defined energy space over their interval of existence; they may however explode in finite time due to higher-order derivatives becoming unbounded.