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PARTIAL REGULARITY FOR MINIMIZERS OF QUASI-CONVEX FUNCTIONALS WITH GENERAL GROWTH*

L. DIENING†, D. LENGELER‡, B. STROFFOLINI§, AND A. VERDE§

Abstract. We prove a partial regularity result for local minimizers of quasi-convex variational integrals with general growth. The main tool is an improved $\mathcal{A}$-harmonic approximation, which should be interesting also for classical growth.

Key words. quasi-convex, partial regularity, harmonic approximation, Lipschitz truncation

AMS subject classifications. 35J60, 35J70, 49N60, 26B25

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1. Introduction.

In this paper we study partial regularity for vector-valued minimizers $u : \Omega \to \mathbb{R}^N$ of variational integrals:

$$\mathcal{F}(u) := \int_\Omega f(\nabla u)\, dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a domain and $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is a continuous function.

Let us recall Morrey's notion of quasi convexity [28].

DEFINITION 1. *f is called quasi-convex if and only if*

$$\fint_{B_1} f(A + \nabla \boldsymbol{\xi})\, dx \geq f(A) \tag{1.2}$$

holds for every $A \in \mathbb{R}^{nN}$ and every smooth $\boldsymbol{\xi} : B_1 \to \mathbb{R}^N$ with compact support in the open unit ball B_1 in $\mathbb{R}^n$.

By Jensen's inequality, quasi convexity is a generalization of convexity. It was originally introduced as a notion for proving the lower semicontinuity and the existence of minimizers of variational integrals. In fact, assuming a power growth condition, quasi convexity is proved to be a necessary and sufficient condition for the sequential weak lower semicontinuity on $W^{1,p}(\Omega.\mathbb{R}^N)$, $p > 1$; see [26] and [1]. For general growth conditions, see [21] and [33]. In the regularity issue, a stronger definition comes into play. In the fundamental paper [20] Evans considered strictly quasi-convex integrands f in the quadratic case and proved that if f is of class C^2 and has bounded second derivatives, then any minimizing function $\mathbf{u}$ is of class $C^{1,\alpha}(\Omega \setminus \Sigma)$, where Σ has n-dimensional Lebesgue measure zero. In [1], this result was generalized to integrands f of p-growth with $p \geq 2$, while the subquadratic growth was considered in [7].

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†Institute of Mathematics, LMU Munich, 80333 Munich, Germany (diening@math.lmu.de). This author's work was partially supported by Gnampa.

‡Institute of Mathematics, University of Freiburg, 79104 Freiburg, Germany (daniel.lengeler@mathematik.uni-freiburg.de). This author's work was partially supported by the European Research Council under FP7, the ERC grant 226234 "Analytic Techniques for Geometric and Functional Inequalities."

§Dipartimento di Matematica, Università di Napoli, Federico II, Via Cintia, 80126 Naples, Italy (bstroffo@unina.it, anverde@unina.it). This third author's work was partially supported by PRIN project "Calcolo delle variazioni e Teoria Geometrica della Misura."

In order to treat the general growth case, we introduce the notion of the *strictly* $W^{1,\varphi}$*-quasi-convex* function, where φ is a suitable N-function; see Assumption 6 (see also [6]).

DEFINITION 2. *The function f is* strictly $W^{1,\varphi}$-quasi-convex *if and only if*

$$\int_{\mathrm{B}} f(\mathbf{Q}+\nabla\mathbf{w})-f(\mathbf{Q})\,dx \geq k\int_B \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{w}|)\,dx$$

for all balls $B\subset\Omega$, all $\mathbf{Q}\in\mathbb{R}^{N\times n}$ and all $\mathbf{w}\in C_0^1(B)$, where $\varphi_a(t)\sim\varphi''(a+t)\,t^2$ for $a,t\geq 0$. A precise definition of φ_a is given in section 2.

We will work with the following set of assumptions:

(H1) $f\in C^1(\mathbb{R}^n)\cap C^2(\mathbb{R}^n\setminus\{0\})$.

(H2) For all $\mathbf{Q}\in\mathbb{R}^{N\times n}$, it holds that

$$|f(\mathbf{Q})|\leq K\varphi(|\mathbf{Q}|).$$

(H3) The function f is *strictly $W^{1,\varphi}$-quasi-convex*.

(H4) For all $\mathbf{Q}\in\mathbb{R}^{N\times n}\setminus\{\mathbf{0}\}$,

$$\big|(D^2f)(\mathbf{Q})\big|\leq c\,\varphi''(|\mathbf{Q}|).$$

(H5) The following Hölder continuity of D^2f away from $\mathbf{0}$ holds for all $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{N\times n}$ such that $|\mathbf{P}|\leq\frac{1}{2}|\mathbf{Q}|$:

$$\big|D^2f(\mathbf{Q})-D^2f(\mathbf{Q}+\mathbf{P})\big|\leq c\,\varphi''(|\mathbf{Q}|)|\mathbf{Q}|^{-\beta}|\mathbf{P}|^{\beta}.$$

Due to (H2), $\mathcal{F}$ is well defined on the Sobolev–Orlicz space $W^{1,\varphi}(\Omega,\mathbb{R}^N)$; see section 2. Let us observe that assumption (H5) has been used to show everywhere regularity of radial functionals with φ-growth [13]. Following the argument given in [24] it is possible to prove that (H3) implies the following *strong Legendre–Hadamard condition*:

$$(D^2f)(\mathbf{Q})(\boldsymbol{\eta}\otimes\boldsymbol{\xi},\boldsymbol{\eta}\otimes\boldsymbol{\xi})\geq c\,\varphi''(|\mathbf{Q}|)|\boldsymbol{\eta}|^2|\boldsymbol{\xi}|^2$$

for all $\boldsymbol{\eta}\in\mathbb{R}^N$, $\boldsymbol{\xi}\in\mathbb{R}^n$, and $\mathbf{Q}\in\mathbb{R}^{N\times n}\setminus\{\mathbf{0}\}$. Furthermore, (H3) implies that the functional

$$\mathcal{J}(t):=\int_B f(\mathbf{Q}+t\nabla\mathbf{w})-f(\mathbf{Q})-k\varphi_{|\mathbf{Q}|}(t|\nabla\mathbf{w}|)\,dx$$

attains its minimal value at $t=0$. Hence $\mathcal{J}''(0)\geq 0$, that is,

$$\int_B (D^2f)(\mathbf{Q})(\nabla\mathbf{w},\nabla\mathbf{w})\,dx\geq k\int_B\varphi''_{|\mathbf{Q}|}(0)|\nabla\mathbf{w}|^2\,dx\geq c\,\varphi''(|\mathbf{Q}|)\int_B|\nabla\mathbf{w}|^2\,dx. \tag{1.3}$$

As usual, the strategy for proving partial regularity consists in showing an excess decay estimate, where the *excess* function is

$$\Phi_s(B,\mathbf{u}):=\bigg(\fint_{\mathrm{B}}|\mathbf{V}(\nabla\mathbf{u})-\langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^{2s}\,dx\bigg)^{\frac{1}{s}} \tag{1.4}$$

with $\mathbf{V}(\mathbf{Q}) = \sqrt{\frac{\varphi'(|\mathbf{Q}|)}{|\mathbf{Q}|}}\mathbf{Q}$ and $s \geq 1$. We write $\Phi := \Phi_1$. Note that $\Phi_{s_1}(B, \mathbf{u}) \leq \Phi_{s_2}(B, \mathbf{u})$ for $1 \leq s_1 \leq s_2$ and $|\mathbf{V}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}|)$.

Our regularity theorem states the following.

THEOREM 3 (main theorem). *Let* $\mathbf{u}$ *be a local minimizer of the quasi-convex functional* (1.1)*, with* f *satisfying* (H1)–(H5) and fix some $\beta \in (0,1)$. Then there exists $\delta = \delta(\beta) > 0$ such that the following holds: If

$$\Phi(2B, \mathbf{u}) \leq \delta \fint_{2\mathrm{B}} |\mathbf{V}(\nabla \mathbf{u})|^2 \, dx \tag{1.5}$$

for some ball $B \subset \mathbb{R}^n$ with $2B \subset \Omega$, then $\mathbf{V}(\nabla \mathbf{u})$ is β-Hölder continuous on B.

The proof of this theorem can be found at the end of section 6. We define the set of regular points $\mathcal{R}(\mathbf{u})$ by

$$\mathcal{R}(\mathbf{u}) = \big\{x_0 \in \Omega : \liminf_{r \to 0} \Phi(B(x_0, r), \mathbf{u}) = 0\big\}. \tag{1.6}$$

As an immediate consequence of Theorem 3 we have: The following.

COROLLARY 4. *Let* $\mathbf{u}$ *be as in Theorem* 3 *and let* $x_0 \in \mathcal{R}(\mathbf{u})$ *with* $\nabla \mathbf{u} \neq 0$*. Then for every* $\beta \in (0,1)$ *the function* $\mathbf{V}(\nabla \mathbf{u})$ *is* β*-Hölder continuous on a neighborhood of* x_0*.*

Note that the Hölder continuity of $\mathbf{V}(\nabla \mathbf{u})$ implies the Hölder continuity of $\nabla \mathbf{u}$ with a different exponent depending on φ. Consider, for example, the situation $\varphi(t) = t^p$ with $1 < p < \infty$. Therefore, β-Hölder continuity of $\mathbf{V}(\nabla \mathbf{u})$ implies for $p \leq 2$ that $\nabla \mathbf{u}$ is β-Hölder continuous and for $p > 2$ that $\nabla \mathbf{u}$ is $\beta \frac{2}{p}$-Hölder continuous.

The proofs of the regularity results for local minimizers in [20], [1], [7] are based on a blow-up technique originally developed by De Giorgi [8] and Almgren [3], [4] in the setting of the geometric measure theory, and by Giusti and Miranda for elliptic systems [23].

Another more recent approach for proving partial regularity for local minimizers is based on the so-called $\mathcal{A}$-harmonic approximation method. This technique has its origin in Simon's proof of the regularity theorem [32] (see also Allard [2]). The technique has been successfully applied in the framework of the geometric measure theory, and to obtain partial-regularity results for general elliptic systems in a series of papers by Duzaar, Grotowski, Kronz, and Mingione [17], [16], [18], [19] (see also [27] for a good survey on the subject). More precisely, we consider a bilinear form on $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ which is (strongly) elliptic in the sense of Legendre–Hadamard, i.e., if for all $\mathbf{a} \in \mathbb{R}^N, \mathbf{b} \in \mathbb{R}^n$ it holds that

$$\mathcal{A}_{ij}^{\alpha\beta} a^i b_\alpha a^j b_\beta \geq \kappa_{\mathcal{A}} |\mathbf{a}|^2 |\mathbf{b}|^2$$

for some $\kappa_{\mathcal{A}} > 0$. The method of $\mathcal{A}$-harmonic approximation consists in obtaining a good approximation of functions $\mathbf{u} \in W^{1,2}(B)$, which are *almost* $\mathcal{A}$*-harmonic* (in the sense of Theorem 14) by $\mathcal{A}$-harmonic functions $\mathbf{h} \in W^{1,2}(B)$ in both the L^2-topology and the weak topology of $W^{1,2}$. Let us recall that $\mathbf{h} \in W^{1,2}(B)$ is called $\mathcal{A}$-harmonic on B if

$$\int_B \mathcal{A}(D\mathbf{h}, D\boldsymbol{\eta}) \, dx = 0 \quad \text{For all } \boldsymbol{\eta} \in C_0^\infty(B) \tag{1.7}$$

holds. Here, in order to prove the result, we will follow the second approach.

As in the situations considered in the above-mentioned papers, the required approximate $\mathcal{A}$-harmonicity of a local minimizer $\mathbf{u} \in W^{1,\varphi}(\Omega \setminus \Sigma)$ is a consequence of the minimizing property and of the *smallness* of the *excess*.

Next, having proven the $\mathcal{A}$-harmonic approximation lemma and the corresponding approximate $\mathcal{A}$-harmonicity of the local minimizer $\mathbf{u}$, the other steps are quite standard. We prove a Caccioppoli-type inequality for minimizers $\mathbf{u}$, and thus we compare $\mathbf{u}$ with the $\mathcal{A}$-harmonic approximation $\mathbf{h}$ to obtain, via our Caccioppoli-type inequality, the desired excess decay estimate.

Thus, the main difficulty is to establish a suitable version of the $\mathcal{A}$-harmonic approximation lemma in this general setting. However, let us point out that our $\mathcal{A}$-harmonic approximation lemma differs also in the linear or p-growth situation from the classical one in [18]. First, we use a direct approach based on the Lipschitz truncation technique which requires no contradiction argument. This allows for a precise control of the constants, which will depend only on the Δ_2-condition for φ and its conjugate. In fact, we will apply the approximation lemma to the family of shifted N-functions that inherit the same Δ_2 constants of φ. Second, we are able to preserve the boundary values of our original function, so $\mathbf{u}-\mathbf{h}$ is a valid test function. Third, we show that $\mathbf{h}$ and $\mathbf{u}$ are close with respect to the gradients rather than just the functions. The main tools in the proof is a Lipschitz approximation of the Sobolev functions as in [12], [5]. However, since $\mathcal{A}$ is only strongly elliptic in the sense of Legendre–Hadamard, we will not be able to apply the Lipschitz truncation technique directly to our almost $\mathcal{A}$-harmonic function $\mathbf{u}$. Instead, we need to use duality and apply the Lipschitz truncation technique to the test functions.

Let us conclude by observing that here we are able to present a unified approach for both cases: superquadratic and subquadratic growth.

2. Notation and preliminary results. We use c, C as generic constants, which may change from line to line, but does not depend on the crucial quantities. Moreover we write $f \sim g$ if and only if there exist constants $c, C > 0$ such that $c\,f \le g \le C\,f$. For $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball $B \subset \mathbb{R}^n$ we define

$$\langle w \rangle_B := \fint_B w(x)\,dx := \frac{1}{|B|}\int_B w(x)\,dx, \tag{2.1}$$

where $|B|$ is the n-dimensional Lebesgue measure of B. For $\lambda > 0$ we denote by λB the ball with the same center as B but λ-times the radius. For $U, \Omega \subset \mathbb{R}^n$ we write $U \Subset \Omega$ if the closure of U is a compact subset of Ω.

The following definitions and results are standard in the context of N-functions; see, for example, [25], [30]. A real function $\varphi : \mathbb{R}^{\ge 0} \to \mathbb{R}^{\ge 0}$ is said to be an N-function if it satisfies the following conditions: $\varphi(0) = 0$ and there exists the derivative φ' of φ. This derivative is right continuous, nondecreasing, and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$. Especially, φ is convex.

We say that φ satisfies the Δ_2 condition if there exists $c > 0$ such that for all $t \ge 0$ it holds that $\varphi(2t) \le c\,\varphi(t)$. We denote the smallest possible constant by $\Delta_2(\varphi)$. Since $\varphi(t) \le \varphi(2t)$ the Δ_2 condition is equivalent to $\varphi(2t) \sim \varphi(t)$.

By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev–Orlicz spaces, i.e., $f \in L^\varphi$ if and only if $\int \varphi(|f|)\,dx < \infty$ and $f \in W^{1,\varphi}$ if and only if $f, \nabla f \in L^\varphi$. By $W^{1,\varphi}_0(\Omega)$ we denote the closure of $C^\infty_0(\Omega)$ in $W^{1,\varphi}(\Omega)$.

By $(\varphi')^{-1} : \mathbb{R}^{\ge 0} \to \mathbb{R}^{\ge 0}$ we denote the function

$$(\varphi')^{-1}(t) := \sup\{s \in \mathbb{R}^{\ge 0} : \varphi'(s) \le t\}.$$

If φ' is strictly increasing, then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^* : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\,ds$$

is again an N-function and $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$. It is the complementary function of φ. Note that $\varphi^*(t) = \sup_{s\geq 0}(st - \varphi(s))$ and $(\varphi^*)^* = \varphi$. For all $\delta > 0$ there exists c_δ (depending only on $\Delta_2(\varphi, \varphi^*)$ such that for all $t, s \geq 0$ it holds that

$$t\,s \leq \delta\,\varphi(t) + c_\delta\,\varphi^*(s). \tag{2.2}$$

For $\delta = 1$ we have $c_\delta = 1$. This inequality is called *Young's inequality*. For all $t \geq 0$

$$\begin{gathered} \frac{t}{2}\varphi'\Big(\frac{t}{2}\Big) \leq \varphi(t) \leq t\,\varphi'(t), \\ \varphi\bigg(\frac{\varphi^*(t)}{t}\bigg) \leq \varphi^*(t) \leq \varphi\bigg(\frac{2\,\varphi^*(t)}{t}\bigg). \end{gathered} \tag{2.3}$$

Therefore, uniformly in $t \geq 0$,

$$\varphi(t) \sim \varphi'(t)\,t, \qquad \varphi^*\big(\varphi'(t)\big) \sim \varphi(t), \tag{2.4}$$

where the constants depend only on $\Delta_2(\varphi, \varphi^*)$.

We say that an N-function ψ is of type (p_0, p_1) with $1 \leq p_0 \leq p_1 < \infty$ if

$$\psi(st) \leq C\,\max\{s^{p_0}, s^{p_1}\}\psi(t) \qquad \text{for all } s, t \geq 0. \tag{2.5}$$

We also write $\psi \in \mathfrak{T}(p_0, p_1, C)$.

Lemma 5. *Let ψ be an N-function with $\psi \in \Delta_2$ together with its conjugate. Then $\psi \in \mathfrak{T}(p_0, p_1, C_1)$ for some $1 < p_0 < p_1 < \infty$ and $C_1 > 0$, where p_0, p_1, and C_1 depend only on $\Delta_2(\psi, \psi^*)$. Moreover, ψ has the representation*

$$\psi(t) = t^{p_0}\big(h(t)\big)^{p_1 - p_0} \qquad \textit{for all } t \geq 0, \tag{2.6}$$

where h is a quasi-concave function, i.e.,

$$h(\lambda t) \leq C_2 \max\{1, \lambda\}h(t) \qquad \textit{for all } \lambda, t \geq 0,$$

where C_2 depends only on $\Delta_2(\psi, \psi^*)$.

Proof. Let $K := \Delta_2(\psi)$ and $K_* := \max\{\Delta_2(\psi^*), 3\}$. Then $\psi^*(2t) \leq K_*\psi^*(t)$ for all $t \geq 0$ implies $\psi(t) \leq K_*\psi(2t/K_*)$ for all $t \geq 0$. Now, choose p_0, p_1 such that $1 < p_0 < p_1 < \infty$ and $K \leq 2^{p_0}$ and $(K_*/2)^{p_0} \leq K_*$. We claim that

$$\psi(st) \leq C\,\max\{s^{p_0}, s^{p_1}\}\psi(t) \qquad \text{for all } s, t \geq 0, \tag{2.7}$$

where C depends only on K and K_*. Indeed, if $s \geq 1$, then choose $m \geq 0$ such that $2^m \leq s \leq 2^{m+1}$. Using $\psi \in \Delta_2$, we get

$$\psi(st) \leq \psi(2^{m+1}t) \leq K^{m+1}\psi(t) \leq K(2^{p_1})^m\psi(t) \leq K s^{p_1}\psi(t). \tag{2.8}$$

If $s \leq 1$, then we choose $m \in \mathbb{N}_0$ such that $(K_*/2)^m s \leq 1 \leq (K_*/2)^{m+1}s$, so that

$$\psi(st) \leq K_*^m \psi\Bigg(\bigg(\frac{2}{K_*}\bigg)^m st\Bigg) \leq K_*\bigg(\frac{K_*}{2}\bigg)^{p_0(m-1)}\psi(t) \leq K_* s^{p_0}\psi(t).$$

This proves (2.7).

Now, let us define

$$h(u) := \psi\Big(u^{\frac{1}{p_1-p_0}}\Big)u^{-\frac{p_0}{p_1-p_0}};$$

then ψ satisfies (2.6). It remains to show that h is quasi-concave. We estimate with (2.7)

$$h(su) \le K\,\psi\left(u^{\frac{1}{p_1-p_0}}\right)\max\left\{s^{\frac{p_1}{p_1-p_0}}, s^{\frac{p_0}{p_1-p_0}}\right\}(su)^{\frac{-p_0}{p_1-p_0}} = K\psi(u)\max\{s,1\}$$

for all $s, u \ge 0$. ∎

Throughout the paper we will assume that φ satisfies the following assumption.

ASSUMPTION 6. *Let φ be an N-function such that φ is C^1 on $[0,\infty)$ and C^2 on $(0,\infty)$. Further assume that*

$$\varphi'(t) \sim t\,\varphi''(t) \tag{2.9}$$

uniformly in $t > 0$. The constants in (2.9) *are called the* characteristics of φ*.*

We remark that under these assumptions $\Delta_2(\varphi,\varphi^*) < \infty$ will be automatically satisfied, where $\Delta_2(\varphi,\varphi^*)$ depends only on the characteristics of φ.

For given φ we define the associated N-function ψ by

$$\psi'(t) := \sqrt{\varphi'(t)\,t}\,. \tag{2.10}$$

It is shown in [9, Lemma 25] that if φ satisfies Assumption 6, then also φ^*, ψ, and ψ^* satisfy this assumption.

Define $\mathbf{A}, \mathbf{V} : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ in the following way:

$$\mathbf{A}(\mathbf{Q}) = \varphi'(|\mathbf{Q}|)\frac{\mathbf{Q}}{|\mathbf{Q}|}, \tag{2.11a}$$

$$\mathbf{V}(\mathbf{Q}) = \psi'(|\mathbf{Q}|)\frac{\mathbf{Q}}{|\mathbf{Q}|}. \tag{2.11b}$$

Another important set of tools are the shifted N-functions $\{\varphi_a\}_{a\ge 0}$ introduced in [9]; see also [11], [31]. We define for $t \ge 0$

$$\varphi_a(t) := \int_0^t \varphi_a'(s)\,ds \qquad \text{with} \qquad \varphi_a'(t) := \varphi'(a+t)\frac{t}{a+t}. \tag{2.12}$$

Note that $\varphi_a(t) \sim \varphi_a'(t)\,t$. Moreover, for $t \ge a$ we have $\varphi_a(t) \sim \varphi(t)$ and for $t \le a$ we have $\varphi_a(t) \sim \varphi''(a)t^2$. This implies that $\varphi_a(s\,t) \le c\,s^2\varphi_a(t)$ for all $s \in [0,1]$, $a \ge 0$ and $t \in [0,a]$. The families $\{\varphi_a\}_{a\ge 0}$ and $\{(\varphi_a)^*\}_{a\ge 0}$ satisfy the Δ_2 condition uniformly in $a \ge 0$.

The connection between $\mathbf{A}$, $\mathbf{V}$, and the shifted functions of φ is best reflected in the following lemma [13, Lemma 2.4]; see also [9].

LEMMA 7. *Let φ satisfy Assumption* 6 *and let* $\mathbf{A}$ *and* $\mathbf{V}$ *be defined by* (2.11)*. Then*

$$\begin{aligned}\big(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big)\cdot\big(\mathbf{P}-\mathbf{Q}\big) \sim \big|\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})\big|^2 &\sim \varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|),\\ \big|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big| &\sim \varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|),\end{aligned}$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ *. Moreover,*

$$\mathbf{A}(\mathbf{Q}) \cdot \mathbf{Q} \sim |\mathbf{V}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}|),$$

uniformly in $\mathbf{Q} \in \mathbb{R}^{N\times n}$. We state a generalization of Lemma 2.1 in [1] to the context of convex functions φ.

LEMMA 8. [9, Lemma 20] *Let* φ *be an N-function with* $\Delta_2(\varphi, \varphi^*) < \infty$. *Then uniformly for all* $\mathbf{P}_0, \mathbf{P}_1 \in \mathbb{R}^{N\times n}$ *with* $|\mathbf{P}_0| + |\mathbf{P}_1| > 0$ *it holds that*

$$\int_0^1 \frac{\varphi'(|\mathbf{P}_\theta|)}{|\mathbf{P}_\theta|}\, d\theta \sim \frac{\varphi'(|\mathbf{P}_0| + |\mathbf{P}_1|)}{|\mathbf{P}_0| + |\mathbf{P}_1|}, \tag{2.13}$$

where $\mathbf{P}_\theta := (1-\theta)\mathbf{P}_0 + \theta\mathbf{P}_1$. *The constants depend only on* $\Delta_2(\varphi, \varphi^*)$.

Note that (H5) and the previous lemma imply that

$$\begin{aligned} \big|(Df)(\mathbf{Q}) - (Df)(\mathbf{P})\big| &= \left|\int_0^1 (D^2 f)(\mathbf{P} + t(\mathbf{Q}-\mathbf{P}))(\mathbf{Q}-\mathbf{P})\, dt\right| \\ &\le c \int_0^1 \varphi''(|\mathbf{P} + t(\mathbf{Q}-\mathbf{P}))|)\, dt|\mathbf{P}-\mathbf{Q}| \\ &\le c\,\varphi''(|\mathbf{P}| + |\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}| \\ &\le c\,\varphi'_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|). \end{aligned} \tag{2.14}$$

The following version of the Sobolev–Poincaré inequality can be found in [9, Lemma 7].

THEOREM 9 (Sobolev–Poincaré). *Let* φ *be an N-function with* $\Delta_2(\varphi, \varphi^*) < \infty$. *Then there exist* $0 < \alpha < 1$ *and* $K > 0$ *such that the following holds. If* $B \subset \mathbb{R}^n$ *is some ball with radius* R *and* $\mathbf{w} \in W^{1,\varphi}(B, \mathbb{R}^N)$, *then*

$$⨍_{\mathrm{B}} \varphi\left(\frac{|\mathbf{w} - \langle \mathbf{w}\rangle_B|}{R}\right) dx \le K \left(⨍_{\mathrm{B}} \varphi^\alpha(|\nabla \mathbf{w}|)\, dx\right)^{\frac{1}{\alpha}}, \tag{2.15}$$

where $\langle \mathbf{w}\rangle_B := ⨍_{\mathrm{B}} \mathbf{w}(x)\, dx$.

3. Caccioppoli estimate. We need the following simple modification of Lemma 3.1 [22, Chapter 5].

LEMMA 10. *Let* ψ *be an N-function with* $\psi \in \Delta_2$, *let* $r > 0$, *and let* $h \in L^\psi(B_{2r}(x_0))$. *Further, let* $f : [r/2, r] \to [0, \infty)$ *be a bounded function such that for all* $\frac{r}{2} < s < t < r$

$$f(s) \le \theta f(t) + A \int_{B_t(x_0)} \psi\left(\frac{|h(y)|}{t-s}\right) dy,$$

where $A > 0$ *and* $\theta \in [0, 1)$. *Then*

$$f\left(\frac{r}{2}\right) \le c(\theta, \Delta_2(\psi))\, A \int_{B_{2r}(x_0)} \psi\left(\frac{|h(y)|}{2r}\right) dy.$$

Proof. Since $\psi \in \Delta_2$, there exist $C_2 > 0$ and $p_1 < \infty$ (both depending only on $\Delta_2(\psi)$) such that $\psi(\lambda u) \le C_2 \lambda^{p_1} \psi(u)$ for all $\lambda \ge 1$ and $u \ge 0$ (compare (2.8) of Lemma 5). This implies

$$f(t) \le \theta f(s) + A \int_{B_s(x_0)} \psi\left(\frac{|h(y)|}{2r}\right) dy\; C_2 (2r)^{p_1}\, (t-s)^{-p_1}.$$

Now Lemma 3.1 in [22] with $\alpha := p_1$ implies

$$f\left(\frac{r}{2}\right) \leq c(\theta, p_1) A \int_{B_s(x_0)} \psi\left(\frac{|h(y)|}{2r}\right) dy\, C_2 (2r)^{p_1} r^{-p_1},$$

which proves the claim. ∎

THEOREM 11. *Let $\mathbf{u} \in W^{1,\varphi}_{loc}(\Omega)$ be a local minimizer of $\mathcal{F}$ and let B be a ball with radius R such that $2B \Subset \Omega$. Then*

$$\int_B \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)\, dx \leq c \int_{2B} \varphi_{|\mathbf{Q}|}\left(\frac{|\mathbf{u} - \mathbf{q}|}{R}\right) dx$$

for all $\mathbf{Q} \in \mathbb{R}^{N\times n}$ and all linear polynomials $\mathbf{q}$ on $\mathbb{R}^n$ with values in $\mathbb{R}^N$ and $\nabla \mathbf{q} = \mathbf{Q}$, where c depends only on n, N, k, K, and the characteristics of φ.

Proof. Let $0 < s < t$. Further, let B_s and B_t be balls in Ω with the same center and with radii s and t, respectively. Choose $\eta \in C_0^\infty(B_t)$ with $\chi_{B_s} \leq \eta \leq \chi_{B_t}$ and $|\nabla \eta| \leq c/(t-s)$. Now, define $\boldsymbol{\xi} := \eta(\mathbf{u} - \mathbf{q})$ and $\mathbf{z} := (1-\eta)(\mathbf{u} - \mathbf{q})$. Then $\nabla \boldsymbol{\xi} + \nabla \mathbf{z} = \nabla \mathbf{u} - \mathbf{Q}$. Consider

$$I := \int_{B_t} f(\mathbf{Q} + \nabla \boldsymbol{\xi}) - f(\mathbf{Q})\, dx.$$

Then by the quasi convexity of f (see (H3)), it follows that

$$I \geq c \int_{B_t} \varphi_{|\mathbf{Q}|}(|\nabla \boldsymbol{\xi}|)\, dx.$$

On the other hand, since $\nabla \boldsymbol{\xi} + \nabla \mathbf{z} = \nabla \mathbf{u} - \mathbf{Q}$ we get

$$\begin{aligned} I &= \int_{B_t} f(\mathbf{Q} + \nabla \boldsymbol{\xi}) - f(\mathbf{Q})\, dx \\ &= \int_{B_t} f(\mathbf{Q} + \nabla \boldsymbol{\xi}) - f(\mathbf{Q} + \nabla \boldsymbol{\xi} + \nabla \mathbf{z})\, dx \\ &\quad + \int_{B_t} f(\nabla \mathbf{u}) - f(\nabla \mathbf{u} - \nabla \boldsymbol{\xi})\, dx \\ &\quad + \int_{B_t} f(\mathbf{Q} + \nabla \mathbf{z}) - f(\mathbf{Q})\, dx \\ &=: II + III + IV. \end{aligned}$$

Since $\mathbf{u}$ is a local minimizer, we know that $(III) \leq 0$. Moreover,

$$\begin{aligned} II + IV &= \int_{B_t} \int_0^1 \big((Df)(\mathbf{Q} + t\nabla \mathbf{z}) - (Df)(\mathbf{Q} + \nabla \boldsymbol{\xi} - t\nabla \mathbf{z})\big) \nabla \mathbf{z}\, dt\, dx \\ &= \int_{B_t} \int_0^1 \big((Df)(\mathbf{Q} + t\nabla \mathbf{z}) - (Df)(\mathbf{Q}) \nabla \mathbf{z}\, dt\, dx \\ &\quad - \int_{B_t} \int_0^1 \big((Df)(\mathbf{Q} + \nabla \boldsymbol{\xi} - t\nabla \mathbf{z}) - (Df)(\mathbf{Q})\big) \nabla \mathbf{z}\, dt\, dx. \end{aligned}$$

This proves

$$\begin{aligned} |II| + |IV| &\leq c \int_{B_t} \int_0^1 \varphi'_{|\mathbf{Q}|}(t|\nabla \mathbf{z}|)\, dt |\nabla \mathbf{z}|\, dx \\ &\quad + c \int_{B_t} \int_0^1 \varphi'_{|\mathbf{Q}|}(|\nabla \boldsymbol{\xi} - t\nabla \mathbf{z}|)\, dt |\nabla \mathbf{z}|\, dx. \end{aligned}$$

Using $\varphi'_{|\mathbf{Q}|}(|\nabla\boldsymbol{\xi} - t\nabla\mathbf{z}|) \le c\,\varphi'_{|\mathbf{Q}|}(|\nabla\boldsymbol{\xi}|) + c\,\varphi'_{|\mathbf{Q}|}(|\mathbf{z}|)$, we get

$$\begin{aligned} |II| + |IV| &\le c \int_{B_t} \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{z}|)\,dx + c\int_{B_t} \varphi'_{|\mathbf{Q}|}(|\nabla\boldsymbol{\xi}|)|\nabla\mathbf{z}|\,dx \\ &\le c\int_{B_t} \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{z}|)\,dx + \frac{1}{2}(I), \end{aligned}$$

where we have used Young's inequality in the last step. Overall, we have shown the a priori estimate

$$\int_{B_t} \varphi_{|\mathbf{Q}|}(|\nabla\boldsymbol{\xi}|)\,dx \le c\int_{B_t} \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{z}|)\,dx. \tag{3.1}$$

Note that $\nabla\mathbf{z} = (1-\eta)(\nabla\mathbf{u} - \mathbf{Q}) - \nabla\eta(\mathbf{u} - \mathbf{q})$, which is zero outside $B_t \setminus B_s$. Hence,

$$\int_{B_t} \varphi_{|\mathbf{Q}|}(|\nabla\boldsymbol{\xi}|)\,dx \le c\int_{B_t\setminus B_s} \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\,dx + c\int_{B_t} \varphi_{|\mathbf{Q}|}\left(\frac{|\mathbf{u}-\mathbf{q}|}{t-s}\right)dx.$$

Since $\eta = 1$ on B_s, we get

$$\int_{B_s} \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\,dx \le c\int_{B_t\setminus B_s} \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\,dx + c\int_{B_t} \varphi_{|\mathbf{Q}|}\left(\frac{|\mathbf{u}-\mathbf{q}|}{t-s}\right)dx.$$

The hole-filling technique proves

$$\int_{B_s} \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\,dx \le \lambda\int_{B_t} \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\,dx + c\int_{B_t} \varphi_{|\mathbf{Q}|}\left(\frac{|\mathbf{u}-\mathbf{q}|}{t-s}\right)dx$$

for some $\lambda \in (0,1)$, which is independent of $\mathbf{Q}$ and $\mathbf{q}$. Now Lemma 10 proves the claim. ∎

Corollary 12. *There exists $0 < \alpha < 1$ such that for all local minimizers $\mathbf{u} \in W^{1,\varphi}_{loc}(\Omega)$ of $\mathcal{F}$, all balls B with $2B \Subset \Omega$, and all $\mathbf{Q} \in \mathbb{R}^{N\times n}$*

$$\fint_{B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,dx \le c\left(\fint_{2B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^{2\alpha}\,dx\right)^{\frac{1}{\alpha}}.$$

Proof. Apply Theorem 11 with $\mathbf{q}$ such that $\langle\mathbf{u} - \mathbf{q}\rangle_{2B} = 0$. Then use Theorem 9 with $\mathbf{w}(x) = \mathbf{u}(x) - \mathbf{Q}x$. ∎

Using Gehring's lemma, we deduce the following assertion.

Corollary 13. *There exists $s_0 > 1$ such that for all local minimizers $\mathbf{u} \in W^{1,\varphi}_{loc}(\Omega)$ of $\mathcal{F}$, all balls B with $2B \Subset \Omega$, and all $\mathbf{Q} \in \mathbb{R}^{N\times n}$*

$$\left(\fint_{B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^{2s_0}\,dx\right)^{\frac{1}{s_0}} \le c\fint_{2B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,dx.$$

4. The $\mathcal{A}$-harmonic approximation.

In this section we present a generalization of the $\mathcal{A}$-harmonic approximation lemma in Orlicz spaces. Basically it says that if a function locally "almost" behaves like an $\mathcal{A}$-harmonic function, then it is close to an $\mathcal{A}$-harmonic function. The proof is based on the Lipschitz truncation technique, which goes back to Acerbi and Fusco [1] but has been refined by many others.

Originally the closeness of the function to its $\mathcal{A}$-harmonic approximation was stated in terms of the L^2-distance and later for the nonlinear problems in terms of the L^p-distance. Based on a refinement of the Lipschitz truncation technique [12], it has been shown in [14] that also the distance in terms of the gradients is small.

Let us consider the elliptic system

$$-\partial_\alpha(\mathcal{A}_{ij}^{\alpha\beta} D_\beta u^j) = -\partial_\alpha H_i^\alpha \quad \text{in } B,$$

where $\alpha, \beta = 1, \dots, n$ and $i, j = 1, \dots, N$. We use the convention that repeated indices are summed. In short we write $-\mathrm{div}(\mathcal{A}\nabla\mathbf{u}) = -\mathrm{div}\mathbf{G}$. We assume that $\mathcal{A}$ is constant. We say that $\mathcal{A}$ is *strongly elliptic in the sense of Legendre–Hadamard* if for all $\mathbf{a} \in \mathbb{R}^N, \mathbf{b} \in \mathbb{R}^n$ it holds that

$$\mathcal{A}_{ij}^{\alpha\beta} a^i b_\alpha a^j b_\beta \geq \kappa_{\mathcal{A}} |\mathbf{a}|^2 |\mathbf{b}|^2$$

for some $\kappa_{\mathcal{A}} > 0$. The biggest possible constant $\kappa_{\mathcal{A}}$ is called the ellipticity constant of $\mathcal{A}$. By $|\mathcal{A}|$ we denote the Euclidean norm of $\mathcal{A}$. We say that a Sobolev function $\mathbf{w}$ on a ball B is $\mathcal{A}$-harmonic if it satisfies $-\mathrm{div}(\mathcal{A}\nabla\mathbf{w}) = 0$ in the sense of distributions.

Given a Sobolev function $\mathbf{u}$ on a ball B, we want to find an $\mathcal{A}$-harmonic function $\mathbf{h}$ which is close the our function $\mathbf{u}$. The way to find $\mathbf{h}$ is very simple: it will be the $\mathcal{A}$-harmonic function with the same boundary values as $\mathbf{u}$. In particular, we want to find a Sobolev function $\mathbf{h}$ which satisfies

$$\begin{aligned} -\mathrm{div}(\mathcal{A}\nabla\mathbf{h}) &= 0 && \text{on } B, \\ \mathbf{h} &= \mathbf{u} && \text{on } \partial B \end{aligned} \tag{4.1}$$

in the sense of distributions.

Let $\mathbf{w} := \mathbf{h} - \mathbf{u}$; then (4.1) is equivalent to finding a Sobolev function $\mathbf{w}$ which satisfies

$$\begin{aligned} -\mathrm{div}(\mathcal{A}\nabla\mathbf{w}) &= -\mathrm{div}(\mathcal{A}\nabla\mathbf{u}) && \text{on } B, \\ \mathbf{w} &= \mathbf{0} && \text{on } \partial B \end{aligned} \tag{4.2}$$

in the sense of distributions.

Our main approximation result is the following.

THEOREM 14. *Let $B \Subset \Omega$ be a ball with radius r_B and let $\widetilde{B} \subset \Omega$ denote either B or $2B$. Let $\mathcal{A}$ be strongly elliptic in the sense of Legendre–Hadamard. Let ψ be an N-function with $\Delta_2(\psi, \psi^*) < \infty$ and let $s > 1$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ depending only on n, N, κ_A, $|\mathcal{A}|$, $\Delta_2(\psi, \psi^*)$, and s such that the following holds: let $\mathbf{u} \in W^{1,\psi}(\widetilde{B})$ be* almost $\mathcal{A}$-harmonic *on B in the sense that*

$$\left| \fint_B \mathcal{A}\nabla\mathbf{u} \cdot \nabla\boldsymbol{\xi}\, dx \right| \leq \delta \fint_{\widetilde{B}} |\nabla\mathbf{u}|\, dx\, \|\nabla\boldsymbol{\xi}\|_{L^\infty(B)} \tag{4.3}$$

for all $\boldsymbol{\xi} \in C_0^\infty(B)$. Then the unique solution $\mathbf{w} \in W_0^{1,\psi}(B)$ of (4.2) satisfies

$$\fint_B \psi\left(\frac{|\mathbf{w}|}{r_B}\right) dx + \fint_B \psi(|\nabla\mathbf{w}|)\, dx \leq \varepsilon \left(\left(\fint_B \left(\psi(|\nabla\mathbf{u}|)\right)^s dx \right)^{\frac{1}{s}} + \fint_{\widetilde{B}} \psi(|\nabla\mathbf{u}|)\, dx \right). \tag{4.4}$$

The proof of this theorem can be found at the end of this section. The distinction between B and $\tilde{B}$ on the right-hand side of (4.4) allows a finer tuning with respect to the exponents. If $B = \tilde{B}$, then only the term involving s is needed.

The following result on the solvability and uniqueness in the setting of classical Sobolev spaces $W_0^{1,q}(B,\mathbb{R}^N)$ can be found in [15, Lemma 2].

LEMMA 15. *Let $B \Subset \Omega$ be a ball, let $\mathcal{A}$ be strongly elliptic in the sense of Legendre–Hadamard, and let $1 < q < \infty$. Then for every $\mathbf{G} \in L^q(B,\mathbb{R}^{N\times n})$, there exists a unique weak solution $\mathbf{u} = T_{\mathcal{A}}\mathbf{G} \in W_0^{1,q}(B,\mathbb{R}^N)$ of*

$$\begin{aligned} -\operatorname{div}(\mathcal{A}\nabla\mathbf{u}) &= -\operatorname{div}\mathbf{G} \qquad && \text{on } B,\\ \mathbf{u} &= \mathbf{0} && \text{on } \partial B. \end{aligned} \tag{4.5}$$

The solution operator $T_{\mathcal{A}}$ is linear and satisfies

$$\|\nabla T_{\mathcal{A}}\mathbf{G}\|_{L^q(B)} \le c\,\|\mathbf{G}\|_{L^q(B)},$$

where c depends only on n, N, $\kappa_{\mathcal{A}}$, $|\mathcal{A}|$, and q.

Remark 16. *Note that our constants do not depend on the size of the ball, since the estimates involved are scaling invariant.*

Let $T_{\mathcal{A}}$ be the solution operator of Lemma 15. Then by the uniqueness of Lemma 15, the operator $T_{\mathcal{A}} : L^q(B,\mathbb{R}^{N\times n}) \to W_0^{1,q}(B,\mathbb{R}^N)$ does not depend on the choice of $q \in (1,\infty)$. Therefore, $T_{\mathcal{A}}$ is uniquely defined from $\bigcup_{1<q<\infty} L^q(B,\mathbb{R}^{N\times n})$ to $\bigcup_{1<q<\infty} W_0^{1,q}(B,\mathbb{R}^N)$.

We need to extend Lemma 15 to the setting of Orlicz spaces. We will do so by means of the following real interpolation theorem of Peetre [29, Theorem 5.1] which states that whenever ψ is of the form (2.6), then L^ψ is an interpolation space between L^{p_0} and L^{p_1}.

THEOREM 17. *Let ψ be an N-function with $\Delta_2(\psi,\psi^*)$ and let p_0, p_1 be as in Lemma 5. Moreover let S be a linear, bounded operator from $L^{p_j} \to L^{p_j}$ for $j = 0, 1$. Then there exist K_2, which depends only on $\Delta_2(\psi,\psi^*)$, and the operator norms of S such that*

$$\begin{gathered} \|Sf\|_\psi \le K_2 \|f\|_\psi,\\ \int \psi(|Sf|/K_2)\,d\mu \le \int \psi(|f|)\,d\mu \end{gathered}$$

for every $f \in L^\psi$.

This interpolation result and Lemma 15 immediately imply the following.

THEOREM 18. *Let $B \subset \Omega$ be a ball, let $\mathcal{A}$ be strongly elliptic in the sense of Legendre–Hadamard, and let ψ be an N-function with $\Delta_2(\psi,\psi^*)$. Then the solution operator $T_{\mathcal{A}}$ of Lemma 15 is continuous from $L^\psi(B,\mathbb{R}^{N\times n})$ to $W_0^{1,\psi}(B,\mathbb{R}^n)$ and*

$$\begin{gathered} \|\nabla T_{\mathcal{A}}\mathbf{G}\|_{L^\psi(B)} \le c\,\|\mathbf{G}\|_{L^\psi(B)},\\ \int_B \psi(|\nabla T_{\mathcal{A}}\mathbf{G}|)\,dx \le c\int_B \psi(|\mathbf{G}|)\,dx \end{gathered} \tag{4.6}$$

for all $\mathbf{G} \in L^\psi(B,\mathbb{R}^{N\times n})$, where c depends only on n, N, $\kappa_{\mathcal{A}}$, $|\mathcal{A}|$, $\Delta_2(\psi,\psi^*)$.

Remark 19. *Since ψ satisfies* (2.5) *for some $1 < p_0 < p_1 < \infty$ it follows easily that $L^\psi(B) \hookrightarrow L^{p_0}(B)$ for every ball $B \subset \Omega$. From this and the uniqueness in Lemma* 15, *the solution of* (4.5) *is also unique in $W_0^{1,\psi}(B,\mathbb{R}^N)$.*

Since $\mathcal{A}$ is only strongly elliptic in the sense of Legendre–Hadamard, we will not be able to apply the Lipschitz truncation technique directly to our almost $\mathcal{A}$-harmonic function $\mathbf{u}$. Instead, we need to use duality and apply the Lipschitz truncation technique to the test functions. For this reason, we prove the following variational inequality.

LEMMA 20. *Let $B \subset \Omega$ be a ball and let $\mathcal{A}$ be strongly elliptic in the sense of Legendre–Hadamard. Then it holds for all $\mathbf{u} \in W_0^{1,\psi}(B)$ that*

$$\|\nabla \mathbf{u}\|_\psi \sim \sup_{\substack{\boldsymbol{\xi} \in C_0^\infty(B) \\ \|\nabla \boldsymbol{\xi}\|_{\psi^*} \le 1}} \int_B \mathcal{A}\nabla \mathbf{u} \cdot \nabla \boldsymbol{\xi}\, dx, \tag{4.7a}$$

$$\int_B \psi(|\nabla \mathbf{u}|)\, dx \sim \sup_{\boldsymbol{\xi} \in C_0^\infty(B)} \left[\int_B \mathcal{A}\nabla \mathbf{u} \cdot \nabla \boldsymbol{\xi}\, dx - \int_B \psi^*(|\nabla \boldsymbol{\xi}|)\, dx \right]. \tag{4.7b}$$

The implicit constants depend only on n, N, $\kappa_{\mathcal{A}}$, $|\mathcal{A}|$, $\Delta_2(\psi, \psi^*)$.

Proof. We begin with the proof of (4.7a). The $\gtrsim$ estimate is a simple consequence of Hölder's inequality, so let us concentrate on $\lesssim$. Since $(L^\psi)^* \cong L^{(\psi^*)}$ (with constants bounded by 2) and $C_0^\infty(B)$ is dense in $L^{(\psi^*)}(\Omega)$, we have

$$\|\nabla \mathbf{u}\|_\psi \le 2 \sup_{\substack{\mathbf{H} \in C_0^\infty(B, \mathbb{R}^{N \times n}) \\ \|\mathbf{H}\|_{\psi^*} \le 1}} \int_B \nabla \mathbf{u} \cdot \mathbf{H}\, dx.$$

Define $\overline{\mathcal{A}}$ by $\overline{\mathcal{A}}_{ij}^{\alpha\beta} := \mathcal{A}_{ji}^{\beta\alpha}$; then $-\mathrm{div}(\overline{\mathcal{A}}\nabla \mathbf{u})$ is the formal adjoint operator of $-\mathrm{div}(\mathcal{A}\nabla \mathbf{u})$. In particular, using (4.5)

$$\begin{aligned} \int_B \nabla \mathbf{u} \cdot \mathbf{H} &= \int_B \nabla \mathbf{u} \cdot \overline{\mathcal{A}} \nabla T_{\overline{\mathcal{A}}} \mathbf{H}\, dx \\ &= \int_B \mathcal{A}\nabla \mathbf{u} \cdot \nabla T_{\overline{\mathcal{A}}} \mathbf{H}\, dx. \end{aligned} \tag{4.8}$$

Hence,

$$\begin{aligned} \|\nabla \mathbf{u}\|_\psi &\le 2 \sup_{\substack{\mathbf{H} \in C_0^\infty(B, \mathbb{R}^{N \times n}) \\ \|\mathbf{H}\|_{\psi^*} \le 1}} \int_B \mathcal{A}\nabla \mathbf{u} \cdot \nabla T_{\overline{\mathcal{A}}} \mathbf{H}\, dx \\ &\le 4 \sup_{\substack{\mathbf{H} \in C_0^\infty(B, \mathbb{R}^{N \times n}) \\ \|\mathbf{H}\|_{\psi^*} \le 1}} \|\mathcal{A}\nabla \mathbf{u}\|_{L^\psi(B)} \|\nabla T_{\overline{\mathcal{A}}} \mathbf{H}\|_{\psi^*} \\ &\le c\, \|\mathcal{A}\nabla \mathbf{u}\|_{L^\psi(B)}, \end{aligned}$$

where we used Theorem 18 (for $T_{\overline{\mathcal{A}}}$ and ψ^*) in the last step of the estimate. This proves (4.7a).

Let us now prove (4.7b). The estimate $\gtrsim$ just follows from

$$\begin{aligned} \int_B \mathcal{A}\nabla \mathbf{u} \cdot \nabla \boldsymbol{\xi}\, dx - \int_B \psi^*(|\nabla \boldsymbol{\xi}|)\, dx &\le \int_B \psi(|\mathcal{A}||\nabla \mathbf{u}|)\, dx \\ &\le c(|\mathcal{A}|) \int_B \psi(|\nabla \mathbf{u}|)\, dx, \end{aligned}$$

where we used $|\mathcal{A}\nabla \mathbf{u} \cdot \nabla \boldsymbol{\xi}| \le |\mathcal{A}||\nabla \mathbf{u}||\nabla \boldsymbol{\xi}|$, Young's inequality, and $\psi \in \Delta_2$.

We turn to $\lesssim$ of (4.7b). Recall that

$$\psi^{**}(t) = \psi(t) = \sup_{u \ge 0} \big(ut - \psi^*(u)\big),$$

where the supremum is attained at $u = \psi'(t)$. Thus the choice $\mathbf{H} := \psi'(|\nabla \mathbf{u}|)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}$ (with $\mathbf{H} = \mathbf{0}$ where $\nabla \mathbf{u} = \mathbf{0}$) implies

$$\int_B \psi(|\nabla \mathbf{u}|)\,dx \le \sup_{\mathbf{H}\in(L^{\psi^*}(B,\mathbb{R}^{N\times n}))} \bigg[\int_B \nabla \mathbf{u}\cdot \mathbf{H}\,dx - \int_B \psi^*(|\mathbf{H}|)\,dx\bigg].$$

Using $T_{\overline{\mathcal{A}}}$ we estimate with (4.8)

$$\int_B \psi(|\nabla \mathbf{u}|)\,dx \le \sup_{\mathbf{H}\in L^{\psi^*}(B,\mathbb{R}^{N\times n})} \bigg[\int_B \mathcal{A}\nabla \mathbf{u}\cdot \nabla T_{\overline{\mathcal{A}}}\mathbf{H}\,dx - \int_B \psi^*(|\mathbf{H}|)\,dx\bigg].$$

By Theorem 18 there exists $c \ge 1$ such that

$$\int_B \psi^*(|\nabla T_{\mathcal{A}}\mathbf{H}|)\,dx \le c\int_B \psi^*(|\mathbf{H}|)\,dx.$$

This proves the following:

$$\begin{aligned}\int_B \psi(|\nabla \mathbf{u}|)\,dx &\le \sup_{\mathbf{H}\in(L^{\psi^*}(B,\mathbb{R}^{N\times n})} \bigg[\int_B \mathcal{A}\nabla \mathbf{u}\cdot \nabla T_{\overline{\mathcal{A}}}\mathbf{H}\,dx - c\int_B \psi^*(|\nabla T_{\overline{\mathcal{A}}}\mathbf{H}|)\,dx\bigg]\\ &\le \sup_{\boldsymbol{\xi}\in L^{\psi^*}(B,\mathbb{R}^{N})} \bigg[\int_B \mathcal{A}\nabla \mathbf{u}\cdot \nabla \boldsymbol{\xi}\,dx - c\int_B \psi^*(|\nabla \boldsymbol{\xi}|)\,dx\bigg].\end{aligned}$$

We replace $\mathbf{u}$ by $c\mathbf{u}$ to get

$$\int_B \psi(c|\nabla \mathbf{u}|)\,dx \le c\sup_{\boldsymbol{\xi}\in L^{\psi^*}(B,\mathbb{R}^{N})} \bigg[\int_B \mathcal{A}\nabla \mathbf{u}\cdot \nabla \boldsymbol{\xi}\,dx - \int_B \psi^*(|\nabla \boldsymbol{\xi}|)\,dx\bigg].$$

Now the claim follows using $\psi \in \Delta_2$ on the left-hand side and the density of $C_0^\infty(B,\mathbb{R}^N)$ in $L^{\psi^*}(B,\mathbb{R}^N)$ (using $\psi^* \in \Delta_2$). ◻

Moreover, we need the following result of [14, Theorem 3.3] about *Lipschitz truncations* in Orlicz spaces.

THEOREM 21 (Lipschitz truncation). *Let $B \subset \Omega$ be a ball and let ψ be an N-function with $\Delta_2(\psi,\psi^*) < \infty$. If $\mathbf{w} \in W_0^{1,\psi}(B,\mathbb{R}^N)$, then for every $m_0 \in \mathbb{N}$ and $\gamma > 0$ there exist $\lambda \in [\gamma, 2^{m_0}\gamma]$ and $\mathbf{w}_\lambda \in W_0^{1,\infty}(B,\mathbb{R}^N)$ (called the Lipschitz truncation) such that*

$$\begin{gathered}\|\nabla \mathbf{w}_\lambda\|_\infty \le c\,\lambda,\\ \fint_B \psi\big(|\nabla \mathbf{w}_\lambda|\chi_{\{\mathbf{w}_\lambda\neq \mathbf{w}\}}\big)\,dx \le c\,\psi(\lambda)\frac{|\{\mathbf{w}_\lambda \neq \mathbf{w}\}|}{|B|} \le \frac{c}{m_0}\fint_B \psi(|\nabla \mathbf{w}|)\,dx\\ \fint_B \psi\big(|\nabla \mathbf{w}_\lambda|\big)\,dx \le c\fint_B \psi\big(|\nabla \mathbf{w}|\big)\,dx.\end{gathered}$$

The constant c depends only on $\Delta_2(\psi,\psi^*)$, n and N.

We are ready to prove Theorem 14.

Proof of Theorem 14. We begin with an application of Lemma 20:

$$\fint_B \psi(|\nabla \mathbf{u}|)\,dx \le c\sup_{\boldsymbol{\xi}\in C_0^\infty(B,\mathbb{R}^{N})} \bigg[\fint_B \mathcal{A}\nabla \mathbf{u}\cdot \nabla \boldsymbol{\xi}\,dx - \fint_B \psi^*(|\nabla \boldsymbol{\xi}|)\,dx\bigg]. \tag{4.9}$$

In the following let us fix $\boldsymbol{\xi} \in C_0^\infty(B)$. Choose $\gamma \geq 0$ such that

$$\psi^*(\gamma) = \fint_B \psi^*(|\nabla \boldsymbol{\xi}|)\, dx \tag{4.10}$$

and let $m_0 \in \mathbb{N}$. Due to Theorem 21 applied to ψ^* we find $\lambda \in [\gamma, 2^{m_0}\gamma]$ and $\boldsymbol{\xi}_\lambda \in W_0^{1,\infty}(B)$ such that

$$\|\nabla \boldsymbol{\xi}_\lambda\|_\infty \leq c\,\lambda, \tag{4.11}$$

$$\psi^*(\lambda)\frac{|\{\boldsymbol{\xi}_\lambda \neq \boldsymbol{\xi}\}|}{|B|} \leq \frac{c}{m_0} \fint_B \psi^*(|\nabla \boldsymbol{\xi}|)\, dx \tag{4.12}$$

$$\fint_B \psi^*\big(|\nabla \boldsymbol{\xi}_\lambda|\big)\, dx \leq c \fint_B \psi^*\big(|\nabla \boldsymbol{\xi}|\big)\, dx. \tag{4.13}$$

Let us point out that the use of the Lipschitz truncation is not a problem of the regularity of $\boldsymbol{\xi}$ as it is C_0^∞. It is the precise estimates above that we need.

We calculate

$$\fint_B \mathcal{A}\nabla \mathbf{u}\cdot\nabla \boldsymbol{\xi}\, dx = \fint_B \mathcal{A}\nabla \mathbf{u}\cdot\nabla \boldsymbol{\xi}_\lambda\, dx + \fint_B \mathcal{A}\nabla \mathbf{u}\cdot\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\lambda)\, dx =: I + II.$$

Using Young's inequality and (4.13), we estimate

$$\begin{aligned} II &= \fint_B \mathcal{A}\nabla \mathbf{u}\cdot\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\lambda)\chi_{\{\boldsymbol{\xi}\neq\boldsymbol{\xi}_\lambda\}}\, dx \\ &\leq c \fint_B \psi(|\nabla \mathbf{u}|\chi_{\{\boldsymbol{\xi}\neq\boldsymbol{\xi}_\lambda\}})\, dx + \frac{1}{2}\fint_B \psi^*(|\nabla \boldsymbol{\xi}|)\, dx =: II_1 + II_2, \end{aligned}$$

where c depends on $|\mathcal{A}|$, $\Delta_2(\psi, \psi^*)$. With Hölder's inequality we get

$$II_1 \leq c\bigg(\fint_B \big(\psi(|\nabla \mathbf{u}|)\big)^s\, dx\bigg)^{\frac{1}{s}} \bigg(\frac{|\{\boldsymbol{\xi}_\lambda \neq \boldsymbol{\xi}\}|}{|B|}\bigg)^{1-\frac{1}{s}}.$$

It follows from (4.12), (4.10), and $\lambda \geq \gamma$ that

$$\frac{|\{\boldsymbol{\xi}_\lambda \neq \boldsymbol{\xi}\}|}{|B|} \leq \frac{c\psi^*(\gamma)}{m_0\psi^*(\lambda)} \leq \frac{c}{m_0}.$$

Thus

$$II_1 \leq c\bigg(\fint_B \big(\psi(|\nabla \mathbf{u}|)\big)^s\, dx\bigg)^{\frac{1}{s}} \bigg(\frac{c}{m_0}\bigg)^{1-\frac{1}{s}}.$$

We choose m_0 so large such that

$$II_1 \leq \frac{\varepsilon}{2}\bigg(\fint_B \big(\psi(|\nabla \mathbf{u}|)\big)^s\, dx\bigg)^{\frac{1}{s}}.$$

Since $\mathbf{u}$ is almost $\mathcal{A}$-harmonic and $\|\nabla\boldsymbol{\xi}_\lambda\|_\infty \le c\,\lambda \le c\,2^{m_0}\gamma$ we have

$$|I| \le \delta \fint_{\widetilde{B}} |\nabla\mathbf{u}|\,dx\,\|\nabla\boldsymbol{\xi}_\lambda\|_\infty \le \delta \fint_{\widetilde{B}} |\nabla\mathbf{u}|\,dx\,c\,2^{m_0}\gamma.$$

We apply Young's inequality and (4.10) to get

$$\begin{aligned}|I| &\le \delta 2^{m_0} c\bigg(\fint_{\widetilde{B}} \psi(|\nabla\mathbf{u}|)\,dx + \psi^*(\gamma)\bigg)\\ &\le \delta 2^{m_0} c \fint_{\widetilde{B}} \psi(|\nabla\mathbf{u}|)\,dx + \delta 2^{m_0} c \fint_{B} \psi^*(|\nabla\boldsymbol{\xi}|)\,dx.\end{aligned}$$

Now, we choose $\delta>0$ so small such that $\delta 2^{m_0}c \le \varepsilon/2$. Thus

$$|I| \le \frac{\varepsilon}{2}\fint_{\widetilde{B}} \psi(|\nabla\mathbf{u}|)\,dx + \frac{1}{2}\fint_{B} \psi^*(|\nabla\boldsymbol{\xi}|)\,dx.$$

Combining the estimates for I, II, and II_1 we get

$$\fint_{B} \mathcal{A}\nabla\mathbf{u}\cdot\nabla\boldsymbol{\xi}\,dx \le \varepsilon\Bigg(\bigg(\fint_{B} \big(\psi(|\nabla\mathbf{u}|)\big)^s\,dx\bigg)^{\frac{1}{s}} + \fint_{\widetilde{B}} \psi(|\nabla\mathbf{u}|)\,dx\Bigg) + \fint_{B} \psi^*(|\nabla\boldsymbol{\xi}|)\,dx.$$

Now taking the supremum over all $\boldsymbol{\xi}\in C_0^\infty(B)$ and using (4.9), we get

$$\fint_{B} \psi(|\nabla\mathbf{w}|)\,dx \le \varepsilon\Bigg(\bigg(\fint_{B} \big(\psi(|\nabla\mathbf{u}|)\big)^s\,dx\bigg)^{\frac{1}{s}} + \fint_{\widetilde{B}} \psi(|\nabla\mathbf{u}|)\,dx\Bigg).$$

The claim follows by Poincaré inequality; see Theorem 9. $\square$

5. Almost $\mathcal{A}$-harmonicity. The following result is a special case of Lemma A.2 in [10].

LEMMA 22. *Let $B\subset\mathbb{R}^n$ be a ball and let $\mathbf{w}\in W^{1,\varphi}(B)$. Then*

$$\fint_{B} |\mathbf{V}(\nabla\mathbf{w}) - \langle\mathbf{V}(\nabla\mathbf{w})\rangle_B|^2\,dx \sim \fint_{B} |\mathbf{V}(\nabla\mathbf{w}) - \mathbf{V}(\langle\nabla\mathbf{w}\rangle_B)|^2\,dx.$$

The constants are independent of B and $\mathbf{w}$; they depend only on the characteristics of φ.

LEMMA 23. *There exists $\delta>0$, which depends only on the characteristics of φ, such that for every ball B with $B\Subset\Omega$ and every $\mathbf{u}\in W^{1,\varphi}(B)$ the estimate*

$$\fint_{B} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^2\,dx \le \delta \fint_{B} |\mathbf{V}(\nabla\mathbf{u})|^2\,dx \tag{5.1}$$

implies

$$\fint_{B} |\mathbf{V}(\nabla\mathbf{u})|^2\,dx \le 4\,|\mathbf{V}(\langle\nabla\mathbf{u}\rangle_B)|^2, \tag{5.2}$$

$$\fint_{B} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^2\,dx \le 4\,\delta\,|\mathbf{V}(\langle\nabla\mathbf{u}\rangle_B)|^2. \tag{5.3}$$

Proof. It follows from (5.1) and Lemma 22 that

$$\begin{aligned}\fint_B |\mathbf{V}(\nabla\mathbf{u})|^2\,dx &\le 2\fint_B |\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\langle\nabla\mathbf{u}\rangle_B)|^2\,dx + 2\,|\mathbf{V}(\langle\nabla\mathbf{u}\rangle_B)|^2\\ &\le c\fint_B |\mathbf{V}(\nabla\mathbf{u})-\langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^2\,dx + 2\,|\mathbf{V}(\langle\nabla\mathbf{u}\rangle_B)|^2\\ &\le \delta\,c\fint_B |\mathbf{V}(\nabla\mathbf{u})|^2\,dx + 2\,|\mathbf{V}(\langle\nabla\mathbf{u}\rangle_B)|^2.\end{aligned}$$

For small δ we absorb the first term on the right-hand side to get (5.2). The remaining estimate (5.3) is a combination of (5.1) and (5.2). ◻

LEMMA 24. *Let $\mathbf{u}$ be a local minimizer of $\mathcal{F}$. Then for every ball B with $2B \Subset \Omega$ and every $\mathbf{Q} \in \mathbb{R}^{N\times n}$ it holds that*

$$\fint_B \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u}-\mathbf{Q}|)\,dx \le c\,\varphi_{|\mathbf{Q}|}\bigg(\fint_{2B}|\nabla\mathbf{u}-\mathbf{Q}|\,dx\bigg).$$

Proof. From Corollary 12 we get

$$\fint_B \varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u}-\mathbf{Q}|)\,dx \le c\bigg(\fint_{2B}\varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u}-\mathbf{Q}|)^{\alpha}\,dx\bigg)^{\frac{1}{\alpha}}.$$

We can then apply Corollary 3.4 in [10] to conclude. ◻

LEMMA 25. *For all $\varepsilon > 0$ there exists $\delta > 0$, which depends only on ε and the characteristics of φ, such that for every ball B with $B \Subset \Omega$ and every $\mathbf{u} \in W^{1,\varphi}(B)$*

$$\fint_B |\mathbf{V}(\nabla\mathbf{u})-\langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^2\,dx \le \delta\fint_B|\mathbf{V}(\nabla\mathbf{u})|^2\,dx \tag{5.4}$$

implies

$$\fint_B |\nabla\mathbf{u}-\langle\nabla\mathbf{u}\rangle_B|\,dx \le \varepsilon\,|\langle\nabla\mathbf{u}\rangle_B|. \tag{5.5}$$

Proof. Let $\mathbf{Q} = \langle\nabla\mathbf{u}\rangle_B$. Then, by Jensen's inequality and Lemma 23, we get

$$\begin{aligned}\varphi_{|\langle\nabla\mathbf{u}\rangle_B|}\bigg(\fint_B|\nabla\mathbf{u}-\langle\nabla\mathbf{u}\rangle_B|\,dx\bigg) &\le \fint_B \varphi_{|\langle\nabla\mathbf{u}\rangle_B|}(|\nabla\mathbf{u}-\langle\nabla\mathbf{u}\rangle_B|)\,dx\\ &\le c\fint_B |\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\langle\nabla\mathbf{u}\rangle_B)|^2\,dx\\ &\le \delta\,c\,|\mathbf{V}(\langle\nabla\mathbf{u}\rangle_B)|^2\\ &\le \delta\,c\,\varphi(|\langle\nabla\mathbf{u}\rangle_B|)\\ &\le \delta\,c\,\varphi_{|\langle\nabla\mathbf{u}\rangle_B|}(|\langle\nabla\mathbf{u}\rangle_B|).\end{aligned}$$

For the last inequality we used the fact that $\varphi(a) \sim \varphi_a(a)$ for $a \geq 0$. Using the Δ_2 condition of $\varphi_{|\langle \nabla \mathbf{u}\rangle_B|}$, it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\fint_B |\nabla \mathbf{u} - \langle \nabla \mathbf{u}\rangle_B|\,dx \leq \varepsilon\,|\langle \nabla \mathbf{u}\rangle_B|. \qquad \square$$

Note that the smallness assumption in (5.4) automatically implies that $\langle \nabla \mathbf{u}\rangle_B \neq 0$ (unless $\nabla \mathbf{u} = 0$ on B). So the smallness assumption ensures that we are in some sense in the nondegenerate situation.

LEMMA 26. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every local minimizer $\mathbf{u} \in W^{1,\varphi}_{loc}(\Omega)$ of $\mathcal{F}$ and every ball B with $2B \Subset \Omega$ and for*

$$\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^2\,dx \leq \delta \fint_{2B} |\mathbf{V}(\nabla \mathbf{u})|^2\,dx \tag{5.6}$$

there holds

$$\left|\fint_B D^2 f(\mathbf{Q})(\nabla \mathbf{u} - \mathbf{Q}, \nabla \boldsymbol{\xi}\,dx\right| \leq \varepsilon\,\varphi''(|\mathbf{Q}|) \fint_{2B} |\nabla \mathbf{u} - \mathbf{Q}|\,dx \|\nabla \boldsymbol{\xi}\|_\infty. \tag{5.7}$$

for every $\boldsymbol{\xi} \in C_0^\infty(B)$, where $\mathbf{Q} := \langle \nabla \mathbf{u}\rangle_{2B}$. In particular, $\mathbf{u}$ is almost $\mathcal{A}$-harmonic (in the sense of Theorem 14), with $\mathcal{A} = D^2 f(\mathbf{Q})/\varphi''(|\mathbf{Q}|)$.

Proof. Let $\varepsilon > 0$. Without loss of generality we can assume that $\delta > 0$ is so small that Lemmas 23 and 25 give

$$\fint_{2B} |\mathbf{V}(\nabla \mathbf{u})|^2\,dx \leq 4\,|\mathbf{V}(\mathbf{Q})|^2, \tag{5.8}$$

$$\fint_{2B} |\nabla \mathbf{u} - \mathbf{Q}|\,dx \leq \varepsilon\,|\mathbf{Q}|. \tag{5.9}$$

From the last inequality we deduce

$$\varphi''(|\mathbf{Q}|)\left(\fint_{2B} |\nabla \mathbf{u} - \mathbf{Q}|\,dx\right)^2 \sim \varphi_{|\mathbf{Q}|}\left(\fint_{2B} |\nabla \mathbf{u} - \mathbf{Q}|\,dx\right). \tag{5.10}$$

Since the estimate (5.7) is homogeneous with respect to $\|\nabla \boldsymbol{\xi}\|_\infty$, it suffices to show that (5.7) holds for all $\boldsymbol{\xi} \in C_0^\infty(B)$ with $\|\nabla \boldsymbol{\xi}\|_\infty = \fint_{2B} |\nabla \mathbf{u} - \mathbf{Q}|\,dx$. Hence, because of (5.10) it suffices to prove

$$\left|\fint_B D^2 f(\mathbf{Q})(\nabla \mathbf{u} - \mathbf{Q}, \nabla \boldsymbol{\xi})\,dx\right| \leq \varepsilon\,c\,\varphi_{|\mathbf{Q}|}\left(\fint_{2B} |\nabla \mathbf{u} - \mathbf{Q}|\,dx\right) \tag{5.11}$$

for all such $\boldsymbol{\xi}$. We define

$$B^{\geq} := \left\{x \in B : |\nabla \mathbf{u} - \mathbf{Q}| \geq \tfrac{1}{2}|\mathbf{Q}|\right\},$$
$$B^{<} := \left\{x \in B : |\nabla \mathbf{u} - \mathbf{Q}| < \tfrac{1}{2}|\mathbf{Q}|\right\}.$$

From the Euler–Lagrange equation we get $\int_B \big(Df(\nabla \mathbf{v}) - Df(\mathbf{Q})\big) : \nabla \boldsymbol{\xi}\, dx = 0$, and therefore

$$\begin{aligned}
&\fint_B D^2 f(\mathbf{Q})(\nabla \mathbf{u} - \mathbf{Q}, \nabla \boldsymbol{\xi})\, dx \\
&\qquad = \fint_B \int_0^1 \big(D^2 f(\mathbf{Q}) - D^2 f(\mathbf{Q} + \theta(\nabla \mathbf{u} - \mathbf{Q}))\big)(\nabla \mathbf{u} - \mathbf{Q}, \nabla \boldsymbol{\xi})\, d\theta\, dx.
\end{aligned}$$

We split the right-hand side into the integral I over $B^{\geq}$ and the integral II over $B^{<}$. Using (H4), we get

$$\begin{aligned}
|I| &\leq c \fint_B \chi_{B^{\geq}} \int_0^1 \big(\varphi''(|\mathbf{Q}|) + \varphi''(|\mathbf{Q} + \theta(\nabla \mathbf{u} - \mathbf{Q})|)\big)\, d\theta\ |\nabla \mathbf{u} - \mathbf{Q}||\nabla \boldsymbol{\xi}|\, dx \\
&\leq c \fint_B \chi_{B^{\geq}} \big(\varphi''(|\mathbf{Q}|) + \varphi''(|\mathbf{Q}| + |\nabla \mathbf{u} - \mathbf{Q}|)\big)|\nabla \mathbf{u} - \mathbf{Q}||\nabla \boldsymbol{\xi}|\, dx \\
&\leq c \fint_B \chi_{B^{\geq}} \big(|\nabla \mathbf{u} - \mathbf{Q}|\varphi'(|\mathbf{Q}|) + \varphi'_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)|\mathbf{Q}|\big)\, dx\ \frac{\|\nabla \boldsymbol{\xi}\|_\infty}{|\mathbf{Q}|} \\
&\leq \varepsilon\, c \fint_B \chi_{B^{\geq}} \big(|\nabla \mathbf{u} - \mathbf{Q}|\varphi'(|\mathbf{Q}|) + \varphi'_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)|\mathbf{Q}|\big)\, dx.
\end{aligned}$$

We used Lemma 8 for the second, Assumption 6 for the third, and (5.9) for the last estimate. Now, using $|\mathbf{Q}| \leq 2\,|\nabla \mathbf{u} - \mathbf{Q}|$ on $B^{\geq}$ and $\varphi_a(t) \sim \varphi(t)$ for $0 \leq a \leq t$, we get

$$\begin{aligned}
|I| &\leq \varepsilon\, c \fint_B \chi_{B^{\geq}} \big(\varphi(|\nabla \mathbf{u} - \mathbf{Q}|) + \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)\big)\, dx \\
&\leq \varepsilon\, c \fint_B \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)\, dx.
\end{aligned}$$

Let us estimate the modulus of II. Using (H5) and $|\nabla \mathbf{u} - \mathbf{Q}| < \frac{1}{2}|\mathbf{Q}|$ on $B^{<}$, we get

$$|II| \leq c \fint_B \chi_{B^{<}} \varphi''(|\mathbf{Q}|)|\mathbf{Q}|^{-\beta_1}|\nabla \mathbf{u} - \mathbf{Q}|^{1+\beta_1}|\nabla \boldsymbol{\xi}|\, dx,$$

where $\beta_1 := \min\{s_0, \beta\}$ with the constant s_0 from Corollary 13. Using Young's inequality, we get

$$\begin{aligned}
|II| &\leq \gamma \varphi''(|\mathbf{Q}|)\|\nabla \boldsymbol{\xi}\|_\infty^2 + c_\gamma \fint_B \chi_{B^{<}} \varphi''(|\mathbf{Q}|)|\mathbf{Q}|^{-2\beta_1}|\nabla \mathbf{u} - \mathbf{Q}|^{2(1+\beta_1)}\, dx \\
&\leq \gamma\, c\, \varphi_{|\mathbf{Q}|}(\|\nabla \boldsymbol{\xi}\|_\infty) + c_\gamma (\varphi(|\mathbf{Q}|))^{-\beta_1} \fint_B \chi_{B^{<}} \big(\varphi''(|\mathbf{Q}|)|\nabla \mathbf{u} - \mathbf{Q}|^2\big)^{1+\beta_1}\, dx \\
&\leq \gamma\, c \fint_{2B} \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)\, dx + c_\gamma (\varphi(|\mathbf{Q}|))^{-\beta_1} \fint_B \chi_{B^{<}} \big(\varphi_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)\big)^{1+\beta_1}\, dx \\
&\leq \gamma\, c \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\, dx + c_\gamma (\varphi(|\mathbf{Q}|))^{-\beta_1} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^{2(1+\beta_1)}\, dx.
\end{aligned}$$

Here we used (5.10) for the second estimate and Jensen's inequality, $\varphi''(a)t^2 \sim \varphi_a(t)$ for $0 \le t \le a$, and $|\nabla \mathbf{u} - \mathbf{Q}| < \frac{1}{2}|\mathbf{Q}|$ on $B^<$ for the third estimate. With the help of Corollary 13 we get

$$|II| \le \gamma\, c \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\, dx + c_\gamma (\varphi(|\mathbf{Q}|))^{-\beta_1} \bigg(\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\, dx \bigg)^{1+\beta_1}.$$

Using the assumption (5.6), Lemma 22, and (5.8), it follows that

$$|II| \le \gamma\, c \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\, dx + c_\gamma \delta^{\beta_1} \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\, dx.$$

Choosing $\gamma > 0$ and then $\delta > 0$ small enough, we get the assertion. $\square$

6. Excess decay estimate. In this section we will focus on the excess decay estimate. Therefore, we compare the almost harmonic solution with its harmonic approximation.

PROPOSITION 27. *For all $\varepsilon > 0$, there exists $\delta = \delta(\varphi, \varepsilon) > 0$ such that the following is true: if for some ball B with $2B \Subset \Omega$ the smallness assumption* (5.6) *holds true, then for every $\tau \in (0, 1]$*

$$\Phi(\tau B, \mathbf{u}) \le c\, \tau^2 \big(1 + \varepsilon\, \tau^{-n-2}\big)\, \Phi(2B, \mathbf{u}), \tag{6.1}$$

where c depends only on the characteristics of φ and is independent of ε.

Proof. It suffices to consider the case $\tau \le \frac{1}{2}$. Let s_0 be as in Corollary 13. Let $\mathbf{q}$ be a linear function such that $\langle \mathbf{u} - \mathbf{q} \rangle_{2B} = 0$ and $\mathbf{Q} := \nabla \mathbf{q} = \langle \nabla \mathbf{u} \rangle_{2B}$. Define $\mathbf{z} := \mathbf{u} - \mathbf{q}$. Let $\mathbf{h}$ be the harmonic approximation of $\mathbf{z}$ with $\mathbf{h} = \mathbf{z}$ on ∂B. It follows from Lemma 26 that $\mathbf{z}$ is almost $\mathcal{A}$-harmonic with $\mathcal{A} = D^2 f(\mathbf{Q})/\varphi''(|\mathbf{Q}|)$. Thus by Theorem 14 for suitable $\delta = \delta(\varphi, \varepsilon)$ and by Theorem 14 the $\mathcal{A}$-harmonic approximation $\mathbf{h}$ satisfies

$$\fint_{B} \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{z} - \nabla \mathbf{h}|)\, dx \le \varepsilon \Bigg(\bigg(\fint_{B} \varphi^{s_0}_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)\, dx \bigg)^{\frac{1}{s_0}} + \fint_{2B} \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{u} - \mathbf{Q}|)\, dx \Bigg).$$

Now, it follows by Corollary 13 that

$$\fint_{B} \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{z} - \nabla \mathbf{h}|)\, dx \le c\, \varepsilon\, \Phi(2B, \mathbf{u}). \tag{6.2}$$

Since $\nabla \mathbf{z} = \nabla \mathbf{u} - \mathbf{Q}$ and $\langle \nabla \mathbf{z} \rangle_{\tau B} = \langle \nabla \mathbf{u} \rangle_{\tau B} - \mathbf{Q}$, we get

$$\begin{aligned} \Phi(\tau B, \mathbf{u}) &\le c \fint_{\tau B} \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{z} - \langle \nabla \mathbf{z} \rangle_{\tau B}|)\, dx \\ &\le c \fint_{\tau B} \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{h} - \langle \nabla \mathbf{h} \rangle_{\tau B}|)\, dx + c \fint_{\tau B} \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{z} - \nabla \mathbf{h}|)\, dx \\ &=: I + II. \end{aligned}$$

For the second estimate we used Jensen's inequality. Using (6.2) we obtain

$$II \le \tau^{-n} c \fint_{B} \varphi_{|\mathbf{Q}|}(|\nabla \mathbf{z} - \nabla \mathbf{h}|)\, dx \le \tau^{-n} c\, \varepsilon\, \Phi(2B, \mathbf{u}).$$

By the interior regularity of the $\mathcal{A}$-harmonic function $\mathbf{h}$ (see [22]) and $\tau \leq \frac{1}{2}$ it holds that

$$\sup_{\tau B} |\nabla \mathbf{h} - \langle \nabla \mathbf{h} \rangle_{\tau B}| \leq c\,\tau \fint_B |\nabla \mathbf{h} - \langle \nabla \mathbf{h} \rangle_B|\,dx.$$

This proves

$$I \leq c\,\varphi_{|\mathbf{Q}|}\biggl(\tau \fint_B |\nabla \mathbf{h} - \langle \nabla \mathbf{h} \rangle_B|\,dx\biggr).$$

Using the estimate $\psi(st) \leq s\psi(t)$ for any $s \in [0,1]$, $t \geq 0$, and any N-function ψ, we would get a factor τ in the estimate of I. However, to produce a factor τ^2, we have to work differently and use the improved estimate $\varphi_a(s\,t) \leq c\,s^2\varphi_a(t)$ for all $s \in [0,1]$, $a \geq 0$, and $t \in [0,a]$. We begin with

$$\begin{aligned}\fint_B |\nabla \mathbf{h} - \langle \nabla \mathbf{h} \rangle_B|\,dx &\leq \fint_B |\nabla \mathbf{z} - \langle \nabla \mathbf{z} \rangle_B|\,dx + 2\fint_B |\nabla \mathbf{z} - \nabla \mathbf{h}|\,dx \\ &= \fint_B |\nabla \mathbf{u} - \langle \nabla \mathbf{u} \rangle_B|\,dx + 2\fint_B |\nabla \mathbf{z} - \nabla \mathbf{h}|\,dx,\end{aligned}$$

which implies

$$I \leq c\,\varphi_{|\mathbf{Q}|}\biggl(\tau \fint_B |\nabla \mathbf{u} - \langle \nabla \mathbf{u} \rangle_B|\,dx\biggr) + c\,\tau\varphi_{|\mathbf{Q}|}\biggl(\fint_B |\nabla \mathbf{z} - \nabla \mathbf{h}|\,dx\biggr).$$

Due to (5.9), we can use for the first term the improved estimate $\varphi_a(s\,t) \leq c\,s^2\varphi_a(t)$, which gives

$$\begin{aligned}I &\leq c\,\tau^2\,\varphi_{|\mathbf{Q}|}\biggl(\fint_B |\nabla \mathbf{u} - \langle \nabla \mathbf{u} \rangle_B|\,dx\biggr) + c\,\tau\,\varphi_{|\mathbf{Q}|}\biggl(\fint_B |\nabla \mathbf{z} - \nabla \mathbf{h}|\,dx\biggr) \\ &\leq c\,\tau^2 \fint_B \varphi_{|\mathbf{Q}|}\bigl(|\nabla \mathbf{u} - \langle \nabla \mathbf{u} \rangle_B|\bigr)\,dx + c\,\tau \fint_B \varphi_{|\mathbf{Q}|}\bigl(|\nabla \mathbf{z} - \nabla \mathbf{h}|\bigr)\,dx.\end{aligned}$$

Thus using (6.2) we get

$$I \leq c\,\tau^2\,\Phi(B,\mathbf{u}) + c\,\tau\,\varepsilon\,\Phi(2B,\mathbf{u}) \leq c\left(\tau^2 + \varepsilon\,\tau\right)\Phi(2B,\mathbf{u}).$$

Combining the estimates for I and II, we get the claim. ∎

It follows now, by a series of standard arguments, that for any $\beta \in (0,1)$, there exists a suitable small δ that ensures local $C^{0,\beta}$-regularity of $\mathbf{V}(\nabla u)$, which implies Hölder continuity of the gradients as well.

PROPOSITION 28 (decay estimate). *For $0 < \beta < 1$ there exists $\delta = \delta(\varphi,\beta) > 0$ such that the following is true. If for some ball $B \subset \Omega$ the smallness assumption* (5.6) *holds true, then*

$$\Phi(\rho B,\mathbf{u}) \leq c\,\rho^{2\beta}\Phi(2B,\mathbf{u}) \tag{6.3}$$

for any $\rho \in (0,1]$, where $c = c(\varphi)$ depends only on the characteristics of φ.

Proof. Due to our assumption, we can apply Proposition 27 for any τ. Let $\gamma(\varepsilon,\tau) := c\,\tau^2\,(1+\varepsilon\,\tau^{-n-2})$ as in (6.1). Let us fix $\tau>0$ and $\varepsilon>0$, such that $\gamma(\varepsilon,\tau)\le\min\{(\tau/2)^{2\beta},\frac14\}$. Let $\delta=\delta(\varphi,\varepsilon)$ chosen accordingly to Proposition 27 and also so small that $(1+\tau^{-n/2})\delta^{1/2}\le\frac12$. By Proposition 27 we have

$$\Phi(\tau B,\mathbf{u})\le\min\{(\tau/2)^{2\beta},\tfrac14\}\,\Phi(2B,\mathbf{u}).\tag{6.4}$$

We claim that the smallness assumption is inherited from $2B$ to τB, so that we can iterate (6.4). For this we estimate with the help of our smallness assumption

$$\begin{aligned}
&\bigg(\fint_{2\mathrm{B}}|\mathbf{V}(\nabla\mathbf{u})|^2\,dx\bigg)^{\frac12}\\
&\quad\le\big(\Phi(2B,\mathbf{u})\big)^{\frac12}+|\langle\mathbf{V}(\nabla\mathbf{v})\rangle_{2B}-\langle\mathbf{V}(\nabla\mathbf{v})\rangle_{\tau B}|+\bigg(\fint_{\tau B}|\mathbf{V}(\nabla\mathbf{v})|^2\,dx\bigg)^{\frac12}\\
&\quad\le\big(\Phi(2B,\mathbf{u})\big)^{\frac12}+\tau^{-n/2}\big(\Phi(2B,\mathbf{u})\big)^{\frac12}+\bigg(\fint_{\tau B}|\mathbf{V}(\nabla\mathbf{v})|^2\,dx\bigg)^{\frac12}\\
&\quad\le\big(1+\tau^{-n/2}\big)\delta^{1/2}\bigg(\fint_{2\mathrm{B}}|\mathbf{V}(\nabla\mathbf{u})|^2\,dx\bigg)^{\frac12}+\bigg(\fint_{\tau B}|\mathbf{V}(\nabla\mathbf{v})|^2\,dx\bigg)^{\frac12}.
\end{aligned}$$

Using $(1+\tau^{-n/2})\delta^{1/2}\le\frac12$, we get

$$\fint_{2\mathrm{B}}|\mathbf{V}(\nabla\mathbf{u})|^2\,dx\le4\fint_{\tau B}|\mathbf{V}(\nabla\mathbf{v})|^2\,dx.$$

Now (6.4) and the previous estimate imply

$$\Phi(\tau B,\mathbf{u})\le\frac14\Phi(2B,\mathbf{u})\le\frac14\delta\fint_{2\mathrm{B}}|\mathbf{V}(\nabla\mathbf{u})|^2\,dx\le\delta\fint_{\tau B}|\mathbf{V}(\nabla\mathbf{u})|^2\,dx.$$

In particular, the smallness assumption is also satisfied for τB. So by induction we get

$$\Phi((\tau/2)^k2B,\mathbf{u})\le\min\{(\tau/2)^{2\beta k},4^{-k}\}\,\Phi(2B,\mathbf{u}),\tag{6.5}$$

which is the desired claim. ∎

Having the decay estimate, it is easy to prove our main theorem.

Proof of Theorem 3. We can assume that (5.6) is satisfied with a strict inequality. By continuity, (5.6) holds for $B=B(x)$ and all x in some neighborhood of x_0. By Proposition 28 and Campanato's characterization of Hölder continuity, we deduce that $\mathbf{V}(\nabla\mathbf{u})$ is β-Hölder continuous in a neighborhood of x_0. ∎

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