OPTIMAL ERROR ESTIMATES FOR A SEMI-IMPLICIT EULER SCHEME FOR INCOMPRESSIBLE FLUIDS WITH SHEAR DEPENDENT VISCOSITIES

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Abstract. Certain rheological behavior of fluids in engineering sciences is modeled by power law ansatz with $p \in (1, 2]$. In the present paper a semi-implicit time discretization scheme for such fluids is proposed. The main result is the optimal $\mathcal{O}(k^2)$ error estimate, where k is the time step size. This improves results in [L. Diening, A. Prohl, and M. Růžička, *SIAM J. Numer. Anal.,* 44 (2006), pp. 1172–1190], which where suboptimal in terms of the order of convergence. Our results hold in three-dimensional domains (with periodic boundary conditions) for the range $p \in (3/2, 2]$ and are uniform with respect to the degeneracy parameter $\delta \in [0, \delta_0]$ of the extra stress tensor. Additional regularity properties of the solution of the discrete problem are proved.

Key words. Non-Newtonian fluids, shear dependent viscosity, time discretization, error analysis, degenerate parabolic systems.

AMS subject classifications. 76A05, 35K65, 65M06, 35Q35

1. Introduction. We study the time discretization of a homogeneous, incompressible fluid with shear-dependent viscosity, governed by the following system of partial differential equations

$$\begin{aligned} \rho\,\mathbf{u}_t - \operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{u}) + \rho\,[\nabla\mathbf{u}]\mathbf{u} + \nabla\pi &= \rho\,\mathbf{f} && \text{in } I\times\Omega,\\ \operatorname{div}\mathbf{u} &= 0 && \text{in } I\times\Omega,\\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega, \end{aligned} \tag{NS$_p$}$$

where the vector field $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity, $\mathbf{S}$ is the extra stress tensor, the scalar π is the kinematic pressure, the vector $\mathbf{f} = (f_1, f_2, f_3)$ is the external body force, ρ the constant density, and $\mathbf{u}_0$ is the initial velocity. Here we used the notation $([\nabla\mathbf{u}]\mathbf{u})_i = \sum_{j=1}^{3} u_j\partial_j u_i,\ i = 1, 2, 3,$ for the convective term. In the following we divide the equation (NS$_p$) by the constant density ρ and relabel $\mathbf{S}/\rho$ and π/ρ again as $\mathbf{S}$ and π, respectively. Thus we consider from now on (NS$_p$) always with the convention that $\rho = 1$. The term $\mathbf{D}\mathbf{u} := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$ denotes the symmetric part of the gradient $\nabla\mathbf{u}$. Throughout the paper we shall assume that $\Omega = (0, 2\pi)^3 \subset \mathbb{R}^3$ and we endow the problem with space periodic boundary conditions. The latter assumption simplifies the problem, but allows us to concentrate on the difficulties that arise from the structure of the extra stress tensor. As usual $I = [0, T]$ denotes some non-vanishing time interval.

Standard examples of power-law extra stress tensors for $p \in (1, \infty)$ are

$$\mathbf{S}(\mathbf{D}\mathbf{u}) = \mu\,(\delta + |\mathbf{D}\mathbf{u}|^2)^{\frac{p-2}{2}}\mathbf{D}\mathbf{u} \quad \text{or} \quad \mathbf{S}(\mathbf{D}\mathbf{u}) = \mu\,(\delta + |\mathbf{D}\mathbf{u}|)^{p-2}\mathbf{D}\mathbf{u}, \tag{1.1}$$

where $\mu > 0$ and $\delta \geq 0$ are given constants. These models belong to the class of power-law ansatz to model certain non-Newtonian behavior of fluid flows, and they are frequently used in engineering literature. A classical reference (with a detailed

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discussion of power-law models including also early models) is the book by Bird, Armstrong, and Hassager [8]. We also refer to Málek, Rajagopal, and Růžička [22] and Málek and Rajagopal [21] for a discussion of such models. Let us mention that most real fluids that can be modeled by a constitutive law of type (1.1) are shear thinning, which corresponds to a "small" shear exponent p, i.e., $p \in (1, 2]$. However there are also shear thickening fluids, which have a shear exponent $p \geq 2$. Moreover, the case $p = 3$ is very interesting also for the modeling of turbulent flows and known in applied literature as the Smagorinsky model [27].

The mathematical analysis of the problem (NS_p), (1.1) started with the work of Ladyžhenskaya [16], [17] proving existence of global weak solutions. After the papers by Nečas et. al. [20], [5], improving the existence theory for global weak solutions, the problem has been studied intensively and various existence and regularity properties have been proved in the last years. We refer the reader to [19], [21], [14], and [7] for a detailed discussion of the relevant results.

We shall study properties of numerical schemes for the case of shear thinning fluids, i.e., $p \in (1, 2]$. In this paper we shall consider the time discretization, since for parabolic problems it is one of the basic steps for the numerical analysis. In particular, we study the following Euler scheme for (NS_p):

Algorithm (Euler semi-implicit) Let be given a time step size $k := T/M > 0$ with the corresponding net $I^M := \{t_m\}_{m=0}^M$, $t_m := m\,T/M$, and let $\mathbf{u}^0 = \mathbf{u}_0$. For $m \geq 1$ and $\mathbf{u}^{m-1}$ given from the previous time step, compute the iterate $\mathbf{u}^m$ as follows:

$$\begin{aligned} d_t\mathbf{u}^m - \operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{u}^m) + [\nabla\mathbf{u}^m]\mathbf{u}^{m-1} + \nabla\pi^m &= \mathbf{f}(t_m) \qquad \text{in } \Omega, \\ \operatorname{div}\mathbf{u}^m &= 0 \qquad \text{in } \Omega, \end{aligned} \tag{NS_p^k}$$

endowed with periodic boundary conditions, where

$$d_t\mathbf{u}^m := \frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{k}. \tag{1.2}$$

It is important to observe that the convective term is treated semi-implicitly (this allows to prove uniqueness), while the more peculiar non-linear extra stress tensor is treated implicitly.

We shall prove an optimal $\mathcal{O}(k^2)$ error estimate by collecting and improving several results obtained in the last decade. Hence, we think it is worth to briefly introduce the problem and to make a small survey of the previously known results. The benchmark result is clearly that for the heat equation: If $e^m := u(t_m) - u^m$ denotes the difference between the continuous solution evaluated at $t = t_m$ ($m = 0, \dots, M$) and the discrete one at the step m, it is possible to show (see Thomée [28] and Quarteroni and Valli [24]) that there exists a constant c independent of the time step size k such that

$$\max_{0\leq m\leq M}\|e^m\|_{L^2(\Omega)}^2 + k\sum_{m=0}^M\|\nabla e^m\|_{L^2(\Omega)}^2 \leq c\,k^2.$$

The counterpart of this result for strong solutions of the Navier-Stokes equations (Newtonian fluid, i.e., system (NS_p) with $p = 2$) can be found, e.g., in chapter 5 of Girault and Raviart [15].

Existence of discrete-time approximations and their convergence to the solutions to the continuous problem have been addressed for (NS_p) starting from Prohl and

Růžička [23] (cf. [3] for a parabolic problem with p-structure). They considered the *fully-implicit* Euler scheme, where the convective term is discretized by $[\nabla \mathbf{u}^m]\mathbf{u}^m$. Their main convergence result reads as follows: if the time step k is small enough and $p \in \left(\frac{3+\sqrt{29}}{5}, 2\right] \simeq (1.677, 2]$, then

$$\max_{0\le m\le M} \|\mathbf{e}^m\|^2_{L^2(\Omega)} + k\sum_{m=0}^{M} \|\mathbf{D}\mathbf{e}^m\|^2_{L^p(\Omega)} \le c\, k^{2\alpha(p)},$$

with $\alpha(p) = \frac{5p-6}{2p} < 1$.

This approach has been later refined in the sense that the admissible range of p became larger and the convergence rate was improved. Recently, the value of admissible p has been enlarged to $p \in \left(\frac{11+\sqrt{21}}{10}, 2\right] \simeq (1.558, 2]$ with convergence rate $\mathcal{O}(k^{\frac{5p-6}{2(p-1)}})$ (cf. Diening, Prohl, and Růžička [11], [25]). In all these results the convergence rate is determined by regularity of the second-order time derivative of the continuous solution $\mathbf{u}$, which is sub-optimal.

An optimal convergence result has been recently obtained for systems with *p-structure* without both pressure and convective term by Diening, Ebmeyer, and Růžička [9]. Moreover, sub-optimal $\mathcal{O}(k^{\frac{5p-6}{2(p-1)}})$ convergence for the *semi-implicit* Euler scheme has been recently proved in the extended range $p \in (3/2, 2]$ by Diening, Prohl, and Růžička [12], together with existence of strong discrete solutions.

In this paper we improve the above results. In particular, we prove the optimal $\mathcal{O}(k^2)$ convergence for $p \in (3/2, 2]$ for the scheme (NS^k_p). Moreover, by using the results in [7] we are also able to deal with the degenerate case $\delta = 0$ and to show stability for $\delta \in (0, \delta_0]$.

We treat the problem in an appropriate functional setting, since for problems with p-structure, it is important to observe that $l^\infty(I^M; L^2(\Omega)) \cap l^p(I^M; W^{1,p}(\Omega))$ may not be the "best" space to measure the error. An approach with a *quasi-norm*, depending on the solution itself seems more natural (cf. Barrett and Liu [4, 18]). We will use an equivalent setting inspired by the regularity theory of degenerate problems (cf. [1]). For that we introduce

$$\mathbf{F}(\mathbf{B}) = (\delta + |\mathbf{B}^{\mathrm{sym}}|)^{\frac{p-2}{2}} \mathbf{B}^{\mathrm{sym}} \qquad \forall\, \mathbf{B} \in \mathbb{R}^{3\times 3}, \tag{1.3}$$

and measure the error using this quantity (cf. Theorem 1.1). The proofs of this paper make strong use of the techniques introduced in the papers [12], [9], [7]. In addition, *i)* we add a more precise way of dealing with the convective term, based on suitable properties of averaging operators inspired by investigations of "Large Eddy Simulation" models (cf. [6]) and *ii)* we show the stability of the numerical scheme with respect to $\delta > 0$. This allows us to deal also with the degenerate case $\delta = 0$. The main result of this paper is the following.

THEOREM 1.1. *Let $\mathbf{S}$ satisfy Assumption 1 with $p \in \left(\frac{3}{2}, 2\right]$ and $\delta \in [0, \delta_0]$, where $\delta_0 > 0$. Let $\mathbf{f} \in C(I; W^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega))$, where $I = [0, T]$, for some $T > 0$, and let $\mathbf{u}_0 \in W^{2,2}_{\mathrm{div}}(\Omega)$ with $\operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{u}_0) \in L^2(\Omega)$ be given. Let $\mathbf{u}$ be the corresponding strong solution of the (continuous) problem (NS_p) satisfying*

$$\|\mathbf{u}_t\|_{L^\infty(I;L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{W^{1,2}(I\times\Omega)} + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{L^{2\frac{5p-6}{2-p}}(I;W^{1,2}(\Omega))} \le c\,, \tag{1.4}$$

with a constant $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$. Let $\mathbf{u}^m$ be the unique solution of the (discrete) problem (NS^k_p) corresponding to the same data. Then, there exists a time *step size $\overline{k}>0$ such that for $k\in(0,\overline{k})$ the following estimates holds true*

$$\max_{0\le m\le M}\|\mathbf{D}\mathbf{e}^m\|^2_{L^2(\Omega)}+k\sum_{m=0}^{M}\|\mathbf{F}(\mathbf{D}\mathbf{u}(t_m))-\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|^2_{L^2(\Omega)}\le c\,k^2, \tag{1.5}$$

where $\overline{k}$ and the constant c depend only on $\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T$, and Ω. In particular, they are independent of $\delta\in[0,\delta_0]$.

Remark 1.2.

(i) Note that in the above theorem the solutions of the continuous problem $\mathbf{u}$ and of the discrete problem $\mathbf{u}^m$ as well as the quantities $\mathbf{S}$ and $\mathbf{F}$ depend on δ. For a lighter notation we have suppressed this dependence. With a few exception this comment applies to the whole paper.

(ii) Under the regularity assumptions on the data in Theorem 1.1 it is shown in [7] (cf. Theorem 2.6 below) that at least locally in time there exists a strong solution of the problem (NS_p) with the required regularity. Under the same assumption it is shown in [7] (cf. Lemma 3.1 below) that there exists a unique solution of the problem (NS_p^k). Note, that the error estimate (1.5) holds on the whole interval I.

(iii) Using ideas from [7] and [12] one can show that the discrete solutions $\mathbf{u}^m$ have essentially the same regularity properties as the continuous solution $\mathbf{u}$ (cf. Theorem 4.1).

Plan of the paper. In section 2 we recall some features of the extra stress tensor with p-structure and give precise definitions of the various norms and semi-norms we shall use. Moreover, we recall the existence theorem for strong solutions of the continuous problem. In section 3 we prove the error estimate in Theorem 1.1 using a suitable Gronwall-like argument and a retarded-time-averaging to treat the continuous equation. All estimates will be uniform in $\delta\in[0,\delta_0]$. Finally, in section 4 we prove the existence of strong solutions for the discrete problem.

2. Generalities on fluids modeled by systems with p-structure.

In this section we introduce the notation used in the paper and define the class of extra stress tensors we shall consider and recall its most relevant properties.

2.1. Notation.

We shall use the customary Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$ and we do not distinguish between scalar, vector, or tensor function spaces. We shall denote by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ and by $\|\cdot\|_{k,p}$ the norm in $W^{k,p}(\Omega)$. In this paper we are considering the space-periodic case, i.e., $\Omega=(0,2\pi)^3$ and each function f we consider will satisfy $f(x+2\pi\,e_i)=f(x)$, $i=1,2,3$, where $\{e_1,e_2,e_3\}$ is the canonical basis of $\mathbb{R}^3$. Often we will also require that the functions have vanishing mean value, i.e., $\int_\Omega f(x)\,dx=0$. This is a standard request in order to have Poincaré's inequality. We define $\mathcal{V}$ as the space of vector-valued functions on Ω that are smooth, divergence-free, and periodic with zero mean value and set

$$W^{1,p}_{\mathrm{div}}(\Omega):=\big\{\text{closure of }\mathcal{V}\text{ in }W^{1,p}(\Omega)\big\}.$$

Since we deal with a time dependent problem, we shall make use of the spaces $L^p(I;X), 1\le p\le\infty$, where $(X,\|\,.\,\|_X)$ is a Banach space. The subscript "$_t$" denotes differentiation with respect to time. We write $f\simeq g$ if there exist positive constants c_0 and c_1 such that

$$c_0 f\le g\le c_1 f.$$

To deal with discrete problems we shall use the discrete spaces $l^p(I^M;X)$, where $I^M := \{t_m\}_{m=0}^M$ is a net with $t_m := mT/M$. These spaces are the discrete counterparts of $L^p(I;X)$ and they consists of X-valued sequences $\{a_m\}_{m=0}^M$, endowed with the norm

$$\|a_m\|_{l^p(I^M;X)} := \begin{cases} \left(k \sum_{m=0}^{M} \|a_m\|_X^p \right)^{1/p} & \text{if } 1 \le p < \infty \\ \max\limits_{0 \le m \le M} \|a_m\|_X & \text{if } p = \infty. \end{cases}$$

We also use the notation $I_m := (t_{m-1}, t_m)$ for $m = 1, \ldots, M$.

As a general rule in the sequel we shall use the symbol "c" to denote generic constants (possibly different from line to line), which depend only on the data of the problem, but neither on the time step size k, nor on $\delta \in [0, \delta_0]$.

2.2. General properties of the extra stress tensor. Let us now discuss the structure of the extra stress tensor $\mathbf{S}$ and motivate our assumptions for it. Due to the principle of objectivity the extra stress tensor $\mathbf{S}$ depends on the velocity gradient $\nabla \mathbf{u}$ only through its symmetric part $\mathbf{D}\mathbf{u} := \frac{1}{2}\big(\nabla \mathbf{u} + \nabla \mathbf{u}^\top\big)$. Therefore we assume that the extra stress tensor $\mathbf{S}\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\text{sym}}$, where $\mathbb{R}^{3\times 3}_{\text{sym}} := \big\{\mathbf{A} \in \mathbb{R}^{3\times 3} \,\big|\, \mathbf{A} = \mathbf{A}^\top\big\}$ satisfies $\mathbf{S}(\mathbf{A}) = \mathbf{S}\big(\mathbf{A}^{\text{sym}}\big)$ and $\mathbf{S}(\mathbf{0}) = \mathbf{0}$, where $\mathbf{A}^{\text{sym}} := \frac{1}{2}\big(\mathbf{A} + \mathbf{A}^\top\big)$.

Often the extra stress tensor $\mathbf{S}$ is derived from a potential, i.e., there exists a sufficiently smooth convex function $\varPhi\colon \mathbb{R}^{\ge 0} \to \mathbb{R}^{\ge 0}$, which satisfies $\varPhi(0) = \varPhi'(0) = 0$, such that for all $\mathbf{A} \in \mathbb{R}^{3\times 3} \setminus \{\mathbf{0}\}$ and $i, j = 1, 2, 3$ it holds that*

$$S_{ij}(\mathbf{A}) = \partial_{ij}\big(\varPhi(|\mathbf{A}^{\text{sym}}|)\big) = \varPhi'\big(|\mathbf{A}^{\text{sym}}|\big)\frac{A_{ij}^{\text{sym}}}{|\mathbf{A}^{\text{sym}}|}\,. \tag{2.1}$$

In many relevant cases the potential $\varPhi$ possesses *p-structure*, or more precisely (p,δ)-structure. This means that there exist $p \in (1,\infty)$, $\delta \in [0,\infty)$, and constants $\nu_0, \nu_1 > 0$ such that for all $t \in \mathbb{R}^{\ge 0}$ holds

$$\nu_0(\delta + t)^{p-2} \le \varPhi''(t) \le \nu_1(\delta + t)^{p-2}\,. \tag{2.2}$$

In this situation one can show (cf. [10, Lemma 6.3], [26, Lemma 6.7, Section 8]) that there are constants $\nu_2, \nu_3 > 0$, which depend only on ν_0, ν_1, and p, such that for all $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{3\times 3}$ with $\mathbf{A}^{\text{sym}} \neq \mathbf{0}$ and $i, j, k, l = 1, 2, 3$ holds

$$\begin{aligned} \sum_{i,j,k,l=1}^{3} \partial_{kl} S_{ij}(\mathbf{A}) C_{ij} C_{kl} &\ge \nu_2\big(\delta + |\mathbf{A}^{\text{sym}}|\big)^{p-2} |\mathbf{C}^{\text{sym}}|^2\,, \\ \big|\partial_{kl} S_{ij}(\mathbf{A})\big| &\le \nu_3\big(\delta + |\mathbf{A}^{\text{sym}}|\big)^{p-2}. \end{aligned} \tag{2.3}$$

These two relations are the basis of our main assumption on the extra stress tensor.

Assumption 1 (extra stress tensor). *We assume that the extra stress tensor* $\mathbf{S}\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\text{sym}}$ *belongs to* $C^1(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3}_{\text{sym}}) \cap C^2(\mathbb{R}^{3\times 3} \setminus \{\mathbf{0}\}, \mathbb{R}^{3\times 3}_{\text{sym}})$ *and satisfies* $\mathbf{S}(\mathbf{A}) = \mathbf{S}\big(\mathbf{A}^{\text{sym}}\big)$ *and* $\mathbf{S}(\mathbf{0}) = \mathbf{0}$. *Moreover, we assume that* $\mathbf{S}$ *has* (p,δ)*-structure, i.e., there exist* $p \in (1,\infty)$, $\delta \in [0,\infty)$, *and constants* $C_0, C_1 > 0$ *such that*

$$\sum_{i,j,k,l=1}^{3} \partial_{kl} S_{ij}(\mathbf{A}) C_{ij} C_{kl} \ge C_0\big(\delta + |\mathbf{A}^{\text{sym}}|\big)^{p-2} |\mathbf{C}^{\text{sym}}|^2\,, \tag{2.4a}$$

$$\big|\partial_{kl} S_{ij}(\mathbf{A})\big| \le C_1\big(\delta + |\mathbf{A}^{\text{sym}}|\big)^{p-2} \tag{2.4b}$$

* For functions $g\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ we use the notation $\partial_{kl} g(\mathbf{A}) := \frac{\partial g(\mathbf{A})}{\partial A_{kl}}$.

is satisfied for all $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{3\times 3}$ *with* $\mathbf{A}^{\mathrm{sym}} \neq \mathbf{0}$ *and for all* $i, j, k, l = 1, 2, 3$.

Closely related to the extra stress tensor $\mathbf{S}$ with p-structure is the function $\mathbf{F}\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ defined through

$$\mathbf{F}(\mathbf{A}) := \big(\delta + |\mathbf{A}^{\mathrm{sym}}|\big)^{\frac{p-2}{2}} \mathbf{A}^{\mathrm{sym}}, \tag{2.5}$$

where $\delta \in [0, \infty)$ is the same as in (2.2) and (2.3). If the dependence on δ is of relevance we write $\mathbf{F}^{\delta}(\mathbf{A})$. Moreover, there is a close relation to Orlicz spaces and N-functions (cf. [26], [7] for a detailed description.)

Remark 2.1. If not stated otherwise we will use the convention that in formulas relating the quantities $\mathbf{S}$ and $\mathbf{F}$ the value of δ is the same in each of the quantities and it is suppressed for shortage of notation.

In the situation of Assumption 1 one can prove the following crucial lemma, which shows the equivalence of several quantities occurring naturally in the analysis of the system (NS$_p$), showing also the strict connection between $\mathbf{S}$ and $\mathbf{F}$ (cf. [9, Lemma 2.1], [10, Lemma 2.3], [26, Lemma 6.16, Section 6].)

Lemma 2.2. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in (1, \infty)$ *and* $\delta \in [0, \infty)$ *and let* $\mathbf{F}$ *be defined by* (2.5)*. Then for all* $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{3\times 3}$ *there holds*

$$\begin{aligned}(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) &\simeq |\mathbf{A}^{\mathrm{sym}} - \mathbf{B}^{\mathrm{sym}}|^2 (\delta + |\mathbf{B}^{\mathrm{sym}}| + |\mathbf{A}^{\mathrm{sym}}|)^{p-2} \\ &\simeq |\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2\end{aligned} \tag{2.6}$$

$$|\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})| \simeq |\mathbf{A}^{\mathrm{sym}} - \mathbf{B}^{\mathrm{sym}}| (\delta + |\mathbf{B}^{\mathrm{sym}}| + |\mathbf{A}^{\mathrm{sym}}|)^{p-2}, \tag{2.7}$$

where the constants depend only on C_0, C_1*, and* p*. In particular, the constants are independent of* δ*.*

Since in the following we shall insert into $\mathbf{S}$ and $\mathbf{F}$ only symmetric tensors, we can drop in the above formula the superscript "sym" and restrict the admitted tensors to symmetric ones.

Lemma 2.2 also clarifies the connection with the quasinorm $\|\mathbf{C}\|_{(\mathbf{B})}$ introduced by Barrett and Liu [2] through

$$\|\mathbf{C}\|_{(\mathbf{B})} := \left(\int_{\Omega} (\delta + |\mathbf{B}(x)| + |\mathbf{C}(x)|)^{p-2} |\mathbf{C}(x)|^2 \, dx\right)^{1/2},$$

where $\mathbf{B}$, $\mathbf{C} : \Omega \to \mathbb{R}^{3\times 3}$.

From Lemma 2.2 and the definition of quasi-norm $\|\mathbf{C}\|_{(\mathbf{B})}$ follows immediately:

Lemma 2.3. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in (1, \infty)$ *and* $\delta \in [0, \infty)$ *and let* $\mathbf{F}$ *be defined by* (2.5)*. Then*

$$\begin{aligned}\|\mathbf{D}\mathbf{v} - \mathbf{D}\mathbf{w}\|^2_{(\mathbf{D}\mathbf{w})} &\simeq \int_{\Omega} (\mathbf{S}(\mathbf{D}\mathbf{v}) - \mathbf{S}(\mathbf{D}\mathbf{w})) \cdot (\mathbf{D}\mathbf{v} - \mathbf{D}\mathbf{w}) \, dx \\ &\simeq \|\mathbf{F}(\mathbf{D}\mathbf{v}) - \mathbf{F}(\mathbf{D}\mathbf{w})\|^2_{L^2(\Omega)}, \qquad \forall\, \mathbf{v}, \mathbf{w} \in W^{1,p}(\Omega),\end{aligned}$$

where the constants depend only on C_0, C_1*, and* p*. In particular, the constants are independent of* δ*.*

The following lemma is a version of Young's inequality and will be used frequently in the sequel.

LEMMA 2.4 (Quasi-norm trick). *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in (1,\infty)$ *and* $\delta \in [0,\infty)$ *and let* $\mathbf{F}$ *be defined by* (2.5). *Then for each* $\varepsilon > 0$ *there exists* $c_\varepsilon(p) > 0$, *such that for all* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ *there holds*

$$\begin{aligned}&\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big) \cdot \big(\mathbf{A} - \mathbf{C}\big)\\ &\qquad \le \varepsilon \,\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big) \cdot \big(\mathbf{A} - \mathbf{B}\big) + c_\varepsilon \,\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{C})\big) \cdot \big(\mathbf{A} - \mathbf{C}\big)\end{aligned}$$

and equivalently

$$\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big) \cdot \big(\mathbf{A} - \mathbf{C}\big) \le \varepsilon \,\big|\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})\big|^2 + c_\varepsilon \,\big|\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{C})\big|^2 .$$

Proof. Cf. [3, Lemma 2.2], [7, Lemma 3.5]. □

Especially, for $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2 \in W^{1,p}(\Omega)$ we easily deduce from Lemma 2.4 the following useful inequality.

$$\begin{aligned}&\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{v}) - \mathbf{S}(\mathbf{D}\mathbf{w}_1)) \cdot (\mathbf{D}\mathbf{v} - \mathbf{D}\mathbf{w}_2)\, dx\\ &\qquad \le \varepsilon \,\|\mathbf{F}(\mathbf{D}\mathbf{v}) - \mathbf{F}(\mathbf{D}\mathbf{w}_1)\|_2^2 + c_\varepsilon \,\|\mathbf{F}(\mathbf{D}\mathbf{v}) - \mathbf{F}(\mathbf{D}\mathbf{w}_2)\|_2^2.\end{aligned} \tag{2.8}$$

We also recall the following result, taken from [13, Lemma 8], [7, Lemma 4.1].

LEMMA 2.5. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in (1,2]$ *and* $\delta \in (0,\infty)$, *and let* $\mathbf{F}$ *be defined by* (2.5). *Then, for sufficiently smooth* $\mathbf{u}$, $\mathbf{v}$ *and* $q \in [1,2]$ *holds*

$$\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_q^2 \le c\,\|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\mathbf{v})\|_2^2 \,\big\|(\delta + |\mathbf{D}\mathbf{u}| + |\mathbf{D}\mathbf{v}|)^{2-p}\big\|_{\frac{q}{2-q}} ,$$

where the constant c *depends only on* C_0, C_1, *and* p. *Moreover,* $\frac{q}{2-q} = \infty$ *for* $q = 2$. *For* $p \in (1,2]$, $\delta \in [0,\infty)$, $r \in [1,\infty]$, *and* $\delta + \|\mathbf{D}\mathbf{u}\|_r + \|\mathbf{D}\mathbf{v}\|_r > 0$ *we can formulate this result also as follows*

$$\begin{aligned}&\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{u}) - \mathbf{S}(\mathbf{D}\mathbf{v})) \cdot \mathbf{D}(\mathbf{u}-\mathbf{v})\, dx\\ &\qquad \ge c\,\|\mathbf{D}\mathbf{u} - \mathbf{D}\mathbf{v}\|^2_{\frac{2r}{2-p+r}} (\delta + \|\mathbf{D}\mathbf{u}\|_r + \|\mathbf{D}\mathbf{u} - \mathbf{D}\mathbf{v}\|_r)^{p-2}.\end{aligned}$$

2.3. Existence of strong solutions. Let us recall the main existence theorem of [7].

THEOREM 2.6. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in \big(\frac{7}{5}, 2\big]$ *and* $\delta \in [0,\delta_0]$ *where* $\delta_0 > 0$. *Assume that* $\mathbf{f} \in L^\infty(I; W^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega))$, *where* $I = [0,T]$, *and* $\mathbf{u}_0 \in W^{2,2}_{\mathrm{div}}(\Omega)$, $\operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{u}_0) \in L^2(\Omega)$. *Then there exists a time* $T' = T'(\delta_0, p, C_0, \mathbf{f}, \mathbf{u}_0, T, \Omega)$, *with* $0 < T' \le T$, *such that the system* (NS_p) *has a strong solution* $\mathbf{u}$ *belonging to* $L^p(I'; W^{1,p}_{\mathrm{div}}(\Omega))$, $I' = [0,T']$, *satisfying for a.e.* $t \in I'$ *and for all* $\boldsymbol{\varphi} \in W^{1,p}_{\mathrm{div}}(\Omega)$

$$\int_\Omega \mathbf{u}_t(t) \cdot \boldsymbol{\varphi} + \mathbf{S}(\mathbf{D}\mathbf{u}(t)) \cdot \mathbf{D}\boldsymbol{\varphi} + [\nabla \mathbf{u}(t)]\mathbf{u}(t) \cdot \boldsymbol{\varphi}\, dx = \int_\Omega \mathbf{f}(t) \cdot \boldsymbol{\varphi}\, dx, \tag{2.9}$$

and

$$\|\mathbf{u}_t\|_{L^\infty(I';L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{W^{1,2}(I'\times\Omega)} + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{L^{2\frac{5p-6}{2-p}}(I';W^{1,2}(\Omega))} \le c_0 \,, \tag{2.10}$$

with $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$*. In particular, we have*

$$\mathbf{u} \in L^{\frac{p(5p-6)}{2-p}}(I'; W^{2,\frac{3p}{p+1}}(\Omega)) \cap C(I'; W^{1,r}(\Omega)) \qquad 1 \le r < 6(p-1) \tag{2.11a}$$

$$\mathbf{u}_t \in L^\infty(I'; L^2(\Omega)) \cap L^{\frac{p(5p-6)}{(3p-2)(p-1)}}(I'; W^{1,\frac{3p}{p+1}}(\Omega)), \tag{2.11b}$$

with the corresponding norms of $\mathbf{u}$ *and* $\mathbf{u}_t$ *bounded by constants* $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega, r)$ *and* $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$*, respectively.*

Due to (2.11a) *and* $p > \frac{7}{5}$ *we have* $\mathbf{u} \in C(I; W^{1,\frac{12}{5}}(\Omega))$ *and the solution* $\mathbf{u}$ *is unique within this class* $C(I; W^{1,\frac{12}{5}}(\Omega))$*.*

Remark 2.7. For $\delta > 0$ there exists a pressure π satisfying

$$\nabla\pi \in L^{\frac{2(5p-6)}{2-p}}(I'; L^2(\Omega)) \tag{2.12}$$

and the second time derivative satisfies

$$\mathbf{u}_{tt} \in L^2(I'; \big(W^{1,2}_{\operatorname{div}}(\Omega)\big)^*), \tag{2.13}$$

with both norms bounded by a constant $c = c(\delta, p, C_0, C_1, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$, which may explode as $\delta \to 0^+$.

Remark 2.8. By parabolic interpolation it follows from (2.11b) that

$$\mathbf{u}_t \in L^{\frac{11p-12}{3(p-1)}}(I' \times \Omega). \tag{2.14}$$

Note that this regularity is one of the reasons that leads to the restriction $p \in (\frac{3}{2}, 2]$ in Theorem 1.1.

3. Proof of Theorem 1.1. In this section we prove our optimal convergence result. This requires a few auxiliary results. We begin with the existence result for solutions to the discrete problem (NS^k_p), which is proved in [7, Theorem 6.3].

Lemma 3.1. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in \big(\frac{3}{2}, 2\big]$ *and* $\delta \in [0, \delta_0]$*, where* $\delta_0 > 0$*. Let* $\mathbf{f} \in C(I; W^{1,2}(\Omega))$*, where* $I = [0, T]$*, for some* $T > 0$*, and* $\mathbf{u}_0 \in W^{2,2}_{\operatorname{div}}(\Omega)$ *be given. Then, there exists a unique strong solution* $\mathbf{u}^m$ *of the problem* (NS^k_p) *satisfying the weak formulation*

$$\int_\Omega d_t\mathbf{u}^m \cdot \boldsymbol{\varphi} + \mathbf{S}(\mathbf{D}\mathbf{u}^m) \cdot \mathbf{D}\boldsymbol{\varphi} + [\nabla\mathbf{u}^m]\mathbf{u}^{m-1} \cdot \boldsymbol{\varphi}\,dx = \int_\Omega \mathbf{f}(t_m) \cdot \boldsymbol{\varphi}\,dx \tag{3.1}$$

for all $\boldsymbol{\varphi} \in W^{1,p}_{\operatorname{div}}(\Omega)$ *with*

$$\max_{0 \le m \le M} \|\mathbf{u}^m\|_2^2 + k \sum_{m=0}^{M} \|\mathbf{D}\mathbf{u}^m\|_p^p \le c, \tag{3.2}$$

$$\|\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_{W^{1,2}(\Omega)} \le c(k), \tag{3.3}$$

where the constants also depend on δ_0*,* p*,* C_0*,* C_1*,* $\|\mathbf{f}\|$*,* $\|\mathbf{u}_0\|$*,* T*, and* Ω*.*

3.1. A generalized discrete Gronwall's lemma. We now prove a generalized discrete Gronwall's lemma, which follows the same lines as [12, Lemma 2.15]. Here we denote the degeneracy parameter by $\lambda \in [0, \Lambda]$ instead of $\delta \in [0, \delta_0]$ as in the other sections. Of course, λ and Λ are related to δ and δ_0, respectively (cf. (3.15)). The main new point (which requires some additional care) is that we check that all constants are "weakly" dependent on λ. With weakly dependence we mean that all constants are uniformly bounded if λ belongs to $[0,\Lambda]$, for some $\Lambda > 0$. In particular, we can keep under control the various constants in the interesting case that λ is asymptotically small. The upper bound Λ for λ will be later derived from a-priori estimates on strong solutions and will depend on the data of the problem. The constant M below will be later the number of time steps with time step size k, such that $kM = T$.

LEMMA 3.2. *Let $p \in (1,2]$ and let $\{a_m\}_m$, $\{b_m\}_m$, $\{r_m\}_m$, and $\{s_m\}_m$ be non-negative sequences with $a_0 = b_0 = 0$ such that*

$$\exists\, c > 0: \qquad k\sum_{m=0}^{M} r_m^2 \le c\,k^2 \quad \text{and} \quad k\sum_{m=0}^{M} s_m^2 \le c\,k^2. \tag{3.4}$$

Further, let there exist constants $\gamma_1,\ \gamma_2, \gamma_3 > 0$, and some $\theta \in (0,1)$ such that for all $\lambda \in [0,\Lambda]$ the following two inequalities are satisfied for all $k \in (0,k_0)$ and all $m \ge 1$:†

$$d_t a_m^2 + \gamma_1(\lambda + b_m)^{p-2} b_m^2 \le b_m r_m + \gamma_2 b_{m-1} b_m + s_m^2, \tag{3.5}$$

$$d_t a_m^2 + \gamma_1(\lambda + b_m)^{p-2} b_m^2 \le b_m r_m + \gamma_3 b_{m-1} b_m^{1-\theta} a_m^{\theta} + s_m^2. \tag{3.6}$$

Then, there exists $\overline{k} \in (0,k_0]$ and constants $\gamma_4,\ \gamma_5 > 0$ such that for all $k \in (0,\overline{k})$

$$\max_{0\le m\le M} b_m \le 1 \tag{3.7}$$

$$\max_{0\le m\le M} a_m^2 + \gamma_1(\Lambda+1)^{p-2} k \sum_{m=0}^{M} b_m^2 \le \gamma_4 k^2 \exp(2\gamma_5\, kM). \tag{3.8}$$

The explicit expressions for $\overline{k}$, γ_4, and γ_5 will be given throughout the proof (cf. (3.11), (3.12), (3.10)*).*

Proof. The proof of this result is very similar to that of [12, Lemma 2.15], but here we need to take more care to trace the precise behavior of all constants with respect to λ. The proof proceeds by induction on $N \le M$. The starting step holds true, since if $N = 0$, then (3.7)-(3.8) are both trivially satisfied. To continue the inductive procedure we need to show that by assuming (3.7)-(3.8) for all $0 \le m \le N-1$, we can conclude (3.7)-(3.8) also at the next step $m = N$. We start showing (3.7), i.e., that $b_N \le 1$. If $b_N \le 1$ there is nothing to prove. Consequently, let us suppose *per absurdum* that $b_N > 1$. We multiply (3.5) by k and we sum over m, for $m = 1,\dots,N$. It follows that:‡

$$\begin{aligned} a_N^2 + \gamma_1 k \sum_{m=1}^{N} (\lambda+b_m)^{p-2} b_m^2 &\le k\sum_{m=1}^{N} b_m (r_m + \gamma_2 b_{m-1}) \frac{\sqrt{\gamma_1}(\lambda+b_m)^{\frac{p-2}{2}}}{\sqrt{\gamma_1}(\lambda+b_m)^{\frac{p-2}{2}}} + k\sum_{m=1}^{N} s_m^2 \\ &\le \frac{\gamma_1}{2} k \sum_{m=1}^{N} (\lambda+b_m)^{p-2} b_m^2 + \frac{1}{\gamma_1} k \sum_{m=1}^{N} (\lambda+b_m)^{2-p}(r_m^2 + \gamma_2^2 b_{m-1}^2) + k\sum_{m=0}^{N} s_m^2. \end{aligned}$$

We absorb the first term on the right-hand side in the left-hand side. Concerning the second term, we observe that since $p \le 2$, and $0 \le b_m \le 1 < b_N$ for $m = 1,\dots,N-1$ it follows that

$$(\lambda+b_m)^{2-p} \le (\lambda+b_N)^{2-p} \le (\lambda+b_N)^{2(2-p)}, \tag{3.9}$$

†Here we use the convention that for $b_m = \lambda = 0$ we set $(\lambda+b_m)^{p-2} b_m^2 = 0$. This is consistent with the notation we introduced in Lemma 2.5

‡Note that the manipulation in the first sum on the right-hand side is done only for $b_m \neq 0$.

regardless of the value of $\lambda \geq 0$. We have now the following inequality

$$a_N^2 + \frac{\gamma_1}{2} k \sum_{m=1}^{N} (\lambda + b_m)^{p-2} b_m^2 \leq \frac{(\lambda + b_N)^{2-p}}{\gamma_1} k \sum_{m=1}^{N} (r_m^2 + \gamma_2^2 b_{m-1}^2) + k \sum_{m=0}^{N} s_m^2.$$

Neglecting all terms on the left-hand side, except the one with $m = N$, dividing both sides by $\frac{\gamma_1}{2} k(\lambda + b_N)^{p-2} \neq 0$, and using again (3.9) we get

$$b_N^2 \leq \frac{2(\lambda + b_N)^{2(2-p)}}{k\gamma_1^2} k \sum_{m=1}^{N} (r_m^2 + \gamma_2^2 b_{m-1}^2) + \frac{2(\lambda + b_N)^{2(2-p)}}{k\gamma_1} k \sum_{m=0}^{N} s_m^2.$$

By using now (3.4) and the estimate (3.8) (valid by assumption for $0 \leq m \leq N-1$) we get

$$b_N^2 \leq k\,(\lambda + b_N)^{2(2-p)} \left[\frac{2}{\gamma_1^2} \left[c + \frac{\gamma_2^2 \gamma_4}{\gamma_1 (1+\Lambda)^{p-2}} \exp(2\gamma_5 k\, N) \right] + \frac{2}{\gamma_1} c \right].$$

We divide both sides by $(\lambda + b_N)^{2(2-p)} \neq 0$ and use the inequality, valid for all $\lambda \geq 0$ and all $x \geq 1$

$$\frac{x^{2(p-1)}}{(\lambda+1)^{2(2-p)}} \leq \frac{x^2}{(\lambda + x)^{2(2-p)}}$$

with $x = b_N$. Next, we multiply both sides by $(1+\Lambda)^{2(2-p)}$ and we finally get

$$1 < b_N^{2(p-1)} \leq k\,(1+\Lambda)^{2(2-p)} \left[\frac{2c}{\gamma_1^2} + \frac{2\gamma_2^2 \gamma_4}{\gamma_1^3 (1+\Lambda)^{p-2}} \exp(2\gamma_5 k\, N) + \frac{2c}{\gamma_1} \right].$$

This gives a contradiction, provided that

$$0 < k < k_1 := \left[(1+\Lambda)^{2(2-p)} \left(\frac{2c}{\gamma_1^2} + \frac{2\gamma_2^2 \gamma_4}{\gamma_1^3 (1+\Lambda)^{p-2}} \exp(2\gamma_5 k\, N) + \frac{2c}{\gamma_1} \right) \right]^{-1}.$$

This proves that $b_N \leq 1$. Note that the value of k_1 depends only on $kN \leq kM$.

We pass now to prove the second part of the result, namely estimate (3.8) for $m = N$. Observe that, if $0 < b_m \leq 1$ and $1 < p \leq 2$, then

$$(1+\Lambda)^{p-2} \leq (1+\lambda)^{p-2} \leq (b_m + \lambda)^{p-2} \qquad \forall\, \lambda \in [0, \Lambda],$$

which is used to bound the second term on the left-hand side of (3.6) from below. We use several times Young's inequality to estimate the terms on the right-hand side of (3.6) for $b_m > 0$ by

$$\begin{aligned} b_m r_m &\leq \frac{\gamma_1 (1+\Lambda)^{p-2}}{6} b_m^2 + \frac{3}{2\gamma_1 (1+\Lambda)^{p-2}} r_m^2, \\ \gamma_3 b_{m-1} b_m^{1-\theta} a_m^\theta &\leq \frac{\gamma_1 (1+\Lambda)^{p-2}}{6} b_{m-1}^2 + \frac{\gamma_1 (1+\Lambda)^{p-2}}{6} b_m^2 + \gamma_5 a_m^2 \end{aligned}$$

with -for the sake of completeness-

$$\gamma_5 = \frac{\theta}{2} \gamma_3^{2/\theta} (1-\theta)^{\frac{1-\theta}{\theta}} \left(\frac{3(1+\Lambda)^{2-p}}{\gamma_1} \right)^{\frac{2-\theta}{\theta}}. \tag{3.10}$$

After these manipulations we multiply inequality (3.6) by k, sum over m, and absorb in the left-hand side the terms with b_m and b_{m-1}. This yields

$$a_N^2 + \frac{\gamma_1(1+\Lambda)^{p-2}}{2} k \sum_{m=1}^{N} b_m^2 \leq k \sum_{m=1}^{N} \frac{3r_m^2}{2\gamma_1(1+\Lambda)^{p-2}} + k \sum_{m=1}^{N} s_m^2 + \gamma_5 k \sum_{m=1}^{N} a_m^2.$$

If we define k_2 as $k_2 := (2\,\gamma_5)^{-1}$, then for all

$$0 < k < \overline{k} := \min\{k_1, k_2\} \tag{3.11}$$

we can absorb $k\gamma_5 a_N^2$ (the last term of the sum $\gamma_5 k \sum_{m=0}^{M} a_m^2$) in the left-hand side, to obtain

$$a_N^2 + \gamma_1(1+\Lambda)^{p-2} k \sum_{m=1}^{N} b_m^2 \leq k \sum_{m=1}^{N} \frac{3r_m^2}{\gamma_1(1+\Lambda)^{p-2}} + 2k \sum_{m=1}^{N} s_m^2 + 2\,\gamma_5 k \sum_{m=1}^{N-1} a_m^2.$$

Now we can apply the "standard" discrete Gronwall's lemma to deduce

$$\begin{aligned} a_N^2 + \gamma_1(1+\Lambda)^{p-2} k \sum_{m=1}^{N} b_m^2 &\leq \left(\frac{3(1+\Lambda)^{2-p}}{\gamma_1} k \sum_{m=1}^{N} r_m^2 + 2k \sum_{m=1}^{N} s_m^2 \right) \exp(2\,\gamma_5\, k\, N) \\ &\leq c\, k^2 \left(\frac{3(1+\Lambda)^{2-p}}{\gamma_1} + 2 \right) \exp(2\,\gamma_5\, k\, N), \end{aligned}$$

hence the assertion follows with

$$\gamma_4 := c \left(\frac{3(1+\Lambda)^{2-p}}{\gamma_1} + 2 \right). \tag{3.12}$$

This finishes the proof of the lemma. □

Lemma 3.2 is one of the main building blocks of the proof of Theorem 1.1. The strategy to prove the optimal $\mathcal{O}(k^2)$-error estimate for the Euler scheme (NS_p^k) is to study a proper error equation: We use the tool employed in [9] and average the equations over the net I^M, but additional new ideas will be needed in order to fit the resulting discrete estimates with the hypotheses of Lemma 3.2.

Definition 3.3 (Retarded-time-averaging). *Let be given a net I^M on $[0,T]$ and a function $\mathbf{v} \in L^1(\Omega_T)$, with $\Omega_T = (0,T) \times \Omega$. We define $\{\overline{\mathbf{v}}^m\}_{m=0}^M$, the sequence that is the retarded-time-averaging of $\mathbf{v}$, as follows:*

$$\overline{\mathbf{v}}^0(x) := \mathbf{v}(x,0) \quad \text{and} \quad \overline{\mathbf{v}}^m(x) := \frac{1}{k}\int_{t_{m-1}}^{t_m} \mathbf{v}(\sigma,x)\,d\sigma = \fint_{I_m} \mathbf{v}(\sigma,x)\,d\sigma, \qquad m \geq 1.$$

This kind of time-averaging seems crucial to obtain the optimal estimates and its employment has been suggested by well-known techniques of Large Eddy Simulation for turbulent flows (cf. Berselli, Iliescu, and Layton [6]), together with previous results in [9].

By averaging (NS_p) over I_m we obtain the following discrete system (in weak form):

$$\int_\Omega d_t \mathbf{u}(t_m) \cdot \boldsymbol{\varphi}\,dx + \int_\Omega \overline{\mathbf{S}(\mathbf{D}\mathbf{u})}^m \cdot \mathbf{D}\boldsymbol{\varphi}\,dx + \int_\Omega \overline{[\nabla \mathbf{u}]\mathbf{u}}^m \boldsymbol{\varphi}\,dx = \int_\Omega \overline{\mathbf{f}}^m \cdot \boldsymbol{\varphi}\,dx$$

for each vector-valued function $\boldsymbol{\varphi} \in W^{1,p}_{\operatorname{div}}(\Omega)$. Taking the difference with (3.1) we obtain the error equation

$$\int_\Omega d_t \mathbf{e}^m \cdot \boldsymbol{\varphi}\, dx + \int_\Omega \big(\overline{\mathbf{S}(\mathbf{D}\mathbf{u})}^m - \mathbf{S}(\mathbf{D}\mathbf{u}^m)\big) \cdot \mathbf{D}\boldsymbol{\varphi}\, dx \\ + \int_\Omega \big(\overline{[\nabla \mathbf{u}]\mathbf{u}}^m - [\nabla \mathbf{u}^m]\mathbf{u}^{m-1}\big) \cdot \boldsymbol{\varphi}\, dx = \int_\Omega \big(\overline{\mathbf{f}}^m - \mathbf{f}(t_m)\big) \cdot \boldsymbol{\varphi}\, dx,$$

for all $\boldsymbol{\varphi} \in W^{1,p}_{\operatorname{div}}(\Omega)$. We use as test function $\mathbf{e}^m$ and perform suitable integration by parts. The first resulting term in the left-hand side is treated by recalling that

$$\int_\Omega d_t \mathbf{e}^m \cdot \mathbf{e}^m\, dx = \frac{1}{2} d_t \|\mathbf{e}^m\|_2^2 + \frac{k}{2}\|d_t \mathbf{e}^m\|_2^2.$$

The term with p-structure is treated as follows:

$$\int_\Omega \big(\overline{\mathbf{S}(\mathbf{D}\mathbf{u})}^m - \mathbf{S}(\mathbf{D}\mathbf{u}^m)\big) \cdot \mathbf{D}\mathbf{e}^m\, dx \\ = \fint_{I_m}\int_\Omega \big(\mathbf{S}(\mathbf{D}\mathbf{u}(\sigma)) - \mathbf{S}(\mathbf{D}\mathbf{u}(t_m))\big) \cdot (\mathbf{D}\mathbf{u}(t_m) - \mathbf{D}\mathbf{u}^m)\, dx d\sigma \\ + \int_\Omega \big(\mathbf{S}(\mathbf{D}\mathbf{u}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{u}^m)\big) \cdot (\mathbf{D}\mathbf{u}(t_m) - \mathbf{D}\mathbf{u}^m)\, dx.$$

The second term from the right-hand side is estimated from below by using (2.6). The integrand of the first term from the right-hand side is estimated by using the quasi-norm trick of Lemma 2.4 (cf. (2.8)) as follows: for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\big(\mathbf{S}(\mathbf{D}\mathbf{u}(\sigma)) - \mathbf{S}(\mathbf{D}\mathbf{u}(t_m))\big) \cdot (\mathbf{D}\mathbf{u}(t_m) - \mathbf{D}\mathbf{u}^m) \\ \le \varepsilon |\mathbf{F}(\mathbf{D}\mathbf{u}(t_m)) - \mathbf{F}(\mathbf{D}\mathbf{u}^m)|^2 + c_\varepsilon |\mathbf{F}(\mathbf{D}\mathbf{u}(t_m)) - \mathbf{F}(\mathbf{D}\mathbf{u}(\sigma)|^2.$$

By Lemma 2.4 there exist constants $c, C > 0$ (independent of δ) such that

$$\int_\Omega \big(\overline{\mathbf{S}(\mathbf{D}\mathbf{u})}^m - \mathbf{S}(\mathbf{D}\mathbf{u}^m)\big) \cdot (\mathbf{D}\mathbf{u}(t_m) - \mathbf{D}\mathbf{u}^m)\, dx \\ \ge c\, \|\mathbf{F}(\mathbf{D}\mathbf{u}(t_m)) - \mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2 - C \fint_{I_m} \|\mathbf{F}(\mathbf{D}\mathbf{u}(t_m)) - \mathbf{F}(\mathbf{D}\mathbf{u}(\sigma))\|_2^2\, d\sigma.$$

Next we consider the convective term. We need to estimate the following expression:

$$\int_\Omega \overline{[\nabla \mathbf{u}]\mathbf{u}}^m \cdot \mathbf{e}^m - [\nabla \mathbf{u}^m]\mathbf{u}^{m-1} \cdot \mathbf{e}^m\, dx.$$

By integrating by parts, adding and subtracting $[\nabla \mathbf{e}^m]\mathbf{u}^{m-1} \cdot \mathbf{u}(t_m)$, and since $\mathbf{u}^m$ and $\mathbf{u}^{m-1}$ are divergence-free we get

$$-\fint_{I_m}\int_\Omega [\nabla \mathbf{e}^m]\mathbf{u}(\sigma) \cdot \mathbf{u}(\sigma)\, dx\, d\sigma + \fint_{I_m}\int_\Omega [\nabla \mathbf{e}^m]\mathbf{u}^{m-1} \cdot \mathbf{u}(t_m)\, dx\, d\sigma.$$

Then, in the first integral we add and subtract $[\nabla \mathbf{e}^m]\mathbf{u}(\sigma) \cdot \mathbf{u}(t_m)$, in the second integral we add and subtract $[\nabla \mathbf{e}^m]\mathbf{u}(t_{m-1}) \cdot \mathbf{u}(t_m)$, and we use some integration by parts to finally arrive at

$$\int_\Omega \overline{[\nabla \mathbf{u}]\mathbf{u}}^m \cdot \mathbf{e}^m - [\nabla \mathbf{u}^m]\mathbf{u}^{m-1} \cdot \mathbf{e}^m\, dx = \alpha_m + \beta_m + \kappa_m,$$

where

$$\begin{aligned}
\alpha_m &:= \fint_{I_m}\int_\Omega [\nabla \mathbf{e}^m]\mathbf{u}(\sigma)\cdot(\mathbf{u}(t_m)-\mathbf{u}(\sigma))\,dx\,d\sigma,\\
\beta_m &:= \fint_{I_m}\int_\Omega [\nabla \mathbf{e}^m]\big(\mathbf{u}(t_{m-1})-\mathbf{u}(\sigma)\big)\cdot\mathbf{u}(t_m)\,dx\,d\sigma,\\
\kappa_m &:= \int_\Omega [\nabla \mathbf{e}^m]\mathbf{e}^{m-1}\cdot\mathbf{u}(t_m)\,dx.
\end{aligned} \tag{3.13}$$

By collecting all previous results we finally get the following discrete inequality:

$$\begin{aligned}
&d_t\|\mathbf{e}^m\|_2^2+\|\mathbf{F}(\mathbf{Du}(t_m))-\mathbf{F}(\mathbf{Du}^m)\|_2^2\\
&\quad\le c\fint_{I_m}\|\mathbf{F}(\mathbf{Du}(t_m))-\mathbf{F}(\mathbf{Du}(\sigma))\|_2^2\,d\sigma+\\
&\qquad+c\left(\int_\Omega|\bar{\mathbf{f}}^m-\mathbf{f}(t_m)|\,|\mathbf{e}^m|\,dx+|\alpha_m|+|\beta_m|+|\kappa_m|\right).
\end{aligned}\tag{3.14}$$

We fit the left-hand side of this inequality in such a way that we can apply Lemma 3.2. First, we observe that from Lemma 2.5, with $r=p$ it follows that

$$(\delta+\|\mathbf{Du}(t_m)\|_p+\|\mathbf{De}^m\|_p)^{p-2}\|\mathbf{De}^m\|_p^2\le c\,\|\mathbf{F}(\mathbf{Du}(t_m))-\mathbf{F}(\mathbf{Du}^m)\|_2^2.$$

By Theorem 2.6 we have

$$\max_{0\le m\le M}\|\mathbf{Du}(t_m)\|_p\le\sup_{0\le t\le T}\|\mathbf{Du}(t)\|_p\le c,$$

with $c=c(\delta_0,p,C_0,\|\mathbf{f}\|,\|\mathbf{u}_0\|,T,\Omega)$. Consequently, we have

$$0\le\lambda:=\delta+\max_{0\le m\le M}\|\mathbf{Du}(t_m)\|_p\le\delta_0+\sup_{0\le t\le T}\|\mathbf{Du}(t)\|_p=:\Lambda. \tag{3.15}$$

So Λ depends on δ_0, p, C_0, $\|\mathbf{f}\|$, $\|\mathbf{u}_0\|$, T, and Ω. Using this we obtain from (3.14) for $m\ge 1$:

$$\begin{aligned}
&d_t\|\mathbf{e}^m\|_2^2+(\lambda+\|\mathbf{De}^m\|_p)^{p-2}\|\mathbf{De}^m\|_p^2\\
&\quad\le c\fint_{I_m}\|\mathbf{F}(\mathbf{Du}(t_m))-\mathbf{F}(\mathbf{Du}(\sigma))\|_2^2\,d\sigma\\
&\qquad+c\left(\int_\Omega|\bar{\mathbf{f}}^m-\mathbf{f}(t_m)|\,|\mathbf{e}^m|\,dx+|\alpha_m|+|\beta_m|+|\kappa_m|\right).
\end{aligned}\tag{3.16}$$

¿From the structure of this inequality it is clear that we will use later Lemma 3.2 with $a_m=\|\mathbf{e}^m\|_2$ and $b_m=\|\mathbf{De}^m\|_p$. Therefore, we are going to estimate the right-hand side in a way suitable for Lemma 3.2.

We start by recalling a result from [9, Lemma 4.6].

LEMMA 3.4. *Let* $\mathbf{u}$ *be such that* $(\mathbf{F}(\mathbf{Du}))_t\in L^2(\Omega_T)$. *Then*

$$\sum_{m=1}^M\int_{I_m}\|\mathbf{F}(\mathbf{Du}(t_m))-\mathbf{F}(\mathbf{Du}(t))\|_2^2\,dt\le k^2\|\big(\mathbf{F}(\mathbf{Du})\big)_t\|_{L^2(\Omega_T)}^2.$$

In order to estimate the second term of the right-hand side of (3.16) we prove a result similar to the classical Poincaré's inequality.

PROPOSITION 3.5. *Let be given* $\mathbf{v} \in L^2(0,T;L^q(\Omega))$*, with* $\mathbf{v}_t \in L^q(\Omega_T)$ *for some* $q \geq 2$*. Then,*

$$k\sum_{m=0}^{M} \|\mathbf{v}(t_m) - \overline{\mathbf{v}}^m\|_q^2 \leq T^{1-\frac{2}{q}}\, k^2 \|\mathbf{v}_t\|^2_{L^q(\Omega_T)}.$$

Proof. Since $\mathbf{v}_t \in L^q(\Omega_T)$, then $\mathbf{v} \in C(0,T;L^q(\Omega))$ and $\mathbf{v}(t_m)$ is well-defined. The proof of Proposition 3.5 follows by direct computation (recall that $\overline{\mathbf{v}}^0(x) = \mathbf{v}(x,0)$):

$$\begin{aligned} k\sum_{m=0}^{M} \|\mathbf{v}(t_m) - \overline{\mathbf{v}}^m\|_q^2 &= k\sum_{m=1}^{M}\left[\int_\Omega \left| \fint_{I_m} \mathbf{v}(x,t_m) - \mathbf{v}(x,\sigma)\, d\sigma\right|^q dx\right]^{\frac{2}{q}} \\ &\leq k\sum_{m=1}^{M}\left[\int_\Omega \left(\fint_{I_m}\int_{I_m} |\mathbf{v}_t(x,\tau)|\, d\tau\, d\sigma\right)^q dx\right]^{\frac{2}{q}}. \end{aligned}$$

Since $|I_m| = t_m - t_{m-1} = k$, by applying Hölder's inequality it follows that

$$\begin{aligned} k\sum_{m=0}^{M} \|\mathbf{v}(t_m) - \overline{\mathbf{v}}^m\|_q^2 &\leq k\sum_{m=1}^{M}\left[\int_\Omega \left(\int_{I_m} |\mathbf{v}_t(x,\tau)|\, d\tau\right)^q dx\right]^{\frac{2}{q}} \\ &\leq k\sum_{m=1}^{M}\left[\int_\Omega k^{q-1}\int_{I_m} |\mathbf{v}_t(x,\tau)|^q\, d\tau\, dx\right]^{\frac{2}{q}} \\ &\leq k^{1+2-\frac{2}{q}}\sum_{m=1}^{M}\left[\int_\Omega \int_{I_m} |\mathbf{v}_t(x,\tau)|^q\, d\tau\, dx\right]^{\frac{2}{q}} \\ &\leq k^{1+2-\frac{2}{q}}\left[\sum_{m=1}^{M}\int_{I_m} \int_\Omega |\mathbf{v}_t(x,\tau)|^q\, dx\, d\tau\right]^{\frac{2}{q}} \left[\sum_{m=1}^{M} 1\right]^{1-\frac{2}{q}} \\ &\leq T^{1-\frac{2}{q}}\, k^2 \|\mathbf{v}_t\|^2_{L^q(\Omega_T)}, \end{aligned} \tag{3.17}$$

where we used $M = T/k$. ☐

By applying Proposition 3.5 with $q = 2$ to the function $\mathbf{f} \in W^{1,2}(I;L^2)$ and by using the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ (valid for $p \geq 6/5$) in combination with Korn's inequality, we have the following result.

LEMMA 3.6. *Let* $\mathbf{f} \in W^{1,2}(I;L^2)$ *and* $p \geq 6/5$*, then*

$$\left|\int_\Omega (\overline{\mathbf{f}}^m - \mathbf{f}(t_m)) \cdot \mathbf{e}^m\, dx\right| \leq c\|\overline{\mathbf{f}}^m - \mathbf{f}(t_m)\|_2 \|\mathbf{D}\mathbf{e}^m\|_p,$$

where $c = c(\Omega, p)$ *and*

$$k\sum_{m=0}^{M} \|\overline{\mathbf{f}}^m - \mathbf{f}(t_m)\|_2^2 \leq k^2\, \|\mathbf{f}_t\|^2_{L^2(\Omega_T)}.$$

Let us now consider the terms α_m, β_m, and κ_m defined in (3.13). We start with κ_m, since it is simpler to treat.

LEMMA 3.7. *There exist* $\theta = \theta(p) \in (0,1]$ *such that*

$$|\kappa_m| \leq \begin{cases} c\,\|\nabla \mathbf{u}\|_{C(I;L^3(\Omega))}\, \|\mathbf{D}\mathbf{e}^{m-1}\|_p \|\mathbf{D}\mathbf{e}^m\|_p^{1-\theta} \|\mathbf{e}^m\|_2^\theta \\ c\,\|\nabla \mathbf{u}\|_{C(I;L^3(\Omega))}\, \|\mathbf{D}\mathbf{e}^m\|_p \|\mathbf{D}\mathbf{e}^{m-1}\|_p. \end{cases}$$

with $c = c(\Omega, p)$.

Proof. For $p > 3/2$ it follows by integration by parts and standard arguments

$$
\begin{aligned}
|\kappa_m| &= \left| \int_\Omega [\nabla \mathbf{u}(t_m)]\mathbf{e}^{m-1} \cdot \mathbf{e}^m \, dx \right| \\
&\le \|\nabla \mathbf{u}(t_m)\|_3 \, \|\mathbf{e}^m\|_{\frac{p}{p-1}} \, \|\mathbf{e}^{m-1}\|_{\frac{3p}{3-p}} \\
&\le c \, \|\nabla \mathbf{u}(t_m)\|_3 \|\mathbf{e}^m\|_{\frac{p}{p-1}} \, \|\mathbf{D}\mathbf{e}^{m-1}\|_p.
\end{aligned}
$$

Next, by using the convex interpolation $L^{\frac{p}{p-1}} = [L^{\frac{3p}{3-p}}, L^2]_\theta$ with $\theta = \frac{4(2p-3)}{5p-6}$, which holds true for $p \in (\frac{3}{2}, 2]$, the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-p}}(\Omega)$, and Korn's inequality we get the first inequality. Now, the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ (which holds true for $p \ge \frac{6}{5}$) implies the second inequality. $\square$

Remark 3.8. The interpolation in the previous lemma between L^2 and $L^{\frac{3p}{3-p}}$ is one of the reasons that the range of p is restricted to $p > \frac{3}{2}$. Indeed, by Theorem 2.6, (2.11a), we have $\nabla \mathbf{u} \in C(I; L^3(\Omega))$ as long as $p > \frac{3}{2}$.

The terms α_m and β_m can be bounded by using the information on the time derivative $\mathbf{u}_t$ and a Poincaré's-type inequality similar to Proposition 3.5. We have the following result.

Lemma 3.9. *Let* $\mathbf{u} \in C(\overline{\Omega_T})$ *be such that* $\mathbf{u}_t \in L^{p'}(\Omega_T)$. *Then*

$$
|\alpha_m| + |\beta_m| \le \rho_m \, \|\mathbf{D}\mathbf{e}^m\|_p,
$$

with

$$
k \sum_{m=0}^{M} \rho_m^2 \le c \, T^{\frac{2-p}{p}} k^2 \, \|\mathbf{u}\|^2_{L^\infty(\Omega_T)} \|\mathbf{u}_t\|^2_{L^{p'}(\Omega_T)},
$$

where $\rho_m := c' \, \|\mathbf{u}\|_{L^\infty(\Omega_T)} \| \fint_{I_m} \mathbf{u}(\tau) - \mathbf{u}(t_m) \, d\tau \|_{p'}$, $c' = c'(\Omega)$, *and* $c = (2c')^2$.

Proof. Let us prove the estimate for α_m. The other term β_m can be treated in the same way. By using Fubini-Tonelli's theorem, Hölder's inequality, and Korn's inequality we obtain

$$
\begin{aligned}
|\alpha_m| &= \left| \fint_{I_m} \int_\Omega \left[\nabla \big(\mathbf{u}(x, t_m) - \mathbf{u}^m(x) \big) \right] \mathbf{u}(x, \tau) \cdot \big(\mathbf{u}(x, \tau) - \mathbf{u}(t_m)(x) \big) \, dx \, d\tau \right| \\
&\le \|\mathbf{u}\|_{L^\infty(\Omega_T)} \int_\Omega |\nabla \mathbf{e}^m| \left| \fint_{I_m} \mathbf{u}(\tau) - \mathbf{u}(t_m) \, d\tau \right| dx \\
&\le c' \, \|\mathbf{u}\|_{L^\infty(\Omega_T)} \left\| \fint_{I_m} \mathbf{u}(\tau) - \mathbf{u}(t_m) \, d\tau \right\|_{p'} \|\mathbf{D}\mathbf{e}^m\|_p.
\end{aligned}
$$

We have, for $q = p'$

$$
\begin{aligned}
\rho_m^2 &\le c \, \|\mathbf{u}\|^2_{L^\infty(\Omega_T)} \left(\int_\Omega \left| \fint_{I_m} (\mathbf{u}(x, \tau) - \mathbf{u}(x, t_m)) \, d\tau \right|^{p'} dx \right)^{\frac{2}{p'}} \\
&\le c \, \|\mathbf{u}\|^2_{L^\infty(\Omega_T)} \left(\int_\Omega \left| \fint_{I_m} \int_{I_m} \mathbf{u}_t(x, s) \, ds \, d\tau \right|^{p'} dx \right)^{\frac{2}{p'}} \\
&\le c \, \|\mathbf{u}\|^2_{L^\infty(\Omega_T)} \left(\int_\Omega \left| \int_{I_m} |\mathbf{u}_t(x, \tau)| \, d\tau \right|^{p'} dx \right)^{\frac{2}{p'}}.
\end{aligned}
$$

With the same arguments as in (3.17) and summation over I_m we obtain

$$k \sum_{m=0}^{M} \rho_m^2 \le c\, T^{\frac{2-p}{p}} k^2 \,\|\mathbf{u}\|^2_{L^\infty(\Omega_T)} \|\mathbf{u}_t\|^2_{L^{p'}(\Omega_T)}.$$

This ends the proof. □

We have now at disposal all the tools needed to prove the optimal error estimate, that is the main result of this paper.

Proof of Theorem 1.1. We will prove the result by an application of Lemma 3.2 to (3.16) with:

$$\begin{aligned}
&a_m = \|\mathbf{e}^m\|_2, \qquad b_m = \|\mathbf{D}\mathbf{e}^m\|_p, \qquad r_m = \rho_m + c\,\|\overline{\mathbf{f}}^m - \mathbf{f}(t_m)\|_2,\\
&s_m^2 = c \fint_{I_m} \|\mathbf{F}(\mathbf{D}\mathbf{u}(t_m)) - \mathbf{F}(\mathbf{D}\mathbf{u}(t))\|_2^2\, dt,\\
&\gamma_1 = 1, \qquad \gamma_2 = \gamma_3 = c\,\|\nabla \mathbf{u}\|_{L^\infty(I;L^3(\Omega))},\\
&\lambda = \delta + \max_{0\le m\le M} \|\mathbf{D}\mathbf{u}(t_m)\|_p, \qquad \Lambda = \delta_0 + \sup_{0\le t\le T} \|\mathbf{D}\mathbf{u}(t)\|_p,
\end{aligned}$$

with c independent of δ. From Lemma 3.6, Lemma 3.9, the hypotheses on $\mathbf{f}$, the regularity (2.11) of the strong solution $\mathbf{u}$, and (2.14) (since $\frac{11p-12}{3(p-1)} > p'$ for $p > \frac{3}{2}$), it follows that

$$k \sum_{m=0}^{M} r^m \le c\, k^2.$$

Next, from Lemma 3.4 and the regularity property (2.10) of strong solutions it also holds that

$$k \sum_{m=0}^{M} s^m \le c\, k^2,$$

In the last two inequalities c depends on δ_0, p, C_0, $\|\mathbf{f}\|$, $\|\mathbf{u}_0\|$, T, and Ω. Hence, from Lemma 3.2 it follows that for $k \in (0, \overline{k})$ (cf. (3.11))

$$\begin{gathered}
\max_{0\le m\le M} \|\mathbf{e}^m\|_2^2 + (1+\Lambda)^{p-2} k \sum_{m=0}^{M} \|\mathbf{D}\mathbf{e}^m\|_p^2 \le \gamma_4 k^2 \exp(2\gamma_5 T)\\
\max_{0\le m\le M} \|\mathbf{D}\mathbf{e}^m\|_p \le 1.
\end{gathered}$$

By using this estimate and coming back to (3.14) all terms in the right-hand side can be easily bounded by $c\,k^2$. It finally follows that

$$\max_{0\le m\le M} \|\mathbf{e}^m\|_2^2 + k \sum_{m=0}^{M} \|\mathbf{F}(\mathbf{D}\mathbf{u}(t_m)) - \mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2 \le c\, k^2,$$

with c depending on δ_0, p, C_0, $\|\mathbf{f}\|$, $\|\mathbf{u}_0\|$, T, and Ω. This ends the proof of the theorem. □

Remark 3.10. The previous result holds also if we replace $\mathbf{f}(t_m)$ by $\overline{\mathbf{f}}^m$ in the scheme (NS_p^k).

4. Regularity properties of discrete solutions. In this section we show additional regularity properties of the discrete solutions. In Lemma 3.1 we proved existence of strong solutions to the discrete system (NS_p^k), but the estimate (3.3) was depending on k. The main goal here is to show uniform estimates with respect to the time step. Note that the result in the case $\delta = 1$ has been previously proved in [12].

THEOREM 4.1. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in \big(\frac{3}{2}, 2\big]$ *and* $\delta \in [0, \delta_0]$, *where* $\delta_0 > 0$. *Let* $\mathbf{f} \in C(I; W^{1,2}(\Omega))$, *where* $I = [0, T]$, *for some* $T > 0$, *and* $\mathbf{u}_0 \in W^{2,2}_{\mathrm{div}}(\Omega)$ *be given. Then, there exists* $\widehat{k} \in (0, \overline{k}\,]$ *with* $\widehat{k} = \widehat{k}(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$ *such that for* $k \in (0, \widehat{k})$ *the discrete solution* $\mathbf{u}^m$ *satisfies*

$$\max_{0\le m\le M} \|d_t\mathbf{u}^m\|_2^2 + k\sum_{m=0}^{M}\left(\|\nabla \mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^{\frac{2(5p-6)}{2-p}} + \|d_t\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2\right) \le c. \tag{4.1}$$

with a constant $c = c(\delta, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$. *In particular, Proposition 4.3 implies the following estimates in terms of usual Sobolev spaces: for each* $1 \le r < 6(p-1)$

$$\mathbf{u}^m \in l^{\frac{p(5p-6)}{2-p}}(I^M; W^{2,\frac{3p}{p+1}}(\Omega)) \cap l^\infty(I^M; W^{1,r}(\Omega)), \tag{4.2a}$$

$$d_t\mathbf{u}^m \in l^{\frac{p(5p-6)}{(3p-2)(p-1)}}(I^M; W^{1,\frac{3p}{p+1}}(\Omega)) \cap l^\infty(I^M; L^2(\Omega)), \tag{4.2b}$$

and the corresponding norms are bounded in terms of δ_0, p, C_0, $\|\mathbf{f}\|$, $\|\mathbf{u}_0\|$, T, *and* Ω. *Moreover, the norm in* $l^\infty(I^M; W^{1,r}(\Omega))$ *depends also on* r. *We also observe that* $\nabla\pi^m \in l^{\frac{2(5p-6)}{2-p}}(I^M; L^2(\Omega))$ *but the norm is bounded by a constant depending on* δ, p, C_0, C_1, $\|\mathbf{f}\|$, $\|\mathbf{u}_0\|$, T, *and* Ω, *which may explode as* $\delta \to 0^+$.

The proof of this result is based on delicate interplay of: a) induction, b) some discrete inequalities, c) the error estimate of the previous section. This is technically due to the fact that a single Gronwall argument seems not enough in order to prove uniform estimates.

REMARK 4.2. In [12] the extra regularity of strong solutions has been used to improve the rate of convergence, but the result was still sub-optimal. In contrast to this, in Theorem 1.1 of the previous section we obtained the optimal $\mathcal{O}(k^2)$-result directly

Interest for strong discrete solutions is also motivated by the fact (see Prohl and Růžička [23]) that they are required to get good error estimates for the space discretization.

Before starting, we recall some useful inequalities. A relevant point checked in [7, Lemma 4.2], is that the constants are independent of $\delta > 0$.

LEMMA 4.3. *Let* $p \in (1, 2]$ *and* $\delta > 0$. *For all sufficiently smooth functions* $\mathbf{v}$ *defined on* Ω_T, *with vanishing mean value over* Ω, *there holds for* $s \in [1, \infty)$

$$\begin{aligned}
\|\mathbf{v}\|^p_{W^{2,\frac{3p}{p+1}}(\Omega)} &\le c\left(\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v})\|_2^2 + \delta^p\right),\\
\|\nabla\mathbf{v}\|^2_{\frac{6s}{6-3p+s}} + \|\nabla^2\mathbf{v}\|^2_{\frac{2s}{2-p+s}} &\le c\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v})\|_2^2\left(\delta + \|\nabla\mathbf{v}\|_s\right)^{2-p},\\
\|\mathbf{v}_t\|^p_{W^{1,\frac{3p}{p+1}}(\Omega)} &\le c\,\|(\mathbf{F}(\mathbf{D}\mathbf{v}))_t\|_2^p\left(\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v})\|_2^2 + \delta^p\right)^{\frac{2-p}{2}},\\
&\le c\left(\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v})\|_2^2 + \|(\mathbf{F}(\mathbf{D}\mathbf{v}))_t\|_2^2 + \delta^p\right),\\
\|\mathbf{v}_t\|^2_{\frac{6s}{6-3p+s}} + \|\nabla\mathbf{v}_t\|^2_{\frac{2s}{2-p+s}} &\le c\,\|(\mathbf{F}(\mathbf{D}\mathbf{v}))_t\|_2^2\left(\delta + \|\nabla\mathbf{v}\|_s\right)^{2-p},
\end{aligned}$$

with constants depending only on Ω, p, and s and independent of $\delta > 0$.

When dealing with time discrete equations the natural quantity which arises taking the discrete time derivative and testing with $d_t\mathbf{u}$ is $\|d_t\mathbf{F}(\mathbf{u}^m)\|_2^2$. By using Lemma 2.2 we get

$$|d_t\mathbf{F}(\mathbf{D}\mathbf{u}^m)|^2 \simeq d_t\mathbf{S}(\mathbf{D}\mathbf{u}^m)\cdot d_t\mathbf{D}\mathbf{u}^m \simeq \big(\delta+|\mathbf{D}\mathbf{u}^m|+|\mathbf{D}\mathbf{u}^{m-1}|\big)^{p-2}|\mathbf{D}(d_t\mathbf{u}^m)|^2.$$

We now show related inequalities suitable to deal with time discrete problems. By using the same techniques as in [7], it is easy to check that also the following inequalities hold true for discrete functions.

PROPOSITION 4.4. *Let $p\in(1,2]$ and $\delta>0$. For all sufficiently smooth functions $\{\mathbf{v}^m\}_m$ (defined on the net I^M), with vanishing mean value over Ω, there holds for $s\in[1,\infty)$*

$$\|\mathbf{v}^m\|^p_{W^{2,\frac{3p}{p+1}}(\Omega)} \le c\,\big(\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v}^m)\|_2^2+\delta^p\,\big)\,, \tag{4.3a}$$

$$\|\nabla\mathbf{v}^m\|^2_{\frac{6s}{6-3p+s}}+\|\nabla^2\mathbf{v}^m\|^2_{\frac{2s}{2-p+s}} \le c\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v}^m)\|_2^2\,\big(\delta+\|\nabla\mathbf{v}^m\|_s\big)^{2-p}\,, \tag{4.3b}$$

$$\|d_t\mathbf{v}^m\|^p_{W^{1,\frac{3p}{p+1}}(\Omega)} \le c\,\|d_t\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^p\,\big(\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v}^m)\|_2^2+\delta^p\,\big)^{\frac{2-p}{2}} \tag{4.3c}$$

$$\le c\,\big(\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v}^m)\|_2^2+\|d_t\mathbf{F}(\mathbf{D}\mathbf{v}^m)\|_2^2+\delta^p\,\big)\,, \tag{4.3d}$$

$$\|d_t\mathbf{v}^m\|^2_{\frac{6s}{6-3p+s}}+\|\nabla d_t\mathbf{v}^m\|^2_{\frac{2s}{2-p+s}} \le c\,\|d_t\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2\,\big(\delta+\|\nabla\mathbf{v}^m\|_s\big)^{2-p}, \tag{4.3e}$$

with constants depending only on Ω, p, and s and independent of $\delta>0$. Moreover, for $1\le r<6(p-1)$ there exist a constant, depending on p, Ω, and r such that

$$\|\nabla\mathbf{v}^m\|_r^p \le c\,\Big(\delta^p+k\sum_{m=0}^M\big(\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{v}^m)\|_2^{\frac{2(5p-6)}{2-p}}+\|d_t\mathbf{F}(\mathbf{D}\mathbf{v}^m)\|_2^2\,\big)\Big). \tag{4.4}$$

Proof. The inequalities follow as in Lemma 4.3. In particular, the main point is to check again that the results in [12, Lemma 2.4] are independent of $\delta>0$. □

Proof of Theorem 4.1. The proof is mainly based on the fact that Theorem 1.1 implies (for a suitable choice of $\overline{k}$, independent of $\delta\in[0,\delta_0]$) that $\|\mathbf{D}\mathbf{e}^m\|_p\le 1$. This observation, together with the fact that $\|\mathbf{D}\mathbf{u}(t)\|_r$, $1\le r<6(p-1)$, is bounded implies then a uniform bound for $\|\mathbf{D}\mathbf{u}^m\|_p$. From this result it is possible to deduce all the uniform estimates claimed in the theorem. We only sketch the main steps of the proof and all missing details can be easily fixed by using the same arguments as [12, Theorem 6.3].

The case $\delta>0$. We have $\mathbf{u}$ a strong solutions of the continuous problem in $[0,T]$ and consequently (cf. Theorem 2.6) there exists $\tilde c_1=\tilde c_1(\delta_0,p,C_0,\mathbf{f},\mathbf{u}_0,T,\Omega,r)$ such that

$$\max_{0\le m\le M}\|\mathbf{D}\mathbf{u}(t_m)\|_p+\max_{0\le m\le M}\|\mathbf{D}\mathbf{u}(t_m)\|_r\le\tilde c_1, \tag{4.5}$$

with $\frac75<p<3<r<6(p-1)$. Clearly we may suppose $\tilde c_1\ge 1$.

¿From Theorem 1.1, it follows that $\max\limits_{0\le m\le M}\|\mathbf{D}\mathbf{e}^m\|_p\le 1$ for all $k\in(0,\overline{k})$, This, together with $\tilde c_1\ge1$, and (4.5) implies

$$\max_{0\le m\le M}\|\mathbf{D}\mathbf{u}^m\|_p\le 2\,\tilde c_1.$$

Then, an induction argument will show that $\{\mathbf{u}^m\}_m$ satisfies the following estimates:

$$\max_{0\le m\le M}\|\mathbf{D}\mathbf{u}^m\|_2^2+k\sum_{m=0}^{M}\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2\le\tilde c_2=\tilde c_2(\tilde c_1), \tag{4.6a}$$

$$\max_{1\le m\le M}\|d_t\mathbf{u}^m\|_2^2+k\sum_{m=0}^{M}\|d_t\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2\le\tilde c_3=\tilde c_3(\tilde c_1), \tag{4.6b}$$

$$k\sum_{m=0}^{M}\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^{\frac{2(5p-6)}{2-p}}\le\tilde c_4=\tilde c_4(\tilde c_1), \tag{4.6c}$$

$$\max_{0\le m\le M}\|\mathbf{D}\mathbf{u}^m\|_r\le\tilde c_5=\tilde c_5(\tilde c_1), \tag{4.6d}$$

$$\max_{0\le m\le M}\|\mathbf{D}\mathbf{u}(t_m)-\mathbf{D}\mathbf{u}^m\|_q\le\tilde c_1, \tag{4.6e}$$

$$\max_{0\le m\le M}\|\mathbf{D}\mathbf{u}^m\|_q\le 2\,\tilde c_1. \tag{4.6f}$$

The main steps are the same as in [12, Section 4], even if we put again some care to trace the behavior of the constants in terms of δ. Moreover, all calculations we perform are justified since by Lemma 3.1 (cf. (3.3)) we know that $\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2$ is finite (even if at that stage it may badly depend on k). The proof proceeds by induction on $0\le N\le M$. The starting step holds true, since if $N=0$, all estimates (4.6) hold true. To continue the inductive procedure we need to show that by assuming (4.6) for all $0\le m\le N-1$, we can conclude (4.6) also at the next step $m=N$.

We start by proving (4.6a). By using $-\Delta\mathbf{u}^m$ as test function, integration by parts, Korn's inequality, using (4.6f) at step $N-1$, and interpolation we get

$$\begin{aligned}
d_t\|\mathbf{D}\mathbf{u}^m\|_2^2+\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2&\le c\,(\|\mathbf{f}\|_{W^{1,2}}^2+\|\nabla\mathbf{u}^{m-1}\|_3\|\nabla\mathbf{u}^m\|_3^2)\\
&\le c\,(\|\mathbf{f}\|_{W^{1,2}}+c(\tilde c_1,\varepsilon)\|\nabla\mathbf{u}^m\|_2^2+\varepsilon\|\nabla\mathbf{u}^m\|_{\frac{3p}{3-p}}^2)\\
&\le c\,(\|\mathbf{f}\|_{W^{1,2}}+c(\tilde c_1,\varepsilon)\|\nabla\mathbf{u}^m\|_2^2+\varepsilon\|\nabla^2\mathbf{u}^m\|_p^2).
\end{aligned}$$

By using (4.3) (with $s=p$) we get

$$\begin{aligned}
d_t\|\mathbf{D}\mathbf{u}^m\|_2^2+\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2\le{}& c(\tilde c_1,\varepsilon)\big(\|\mathbf{f}\|_{W^{1,2}}^2+\|\nabla\mathbf{u}^m\|_2^2\big)\\
&+\varepsilon\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2\,(\delta+2\tilde c_1)^{2-p}
\end{aligned}$$

and by choosing ε small enough we can absorb the term with $\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2$ in the left-hand side and use Gronwall to deduce that there exists[§] $k_3\in(0,\overline{k}]$ with $k_3(\tilde c_1)$ and $\tilde c_2=\tilde c_2(\tilde c_1)$ such that for all $k\in(0,k_3)$

$$\max_{0\le m\le N}\|\mathbf{D}\mathbf{u}^m\|_2^2+k\sum_{m=0}^{N}\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^m)\|_2^2\le\tilde c_2.$$

To prove (4.6b), we take the discrete time derivative and use $d_t\mathbf{u}^m$ as test function.

[§] The positive number $\overline{k}$ is the time step obtained in Theorem 1.1.

With standard calculations one gives meaning to $\mathbf{u}^{-1}$ and obtains (cf. [12, p. 1186])

$$\begin{aligned}\|d_t\mathbf{u}^N\|_2^2 + k\sum_{m=0}^{N}\|d_t\mathbf{F}(\mathbf{Du}^m)\|_2^2 &\le c(\|\mathbf{f}\|,\|\mathbf{u}_0\|)\\ &\quad + k\sum_{m=0}^{N} c(\tilde{c}_1)\big(\|d_t\mathbf{u}^{m-1}\|_2^2 + \|d_t\mathbf{u}^m\|_2^2\big).\end{aligned}$$

Now, Gronwall's inequality implies the existence of $k_4 \in (0,k_3]$ with $k_4(\tilde{c}_1)$ and of $\tilde{c}_3 = \tilde{c}_3(\tilde{c}_1)$ such that (4.6b) holds, provided that $k \in (0,k_4)$.

The proof of (4.6c) is obtained by using again $-\Delta\mathbf{u}^m$ as test function, but keeping it on the right-hand side. Proceeding as in [11] we get

$$\begin{aligned}\|\nabla\mathbf{F}(\mathbf{Du}^m)\|_2^2 &\le c(\|\mathbf{f}\|,\|\mathbf{u}_0\|) + c(\tilde{c}_1)\|\nabla\mathbf{u}^m\|_2^2 + \left|\int d_t\mathbf{u}^m\Delta\mathbf{u}^m\,dx\right|\\ &\le c(\|\mathbf{f}\|,\|\mathbf{u}_0\|) + c(\tilde{c}_1) + \|d_t\mathbf{u}^m\|_{\frac{3p}{2p-1}}\|\nabla^2\mathbf{u}^m\|_{\frac{3p}{p+1}}.\end{aligned}$$

By using again interpolation and Proposition 4.3 we get (after some calculations)

$$k\sum_{m=0}^{N}\|\nabla\mathbf{F}(\mathbf{Du}^m)\|_2^{\frac{2(5p-6)}{2-p}} \le c(\tilde{c}_1)\Big(\delta^p + k\sum_{m=0}^{N}\|d_t\mathbf{F}(\mathbf{Du}^m)\|_2^2\Big) \le \tilde{c}_4(\tilde{c}_1),$$

where in the last step we used (4.6b). The next inequality

$$\max_{0\le m\le N}\|\mathbf{Du}^m\|_r \le \tilde{c}_5 = \tilde{c}_5(\tilde{c}_1)$$

is a direct consequence of the previous inequalities and of (4.3). Next, from (4.5) and (4.6d) we get

$$\max_{0\le m\le N}\|\mathbf{Du}(t_m) - \mathbf{Du}^m\|_r \le c(\tilde{c}_1, r), \tag{4.7}$$

since $1 \le r < 6(p-1)$. Moreover, from the error estimate we know that

$$\max_{0\le m\le M}\|\mathbf{Du}(t_m) - \mathbf{Du}^m\|_p \le c\,k^2.$$

Then, by using interpolation it follows that there exists $\widehat{k} \in (0,k_4]$, depending on the data of the problem such that

$$\max_{0\le m\le N}\|\mathbf{Du}(t_m) - \mathbf{Du}^m\|_q \le \tilde{c}_1,$$

for $\frac{3}{2} < p < q < r < 6(p-1)$ and for all $k \in (0,\widehat{k})$. Finally, by using again (4.5), estimate (4.6f) (with M replaced by N) follows for all $k \in (0,\widehat{k})$. This ends the proof of Theorem 4.1 in the case $\delta \in (0,\delta_0]$.

Moreover, we showed that all estimates are uniform in $\delta \in (0,\delta_0]$.

The case $\delta = 0$. We prove the regularity results for discrete solutions in the degenerate case by approximating the degenerate extra stress tensor with a non-degenerate one. If $\mathbf{S}$ is a degenerate extra stress tensor with (p,δ)-structure satisfying

Assumption 1 with $\delta = 0$ and if $\{\eta_\varepsilon\}_{\varepsilon>0}$, is a classical family of a mollifiers, then the extra stress tensor $\mathbf{S}^\varepsilon$ defined through

$$\mathbf{S}^\varepsilon(\mathbf{B}) := (\eta_\varepsilon * \mathbf{S})(\mathbf{B}) - (\eta_\varepsilon * \mathbf{S})(\mathbf{0}) \qquad \forall\, \mathbf{B} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{4.8}$$

is non-degenerate, satisfies Assumption 1 with $\delta = \varepsilon$, and converges to $\mathbf{S}$ as $\varepsilon \to 0^+$. Details can be found in [7, Section 3.1].

We now consider a family of approximate numerical schemes: Let $\mathbf{u}_\varepsilon^0 = \mathbf{u}_0$ and for $m \geq 1$ and $\mathbf{u}_\varepsilon^{m-1}$ given from the previous time step, compute the iterate $\mathbf{u}_\varepsilon^m$ as follows:

$$\begin{aligned} d_t\mathbf{u}_\varepsilon^m - \operatorname{div}\mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon^m) + [\nabla\mathbf{u}_\varepsilon^m]\mathbf{u}_\varepsilon^{m-1} + \nabla\pi_\varepsilon^m &= \mathbf{f}(t_m) \quad \text{in } \Omega,\\ \operatorname{div}\mathbf{u}_\varepsilon^m &= 0 \qquad\quad \text{in } \Omega. \end{aligned} \tag{4.9}$$

By the previous results $\mathbf{u}_\varepsilon^m$ can be bounded independently of $\varepsilon \in (0, \delta_0]$ as follows

$$\begin{aligned} \mathbf{u}_\varepsilon^m &\in l^{\frac{p(5p-6)}{2-p}}(I^M; W^{2,\frac{3p}{p+1}}(\Omega)) \cap l^\infty(I^M; W^{1,r}(\Omega)),\\ d_t\mathbf{u}_\varepsilon^m &\in l^{\frac{p(5p-6)}{(3p-2)(p-1)}}(I^M; W^{1,\frac{3p}{p+1}}(\Omega)) \cap l^\infty(I^M; L^2(\Omega)) \end{aligned}$$

and

$$k\sum_{m=0}^{M}\left(\|\nabla\mathbf{F}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon^m)\|_2^{\frac{2(5p-6)}{2-p}} + \|d_t\mathbf{F}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon^m)\|_2^2\right) \leq c.$$

Now the limit procedure can be handled exactly as in [7, Section 5] and this shows that

$$\mathbf{u}^m := \lim_{\varepsilon\to 0^+}\mathbf{u}_\varepsilon^m$$

is a strong solution of the (degenerate) limit problem. Finally $\mathbf{u}^m$ inherits from $\mathbf{u}_\varepsilon^m$ all the regularity properties stated above, by lower semi continuity of the norm. $\square$

Acknowledgments. Lars Diening and Michael Růžička have been supported by DFG Forschergruppe "Nonlinear Partial Differential Equations: Theoretical and Numerical Analysis". Lars Diening is also indebted to the Landesstiftung Baden-Württemberg. Luigi C. Berselli thanks the University of Freiburg for the kind hospitality during part of the preparation of the paper.

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