MUCKENHOUPT WEIGHTS IN VARIABLE EXPONENT SPACES

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ABSTRACT. In this article we define an appropriate Muckenhoupt class for variable exponent Lebesgue spaces, in other words, we characterize the set of weights ω for which the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n, \omega)$. The exponent is assumed to satisfy the usual log-Hölder continuity condition.

1. INTRODUCTION

During the last ten years, function spaces with variable exponent and related differential equations have attracted a lot of interest with contributions by over a hundred researchers so far, cf. the recent monograph [22]. Apart from interesting theoretical considerations, these investigations were motivated by a proposed application to modeling electrorheological fluids [2, 64, 66], and, more recently, an application to image restoration [1, 11, 33, 50]. In this article we focus on the function spaces aspect of variable exponent problems. For more information on the PDE aspect see e.g. [3, 5, 6, 8, 9, 26, 30, 34, 54, 70].

The first article on variable exponent Lebesgue spaces is by Orlicz in 1931 [60]. The research that followed dealt with rather general modular spaces, cf. [57]. Starting in the mid-70s, Polish mathematicians such as H. Hudzik, A. Kamińska and J. Musielak pursued a somewhat more concrete line of inquiry, see e.g. the monograph [56] for details. The spaces introduced, now know as Musielak–Orlicz spaces, are still actively studied today. Variable exponent spaces were considered again in 1991 by O. Kováčik and J. Rákosník [46] who obtained results on many basic properties. In 2001 X.-L. Fan and D. Zhao [27] independently reproved the basic results by recourse to the general theory of Musielak–Orlicz spaces.

During the 1990s there appeared a dozen or two papers on variable exponent spaces, but the development of the theory was rather sluggish. A central motivation for studying variable exponent spaces was the hope that many classical results from Lebesgue space theory could be generalized to this setting, but not to general Musielak–Orlicz spaces. Although this proved to be the case, it was often the result of complicated work, see e.g. [24, 25] by D. Edmunds and J. Rákosník on the Sobolev embedding. Progress was in fact being halted by the lack of one central tool: the Hardy–Littlewood maximal operator.

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The problem of the boundedness of the maximal operator in $L^{p(\cdot)}(\Omega)$ was solved in the local case by L. Diening [19], who also showed the importance and geometric significance of the so-called log-Hölder condition in variable exponent spaces. His technique was soon generalized to the unbounded case by D. Cruz-Uribe, A. Fiorenza and C. Neugebauer [13] and, independently, A. Nekvinda [58].

It is not difficult to define variable exponent spaces also in the weighted case or indeed in the case of general measure spaces. The basic properties from [27, 46] hold also in this case [35]. However, it turns out that the maximal operator presents a substantial new challenge. In essence, Diening's method from [19] reduces the problem to a global application of the classical maximal inequality with exponent $p^- := \operatorname{ess\,inf} p$. In the weighted situation one might at best be able to handle an A_{p^-}-weight with this approach which would not be so interesting; further, this would obviously not give a necessary and sufficient condition.

In the absence of a general theory, V. Kokilashvili, S. Samko and their collaborators have proved several boundedness results with particular classes of weights: initially in the case of power-type weights [44, 68, 69, 71] and more recently in the case of weights which are controlled by power-type functions [7, 40, 41, 42, 43, 63, 67]. Other investigations with such weights include [4, 10, 39, 51, 52]; more general metric measure spaces have been studied for instance in [28, 32, 35, 53]; $p(r)$-type Laplacian weighted ODEs have been considered in [72, 73]. Obtaining weighted results was also explicitly mentioned as an open problem by D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Pérez [14].[1]

In this article an appropriate generalization of Muckenhoupt's A_p-weights [55] is introduced for variable exponent spaces; in other words we characterize the class of weights ω for which the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n, \omega)$. We treat ω as a measure. The exact definition of the class $A_{p(\cdot)}$ is given in Section 3. The space $L^{p(\cdot)}(\mathbb{R}^n, \omega)$ and the set $\mathcal{P}^{\log}(\mathbb{R}^n)$ are defined in Section 2. The main result of this paper is the following:

Theorem 1.1. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. Then*

$$M\colon L^{p(\cdot)}(\mathbb{R}^n, \omega) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n, \omega) \quad \text{if and only if} \quad \omega \in A_{p(\cdot)}.$$

The embedding constant depends only on the characteristics of p and on $\|\omega\|_{A_p}$.

The assumption $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ is standard in the variable exponent context, although not strictly speaking necessary even in the unweighted case [20, 47, 49, 59, 62]; see also [21] on the necessity of the assumption $1 < p^-$.

Most of the paper is devoted to the proof of Theorem 1.1. In contrast to the classical case, both the necessity and the sufficiency of the $A_{p(\cdot)}$-condition are non-trivial. We start in the next section with reiterating the necessary background. The key ingredient for the proof, the so-called local-to-global theorem, is introduced at the end of the section. In Section 3 we introduce the class $A_{p(\cdot)}$ and prove several basic properties including monotonicity, duality and reverse factorization. In Section 5 the sufficiency of the $A_{p(\cdot)}$-condition is shown, whereas Section 6 deals with its necessity.

[1]After circulating a preprint of this article, it appeared that D. Cruz-Uribe, A. Fiorenza and C. Neugebauer are currently working on the problem of the boundedness of the maximal operator in the weighted case and are preparing a paper containing certain local versions of the sufficiency in the main theorem of this paper that they have obtained independently from this work. Also, after the completion of this paper the authors and D. Cruz-Uribe have derived similar results in the setting of weights as multipliers [12].

It should be noted that once one knows how to prove the boundedness of the maximal operator, one easily obtains the boundedness of several other operators through extrapolation. Such results have recently been studied e.g. by Cruz-Uribe, Martell, Pérez and collaborators, cf. [14, 15, 16, 17, 18]; the first mentioned paper deals with the variable exponent case. The so-called diagonal case of extrapolation (originally due to Rubio de Francia [65]) is easy to generalize to the variable exponent weighted case. For instance it allows us to obtain the Poincaré inequality

$$\|u - u_{B,\omega}\|_{L^{p(\cdot)}(B,\omega)} \lesssim \operatorname{diam}(B)\, \|\nabla u\|_{L^{p(\cdot)}(B,\omega)},$$

for a ball $B \subset \mathbb{R}^n$, $p \in \mathcal{P}^{\log}_{\pm}(B)$, $\omega \in A_{p(\cdot)}$ and $u \in W^{1,p(\cdot)}(B,\omega)$, where $u_{B,\omega}$ denotes the average with respect to the measure $\omega\, dx$. It appears to be much more challenging to generalize the off-diagonal case (originally due to Harboure, Macías and Segovia [31]). This result would allow us for instance to obtain the correct mapping properties for the Riesz potential. But this case has to be left to future investigations.

Let us conclude the introduction by considering some other recent advances on maximal operators in variable exponent spaces. T. Kopaliani [45], building on Diening [20], recently showed that

$$M\colon L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n) \quad \text{if and only if} \quad \sup_Q \tfrac{1}{|Q|} \|\chi_Q\|_{p(\cdot)}\, \|\chi_Q\|_{p'(\cdot)} < \infty$$

provided p is bounded away from 1 and ∞ and constant outside a large ball. (Incidentally, the latter condition can be weakened to log-Hölder decay using [36].) Kopaliani's condition can be seen as the $A_{p(\cdot)}$-condition for $\omega \equiv 1$, namely, in the constant exponent case we have

$$M\colon L^{p}(\mathbb{R}^n,\omega) \hookrightarrow L^{p}(\mathbb{R}^n,\omega) \quad \text{if and only if} \quad \sup_Q \tfrac{1}{|Q|} \|\omega^{\frac{1}{p}}\chi_Q\|_{p}\, \|\omega^{-\frac{1}{p}}\chi_Q\|_{p'} < \infty.$$

We show in Remark 3.11 that $\sup_Q \frac{1}{|Q|} \|\omega^{\frac{1}{p(\cdot)}}\chi_Q\|_{p(\cdot)}\, \|\omega^{-\frac{1}{p(\cdot)}}\chi_Q\|_{p'(\cdot)}$ is indeed bounded if $\omega \in A_{p(\cdot)}$ and $p \in \mathcal{P}^{\log}_{\pm}$. In view of this and the result of Kopaliani, one could reasonably conjecture that the condition $p \in \mathcal{P}^{\log}$ is not truly important in the weighted case either and might be dropped. However, it has been shown in [22, Theorem 5.3.4] by a counterexample that $\sup_Q \frac{1}{|Q|} \|\chi_Q\|_{p(\cdot)}\, \|\chi_Q\|_{p'(\cdot)} < \infty$ does not imply $M\colon L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$. Thus even in the case $\omega \equiv 1$ some additional condition must be placed on p. In this paper the condition is $p \in \mathcal{P}^{\log}_{\pm}$. The question is whether it can be weakened. It is conceivable that the boundedness of the maximal operator in the unweighted space $L^{p(\cdot)}(\mathbb{R}^n)$ would also play a role. In the classical setting this is a non-issue, since the operator is always bounded in this case. Let us formulate these speculations as a question:

Question 1.2. *Let p be a variable exponent such that $M\colon L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$. Is it then true that*

$$\omega \in A_{p(\cdot)} \quad \textit{if and only if} \quad \sup_Q \tfrac{1}{|Q|} \|\omega^{\frac{1}{p(\cdot)}}\chi_Q\|_{p(\cdot)}\, \|\omega^{\frac{1}{p(\cdot)}}\chi_Q\|_{p'(\cdot)} < \infty$$

$$\textit{if and only if} \quad M\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n,\omega)?$$

A further open question is whether any of the techniques of this paper can be extended to Musielak–Orlicz spaces or other Banach function spaces.

2. Preliminaries

In this section we present background material, mostly relating to variable exponent Lebesgue spaces. For more information on these spaces we refer to the recent monograph [22].

Definitions and conventions. The notation $f \lesssim g$ means that $f \leqslant cg$ for some constant c, and $f \approx g$ means $f \lesssim g \lesssim f$. By c or C we denote a generic constant, whose value may change between appearances even within a single line. By cD we denote the concentric c-fold dilate of the ball or cube D. A measure is *doubling* if $\mu(2B) \leqslant C\mu(B)$ for every ball B. By $\fint_E f\,dx$ we denote the integral average of f over E. The notation $A\colon X \hookrightarrow Y$ means that A is a continuous embedding from X to Y.

By $\Omega \subset \mathbb{R}^n$ we denote an open set. A measurable function $p\colon \Omega \to [1,\infty)$ is called a *variable exponent*, and for $A \subset \Omega$ we denote

$$p_A^+ := \operatorname*{ess\,sup}_{x\in A} p(x), \quad p_A^- := \operatorname*{ess\,inf}_{x\in A} p(x), \quad p^+ := p_\Omega^+ \quad \text{and} \quad p^- := p_\Omega^-.$$

We always assume that $p^+ < \infty$. The conjugate exponent $p'\colon \Omega \to [1,\infty]$ is defined point-wise by $\frac{1}{p} + \frac{1}{p'} = 1$.

The (Hardy–Littlewood) *maximal operator* M is defined on L^1_{loc} by

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy.$$

We will mostly use this centered version over balls, but it is clear that the boundedness of this operator is equivalent to the boundedness of the non-centered maximal operator, or to that of the maximal operator over cubes.

Logarithmic Hölder continuity. We say that p satisfies the *local* log-*Hölder continuity* condition if

$$|p(x) - p(y)| \leqslant \frac{c}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$. If

$$|p(x) - p_\infty| \leqslant \frac{c}{\log(e + |x|)}$$

for some $p_\infty \geqslant 1$, $c > 0$ and all $x \in \Omega$, then we say p satisfies the log-*Hölder decay condition* (*at infinity*). We denote by $\mathcal{P}^{\log}(\Omega)$ the class of variable exponents which are log-*Hölder continuous*, i.e. which satisfy the local log-Hölder continuity condition and the log-Hölder decay condition. Actually, this class is somewhat too weak for us, and we will usually need the class $\mathcal{P}^{\log}_{\pm}(\Omega)$ which consists of those $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. The constant c in the log-Hölder condition and the bounds p^- and p^+ will be called the *characteristics* of p.

The reason that the log-Hölder continuity condition is so central in the study of variable exponent spaces was discovered by L. Diening [19] who noted that it implies that

$$\sup_{x,y\in B} |B|^{-|p(x)-p(y)|} \leqslant \max\left\{1, |B|^{p_B^- - p_B^+}\right\} \lesssim 1$$

for all balls B with radius bounded by a given constant. It follows that when working in small balls we can change the exponent on any quantity which is polynomial in the radius of the ball. Subsequently, D. Cruz-Uribe, A. Fiorenza and C. Neugebauer [13] noticed that the decay condition implies that

$$\sup_{x,y\in B} |B|^{|p(x)-p(y)|} \leqslant |B|^{p_B^+-p_B^-} \lesssim 1$$

if $B = B(z,r)$ is a ball relatively far away from the origin in the sense that $|z| \geqslant Lr$ for some fixed $L > 1$.

Denote by p_B the harmonic average of p over B, i.e.

$$p_B := \left(\fint_B \frac{1}{p(x)}\, dx \right)^{-1}.$$

Since $p_B^- \leqslant p_B \leqslant p_B^+$ it is clear that $|B|^{p_B^-} \approx |B|^{p_B} \approx |B|^{p_B^+}$ for small balls when $p \in \mathcal{P}^{\log}$. Additionally, one easily calculates from the decay condition that

$$|B|^{p_\infty} \approx |B|^{p_B}$$

for all balls with radius larger than some constant. These properties will be used often also in this article. Establishing their analogues with $|B|$ replaced by $\omega(B)$ for $\omega \in A_\infty$ is the first central step when starting to work with $A_{p(\cdot)}$-weights in Section 3.

Since we deal only with bounded exponents, we can change the harmonic mean in a power to an arithmetic mean:

Lemma 2.1. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Then $|B|^{p_B} \approx |B|^{\langle p\rangle_B}$ for every ball $B \subset \mathbb{R}^n$, where $\langle p\rangle_B = \fint_B p(x)\, dx$.*

Proof. Since $p_B^- \leqslant \langle p\rangle_B \leqslant p_B^+$, the equivalence $|B|^{p_B^-} \approx |B|^{p_B} \approx |B|^{p_B^+}$ yields the claim for all balls of radius at most 1, so we assume that $B = B(x,r)$ with $r > 1$. Since p is bounded, it suffices to show that $r^{\frac{\langle p\rangle_B}{p_B}-1} \approx 1$. For this we estimate

$$\Big|\frac{\langle p\rangle_B}{p_B} - 1\Big| = \left| \fint_B \fint_B \frac{p(x)-p(y)}{p(y)}\, dx\, dy \right| \leqslant \fint_B \fint_B |p(x)-p(y)|\, dx\, dy \leqslant 2 \fint_B |p(x)-p_\infty|\, dx.$$

Then we use the decay condition and find that

$$\begin{aligned}\Big|\frac{\langle p\rangle_B}{p_B} - 1\Big| &\lesssim \fint_B \frac{1}{\log(e+|x|)}\, dx \leqslant \fint_{B(0,r)} \frac{1}{\log(e+|x|)}\, dx \\ &= \fint_{B(0,1)} \frac{1}{\log(e+r\,|z|)}\, dz = \frac{c}{\log(e+r)} \int_0^1 \frac{s^{n-1}\log(e+r)}{\log(e+rs)}\, ds.\end{aligned}$$

Now we see that the integrand in the last step is at most 1 for every $s \in [0,1]$, so we conclude that $\log(e+r)\big|\frac{\langle p\rangle_B}{p_B} - 1\big| \lesssim 1$. But this clearly implies that $r^{\frac{\langle p\rangle_B}{p_B}-1} \approx 1$, so we are done. □

The variable exponent Lebesgue space. By ω we always denote a *weight*, i.e. a locally integrable function with range $(0,\infty)$. In a classical Lebesgue space the relation between the modular $\varrho(\cdot)$ and norm $\|\cdot\|$ is very simple:

$$\|f\|_{L^p(\Omega,\omega)} = \big(\varrho_{L^p(\Omega,\omega)}(f)\big)^{\frac{1}{p}} \quad \text{where} \quad \varrho_{L^p(\Omega,\omega)} = \int_\Omega |f(x)|^p\, \omega(x)\, dx.$$

In the variable exponent context we retain the form of the modular, but define the norm in the spirit of the Luxemburg norm in Orlicz spaces (or the Minkowski functional in abstract spaces):

$$\begin{aligned} &\|u\|_{L^{p(\cdot)}(\Omega,\omega)} := \inf\Big\{\lambda>0 :\ \varrho_{L^{p(\cdot)}(\Omega,\omega)}\Big(\frac{u}{\lambda}\Big) \leqslant 1\Big\}, \\ &\text{where} \quad \varrho_{L^{p(\cdot)}(\Omega,\omega)}(u) := \int_\Omega |u(x)|^{p(x)}\, \omega(x)\, dx. \end{aligned} \tag{2.2}$$

We omit ω from the notation of modular and norm if $\omega \equiv 1$.

It is clear that

$$\|u\|_{L^{p(\cdot)}(\Omega,\omega)} = \big\|u\, \omega^{\frac{1}{p(\cdot)}}\big\|_{L^{p(\cdot)}(\Omega)} = \big\|u\, \omega^{\frac{1}{p(\cdot)}}\, \chi_\Omega\big\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The following rather crude relationship between norm and modular is surprisingly useful:

$$\begin{aligned} &\min\Big\{\varrho_{L^{p(\cdot)}(\Omega,\omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega,\omega)}(f)^{\frac{1}{p^+}}\Big\} \\ &\qquad \leqslant \|f\|_{L^{p(\cdot)}(\Omega,\omega)} \leqslant \max\Big\{\varrho_{L^{p(\cdot)}(\Omega,\omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega,\omega)}(f)^{\frac{1}{p^+}}\Big\}. \end{aligned} \tag{2.3}$$

The proof of this well-known fact follows directly from the definition of the norm.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega,\omega)$ consists of all measurable functions $f\colon \Omega \to \mathbb{R}$ for which $\|f\|_{L^{p(\cdot)}(\Omega,\omega)} < \infty$. Equipped with this norm, $L^{p(\cdot)}(\Omega,\omega)$ is a Banach space. The variable exponent Lebesgue space is a Musielak–Orlicz space, and for a constant function p it coincides with the standard Lebesgue space. Basic properties of these spaces can be found in [22, 27, 46].

Hölder's inequality can be written in the form

$$\|fg\|_{L^{s(\cdot)}(\mathbb{R}^n,\omega)} \leqslant 2\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n,\omega)},$$

where $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$ [22, Lemma 3.2.20]. It holds for any weight ω, and indeed for more general measures as well.

The local-to-global method. In [36] a simple and convenient method to pass from local to global results was introduced; it is in some sense a generalization the following property of the Lebesgue norm:

$$\|f\|^p_{L^p(\mathbb{R}^n)} = \sum_i \|f\|^p_{L^p(\Omega_i)} \tag{2.4}$$

for a partition of $\mathbb{R}^n$ into measurable sets Ω_i. By a *partition* we mean that the sets Ω_i are disjoint and cover $\mathbb{R}^n$ up to a set of measure zero. The idea is to obtain global results by summing up a collection of local ones. In the variable exponent case it seems that arbitrary partitions will not do, rather we need to restrict our attention to special kinds of partitions.

Definition 2.5. An *orderly partition* is a partition (Q_j) of $\mathbb{R}^n$ into equal sized cubes, ordered so that $i > j$ if $\operatorname{dist}(0, Q_i) > \operatorname{dist}(0, Q_j)$.

The following result, which is critical for many later proofs, appears as Theorem 2.4 of [36] (cf. Section 4 of the same reference for the inclusion of weights). Note that the claim of the theorem holds trivially if p is constant, by (2.4).

Theorem 2.6 (Local-to-Global Theorem)**.** *If p satisfies the* log*-Hölder decay condition and is bounded and (Q_j) is as in Definition 2.5, then*

$$\big\| \|f\|_{L^{p(\cdot)}(Q_i,\omega)} \big\|_{l^{p_\infty}} \approx \|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}.$$

We conclude the introduction by another useful result which says that we can move between a variable exponent space and a constant exponent space provided we have an appropriate bound on our functions. This is an extension of [36, Lemma 5.1] and [23, Lemma 4.5] which dealt with the case $\beta = 0$. We say that the weight ω has *at most polynomial growth* if there exists $q > 0$ such that $\omega(B(0,r)) \lesssim r^q$ for $r > 1$. Note that this certainly holds if $\omega \in A_\infty$.

Lemma 2.7. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$, $\beta \in \mathbb{R}$, and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a function with $|f(x)| \lesssim (1+|x|)^\beta$. If ω has at most polynomial growth, then $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \approx \|f\|_{L^{p_\infty}(\mathbb{R}^n,\omega)}$.*

Proof. We consider three cases. If $\|f\|_{L^{p_\infty}(\mathbb{R}^n,\omega)} = 0$, then $f \equiv 0$ almost everywhere, and the claim is clear.

If $0 < \|f\|_{L^{p_\infty}(\mathbb{R}^n,\omega)} < \infty$, then we may assume that $\|f\|_{L^{p_\infty}(\mathbb{R}^n,\omega)} = 1$ since the claim is homogeneous in f. Let $\tilde{p} := \min\{p_\infty, p\}$. By Hölder's inequality, $L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow L^{\tilde{p}(\cdot)}(\mathbb{R}^n,\omega)$ if $\|1\|_{L^{r(\cdot)}(\mathbb{R}^n,\omega)} < \infty$, where $\frac{1}{\tilde{p}} = \frac{1}{p} + \frac{1}{r}$. The definition of $\tilde{p}$ and the decay condition imply that

$$\frac{1}{r(x)} = \max\Big\{\frac{1}{p_\infty} - \frac{1}{p(x)}, 0\Big\} \leqslant \frac{c}{\log(e+|x|)}.$$

Hence $r(x) \geqslant c\log(e+|x|)$; denoting by q the exponent from the growth bound of $\omega(B(0,r))$, we conclude that

$$\begin{aligned} \varrho_{L^{r(\cdot)}(\mathbb{R}^n,\omega)}(\lambda) &\leqslant \sum_j \int_{B(0,j+1)\setminus B(0,j)} \lambda^{r(x)}\omega(x)\,dx \lesssim \sum_j (e+j)^{c\log\lambda}\omega(B(0,j+1)) \\ &\leqslant \sum_j (e+j)^{c\log\lambda + q} < \infty, \end{aligned}$$

provided $\lambda \in (0,1)$ is chosen small enough. Therefore, $L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow L^{\tilde{p}(\cdot)}(\mathbb{R}^n,\omega)$.

Since $|f| \lesssim (1+|x|)^\beta$ and $p_\infty \geqslant \tilde{p}(\cdot)$ we conclude that

$$|f(x)|^{p_\infty} = (1+|x|)^{\beta p_\infty}\Big(\frac{|f(x)|}{(1+|x|)^\beta}\Big)^{p_\infty} \lesssim (1+|x|)^{\beta(p_\infty - \tilde{p}(x))}|f(x)|^{\tilde{p}(x)}.$$

It follows from the decay condition on p that $(1+|x|)^{p_\infty - \tilde{p}(x)} \leqslant C$. Hence we obtain that

$$\int_{\mathbb{R}^n} |f(x)|^{\tilde{p}(x)}\omega(x)\,dx \gtrsim \int_{\mathbb{R}^n} |f(x)|^{p_\infty}\omega(x)\,dx = \|f\|^{p_\infty}_{L^{p_\infty}(\mathbb{R}^n,\omega)} = 1.$$

Since p is bounded, it follows that $\|f\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^n,\omega)} \geqslant C$, which combined with the embedding yields $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \geqslant C$. The opposite inequality is proved analogously.

Finally, suppose that $\|f\|_{L^{p_\infty}(\mathbb{R}^n,\omega)} = \infty$ and take a sequence of non-negative functions f_i such that $f_i \nearrow |f|$ and $\|f_i\|_{L^{p_\infty}(\mathbb{R}^n,\omega)} < \infty$. Then it follows by the second case and monotone convergence (cf. Theorem 2.3.17, [22]) that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} = \lim \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \approx \lim \|f_i\|_{L^{p_\infty}(\mathbb{R}^n,\omega)} = \infty$. $\square$

3. The Muckenhoupt class $A_{p(\cdot)}$

Let us define the class $A_{p(\cdot)}$ to consist of those weights ω for which

$$\|\omega\|_{A_{p(\cdot)}} := \sup_{B\in\mathcal{B}} |B|^{-p_B} \|\omega\|_{L^1(B)} \|\tfrac{1}{\omega}\|_{L^{p'(\cdot)/p(\cdot)}(B)} < \infty,$$

where $\mathcal{B}$ denotes the family of all balls in $\mathbb{R}^n$ and $\|\cdot\|_{L^{p'(\cdot)/p(\cdot)}(B)}$ is defined as in (2.2) even when $p'(\cdot)/p(\cdot)$ is not greater or equal to one. (If $p'(\cdot)/p(\cdot)$ takes values also in $(0,1)$, then $\|\cdot\|_{L^{p'(\cdot)/p(\cdot)}(B)}$ is not a norm but only a quasi-norm.) Note that this class is the ordinary Muckenhoupt class A_p if p is a constant function; for properties of A_p we refer to [29, 61]. The classes $A_{p(\cdot)}(D)$ and $A^{\mathrm{loc}}_{p(\cdot)}$ are defined using the same formula, but $\mathcal{B}$ is now the family of all balls in $D \subset \mathbb{R}^n$ and all balls in $\mathbb{R}^n$ with radius at most 1, respectively. When we need some specific family of sets $\mathcal{B}$, we use the notation $A^{\mathcal{B}}_{p(\cdot)}$ and $\|\omega\|_{A^{\mathcal{B}}_{p(\cdot)}}$. In what follows we often write $\omega(B)$ for $\|\omega\|_{L^1(B)}$, i.e. we think of ω also as a measure.

In the classical case, $\|\omega\|_{A_p}$ is called the A_p-constant of the weight and with p it determines the embedding constant of $M\colon L^p(\mathbb{R}^n,\omega) \hookrightarrow L^p(\mathbb{R}^n,\omega)$ (see [61]). In the variable exponent context this is not quite true, as the following example shows. Let $\omega_a \equiv a \in (0,\infty)$. Then $\|\omega_a\|_{A_{p(\cdot)}} = \|1\|_{A_{p(\cdot)}}$ is independent of a. Consider now a variable exponent with $p|_{D_1} \equiv p_1$ and $p|_{D_2} \equiv p_2 \neq p_1$. Suppose that $|D_1|, |D_2| \in (0,\infty)$. Then

$$\|\chi_{D_1}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega_a)} = \|\chi_{D_1}\|_{L^{p_1}(D_1,\omega_a)} = (a\,|D_1|)^{\frac{1}{p_1}}$$

and

$$\|M\chi_{D_1}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega_a)} \geqslant \|M\chi_{D_1}\|_{L^{p_2}(D_2,\omega_a)} \geqslant (a\,|D_2|)^{\frac{1}{p_2}} |D_1|\, r^{-n},$$

where $r = \sup_{x\in D_1, y\in D_2} |x-y|$. Thus we obtain

$$\frac{\|M\chi_{D_1}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega_a)}}{\|\chi_{D_1}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega_a)}} \gtrsim a^{\frac{1}{p_2}-\frac{1}{p_1}}$$

and see that the embedding constant depends on a even though $\|\omega_a\|_{A_{p(\cdot)}}$ does not. In essence, this is just another manifestation of the non-homogeneity of the variable exponent modular. The further the weight is from 1, the greater the problem. In the variable exponent setting we define the $A_{p(\cdot)}$-*constant* of the weight ω to be $\|\omega\|_{A_{p(\cdot)}} + \omega(B(0,1)) + \frac{1}{\omega(B(0,1))}$. It turns out that this quantity, together with the characteristics of the exponent, is sufficient to control the embedding constant.

Let us now derive some results on weights in the class $A_{p(\cdot)}$. We will use the notation C_{incl} throughout the article for the constant appearing in the next lemma.

Lemma 3.1. *Let $p, q \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$. If $q \leqslant p$, then there exists a constant C_{incl} depending on the* log*-Hölder constants of p and q such that $\|\omega\|_{A_{p(\cdot)}} \leqslant C_{\mathrm{incl}} \|\omega\|_{A_{q(\cdot)}}$.*

Proof. Since $q \leqslant p$, we have $\frac{q'}{q} \geqslant \frac{p'}{p}$. Then it follows from Hölder's inequality that

$$\|\tfrac{1}{\omega}\|_{L^{p'(\cdot)/p(\cdot)}(B)} \leqslant 2\|1\|_{L^{\alpha(\cdot)}(B)}\|\tfrac{1}{\omega}\|_{L^{q'(\cdot)/q(\cdot)}(B)},$$

where $\frac{1}{\alpha} = \frac{p}{p'} - \frac{q}{q'} = p - q \geqslant 0$. Note that α is not necessarily bounded. Fortunately, this is not so important, since we recently proved in [22, Theorem 4.5.7] that $\|1\|_{L^{\alpha(\cdot)}(B)} \approx |B|^{1/\alpha_B}$ even for unbounded α, as long as $\frac{1}{\alpha}$ is log-Hölder continuous. Therefore

$$\|1\|_{L^{\alpha(\cdot)}(B)} \approx |B|^{\frac{1}{\alpha_B}} = |B|^{\langle p\rangle_B - \langle q\rangle_B} \approx |B|^{p_B - q_B},$$

where $\langle p\rangle_B := \fint_B p(x)\,dx$. The last equivalence follows from Lemma 2.1. These inequalities imply that $\|\omega\|_{A_{p(\cdot)}} \lesssim \|\omega\|_{A_{q(\cdot)}}$ which yields the claim. □

Remark 3.2. The constant in the previous lemma depends on the exponents involved only through the log-Hölder constant. In particular C_{incl} is independent of q when we apply the lemma with exponents $p(\cdot)$ and q. This will be used several times later on.

As usual we define the class A_∞ as the union of all classes A_p, $p \in [1, \infty)$, similarly for A_∞^{loc}. The class A_1 consists of those weights for which $M\omega \lesssim \omega$ and it is contained in every A_p. In view of the previous lemma we have $A_1 \subset A_{p^-} \subset A_{p(\cdot)} \subset A_{p^+} \subset A_\infty$ for $p \in \mathcal{P}_\pm^{\log}$.

For future reference we make the following observation; in fact, this simple, well-known property of A_{p^+} proves to be crucial in our controlling $\omega(B)$ to various exponents.

Lemma 3.3. *Let $p \in \mathcal{P}_\pm^{\log}(\mathbb{R}^n)$. If $\omega \in A_{p(\cdot)}$, then*

$$\omega(B(x,r)) \gtrsim \omega(B(y,R))\Big(\frac{r^n}{|x-y|^n + r^n + R^n}\Big)^{p^+}$$

for all $x, y \in \mathbb{R}^n$ and $r, R > 0$.

Proof. By the previous lemma we conclude that $\omega \in A_{p^+}$. Then we may use the p^+-maximal inequality to derive

$$\begin{aligned}\omega(B(x,r)) &= \int_{\mathbb{R}^n} \big(\chi_{B(x,r)}(z)\big)^{p^+}\omega(z)\,dz \gtrsim \int_{\mathbb{R}^n} \big(M\chi_{B(x,r)}(z)\big)^{p^+}\omega(z)\,dz \\ &\geqslant \omega(B(y,R))\Big(\frac{r^n}{|x-y|^n + r^n + R^n}\Big)^{p^+}.\end{aligned}$$ □

Using of the previous lemma we can prove the following fundamental estimates which state that the relationship between norm and modular of a characteristic function is unexpectedly nice also in the weighted case, provided the weight is in A_∞. This property is central in many of the later arguments. We start with a local version.

Lemma 3.4. *Let $p \in \mathcal{P}_\pm^{\log}(\mathbb{R}^n)$ and $\omega \in A_\infty$. Then*

$$\|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{p_B^+}} \approx \omega(B)^{\frac{1}{p_B^-}} \approx \omega(B)^{\frac{1}{p(x)}} \approx \omega(B)^{\frac{1}{p_B}}$$

if B is a ball with $\operatorname{diam} B \leqslant 2$ *and $x \in B$. In addition, $\omega(B)^{\frac{1}{p_B}} \approx \omega(B)^{\frac{1}{p_\infty}}$ when* $\operatorname{diam} B \in (\frac{1}{4\sqrt{n}}, 2]$.

Proof. Since $\omega \in A_\infty$, there exists $q \in [1,\infty)$ such that $\omega \in A_q$. Suppose first that $B = B(x,r)$ with $r \leqslant 1$. From Lemma 3.3 we conclude that

$$\Big(\frac{r^n}{1+|x|^n}\Big)^q \omega(B(0,1)) \lesssim \omega(B) \lesssim (1+|x|^n)^q \omega(B(0,1)). \tag{3.5}$$

Thus

$$\omega(B)^{p_B^- - p_B^+} \lesssim \big[1+\omega(B(0,1))^{p^- - p^+}\big](1+|x|^n)^{q|p_B^+ - p_\infty| + q|p_B^- - p_\infty|}\, r^{-nq(p_B^+ - p_B^-)}.$$

Here the first factor is a constant, the second is bounded due to the log-Hölder decay condition, and the third is bounded due to the local log-Hölder continuity condition. Similarly we obtain $\omega(B)^{p_B^- - p_B^+} \gtrsim 1$. By (2.3) we have

$$\min\Big\{\omega(B)^{\frac{1}{p_B^+}}, \omega(B)^{\frac{1}{p_B^-}}\Big\} \leqslant \|1\|_{L^{p(\cdot)}(B,\omega)} \leqslant \max\Big\{\omega(B)^{\frac{1}{p_B^+}}, \omega(B)^{\frac{1}{p_B^-}}\Big\}.$$

Since $\omega(B)^{p_B^- - p_B^+} \approx 1$, the upper bound is equivalent to the lower bound, and the first claim follows.

If $r \in (\frac{1}{8\sqrt{n}}, 1]$, then (3.5) becomes

$$\big(1+|x|^n\big)^{-q}\omega(B(0,1)) \lesssim \omega(B) \lesssim (1+|x|^n)^q \omega(B(0,1))$$

which by the decay condition implies that $\omega(B)^{\frac{1}{p_B}} \approx \omega(B)^{\frac{1}{p_\infty}}$. □

Let us next use the Local-to-Global Theorem to get a large-ball version of the previous lemma. We will use this result several times, so we formulate it in a general form. We say that a measure is *doubling on small balls* if the doubling condition holds for all balls of radius at most 1.

Lemma 3.6. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$ and suppose that ω is doubling on small balls and that $\|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{p_\infty}}$ for all balls with* $\operatorname{diam} B \in (\frac{1}{4\sqrt{n}}, 2)$. *Then*

$$\|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{p_\infty}}$$

also for all balls with diameter at least 2.

Proof. Let (Q_i) be an orderly partition of $\mathbb{R}^n$ into cubes with diameter $\frac{1}{2}$ as in Definition 2.5 and let B be a ball of diameter at least 2. We want to split B into the pieces $B \cap Q_i$ and apply the assumption $\|1\|_{L^{p(\cdot)}(B',\omega)} \approx \omega(B')^{\frac{1}{p_\infty}}$ to each piece. However, $B \cap Q_i$ is not a ball and we have to modify this argument slightly.

Let I be the set of indices for which $B \cap Q_i \neq \emptyset$. We first apply Theorem 2.6 to (Q_i):

$$\|\chi_B\|^{p_\infty}_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \approx \sum \|\chi_B\|^{p_\infty}_{L^{p(\cdot)}(Q_i,\omega)} \leqslant \sum_{i\in I} \|\chi_B\|^{p_\infty}_{L^{p(\cdot)}(2Q_i,\omega)} = \sum_{i\in I} \|1\|^{p_\infty}_{L^{p(\cdot)}(B\cap 2Q_i,\omega)},$$

Let $(\hat{Q}_i)$ be the orderly partition obtained by shifting each cube in (Q_i) half a cube along the $(1,\dots,1)$-direction. Then

$$2Q_i = \bigcup_{j\in J_i} \hat{Q}_j,$$

where J_i is an index set with 2^n elements. Then we can apply Theorem 2.6 to deduce

$$\sum_{i\in I} \|1\|^{p_\infty}_{L^{p(\cdot)}(B\cap 2Q_i,\omega)} \leqslant 2^n \sum \|1\|^{p_\infty}_{L^{p(\cdot)}(B\cap \hat{Q}_i,\omega)} \approx \|\chi_B\|^{p_\infty}_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}.$$

It follows that

$$\|1\|^{p_\infty}_{L^{p(\cdot)}(B,\omega)} = \|\chi_B\|^{p_\infty}_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \approx \sum_{i\in I} \|1\|^{p_\infty}_{L^{p(\cdot)}(B\cap 2Q_i,\omega)}.$$

Let $i \in I$. Then we find balls B^- and B^+ with $B^- \subset B \cap 2Q_i \subset 2Q_i \subset B^+$ such that $\operatorname{diam} B^- = \frac{1}{4\sqrt{n}}$ and $\operatorname{diam} B^+ = 1$. We conclude that

$$\omega(B^-) \approx \|1\|^{p_\infty}_{L^{p(\cdot)}(B^-,\omega)} \leqslant \|1\|^{p_\infty}_{L^{p(\cdot)}(B\cap 2Q_i,\omega)} \leqslant \|1\|^{p_\infty}_{L^{p(\cdot)}(2Q_i,\omega)} \leqslant \|1\|^{p_\infty}_{L^{p(\cdot)}(B^+,\omega)} \approx \omega(B^+).$$

Now the doubling property of the measure implies that the upper and lower bounds are comparable, so that $\|1\|^{p_\infty}_{L^{p(\cdot)}(B\cap 2Q_i,\omega)} \approx \omega(B \cap 2Q_i)$ whenever $B \cap Q_i \neq \emptyset$. Combing the above estimates we have

$$\|1\|^{p_\infty}_{L^{p(\cdot)}(B,\omega)} \approx \sum_{i\,:\,B\cap Q_i\neq\emptyset} \omega(B \cap 2Q_i).$$

Since the $2Q_i$ have finite overlap and cover $\mathbb{R}^n$, it follows that

$$\|1\|^{p_\infty}_{L^{p(\cdot)}(B,\omega)} \approx \omega(B). \qquad \square$$

We are now ready to prove the relationship between norm and modular of a characteristic function of a ball.

Corollary 3.7. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$ and $\omega \in A_\infty$. Then $\|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{p_B}}$ for all balls $B \subset \mathbb{R}^n$. In addition, if $0 \notin 2B$, then $\|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{p(y)}}$ for all $y \in B$.*

Proof. By Lemma 3.4 the claim holds for balls of radius at most 1. The same lemma implies that the conditions of Lemma 3.6 are satisfied, and thus we obtain $\|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{p_\infty}}$ for large balls. To conclude the proof we show that $\omega(B)^{\frac{1}{p_\infty}-\frac{1}{p_B}} \approx 1$ for large balls. As in the proof of Lemma 2.1 we obtain that $|\frac{1}{p_\infty} - \frac{1}{p_B}| \lesssim \big(\log(e + \max\{|x|, r\})\big)^{-1}$, where $B = B(x, r)$ with $r > 1$. Since $\omega \in A_\infty$, there exists $q \in [1,\infty)$ such that $\omega \in A_q$. Hence, Lemma 3.3 implies that

$$(1 + (|x|/r)^n)^{-q}\omega(B(0, 1)) \lesssim \omega(B) \lesssim (r^n + |x|^n)^q\omega(B(0, 1)).$$

Combining these estimates yields $|\log \omega(B)||\frac{1}{p_\infty} - \frac{1}{p_B}| \lesssim 1$, which concludes the proof of the main claim.

Consider then the case $0 \notin 2B$. Now, by the decay condition,

$$|\frac{1}{p_\infty} - \frac{1}{p(y)}| \lesssim \big(\log(e + |y|)\big)^{-1} = \big(\log(e + \max\{|y|, r\})\big)^{-1}$$

for $y \in B$. Then the same steps as in the first case yield the claim. $\square$

For $\omega \in A_{p(\cdot)}$ we define a *dual weight* by $\omega'(y) := \omega(y)^{1-p'(y)}$. In the classical case it is immediate that $\|\omega\|_{A_p} = \|\omega'\|_{A_{p'}}^{p-1}$ so $\omega \in A_p$ if and only if $\omega' \in A_{p'}$. We now prove the analogous result for the variable exponent case. Again, the more complicated relationship between norm and modular causes additional work.

Proposition 3.8. *If $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$ and $\omega \in A_{p(\cdot)}$, then $\omega' \in A_{p'(\cdot)}$ and*

$$|B|^{-p_B} \, \|\omega\|_{L^1(B)} \, \|\tfrac{1}{\omega}\|_{L^{p'(\cdot)/p(\cdot)}(B)} \approx \frac{\omega(B)}{|B|} \left(\frac{\omega'(B)}{|B|} \right)^{p_B-1}.$$

Proof. Let $\omega \in A_{p(\cdot)}$ and suppose first that $B \subset \mathbb{R}^n$ is a ball with $\operatorname{diam} B \leqslant 2$. By definition of $\|\omega\|_{A_{p(\cdot)}}$ we have

$$\frac{\omega(B)}{|B|^{p_B}} \, \|\tfrac{1}{\omega}\|_{L^{p'(\cdot)/p(\cdot)}(B)} \leqslant \|\omega\|_{A_{p(\cdot)}}. \tag{3.9}$$

Since we do not know that $\omega' \in A_\infty$, we cannot directly apply Corollary 3.7 to the norm of $\frac{1}{\omega}$. Let us show that we also have a constant lower bound for the left hand side. We apply Hölder's inequality as in the classical case:

$$|B| = \int_B \omega(y)^{\frac{1}{p(y)}} \, \omega(y)^{-\frac{1}{p(y)}} \, dy \leqslant 2 \, \big\|\omega^{\frac{1}{p(\cdot)}}\big\|_{L^{p(\cdot)}(B)} \, \big\|\omega^{-\frac{1}{p(\cdot)}}\big\|_{L^{p'(\cdot)}(B)} \approx \big\|\omega(B)^{\frac{1}{p_B}} \omega^{-\frac{1}{p(\cdot)}}\big\|_{L^{p'(\cdot)}(B)},$$

where the equivalence is due to Corollary 3.7. Hence the corresponding modular is greater than a constant:

$$\begin{aligned} 1 \lesssim \varrho_{L^{p'(\cdot)}(B)}\Big(\tfrac{\omega(B)^{\frac{1}{p_B}}}{|B|} \omega^{-\frac{1}{p(\cdot)}}\Big) &= \int_B \left(\frac{\omega(B)^{\frac{1}{p_B}}}{|B|} \right)^{p'(y)} \omega(y)^{-\frac{p'(y)}{p(y)}} \, dy \\ &\approx \int_B \left(\frac{\omega(B)}{|B|^{p_B}} \right)^{\frac{p'(y)}{p(y)}} \omega(y)^{-\frac{p'(y)}{p(y)}} \, dy = \varrho_{L^{p'(\cdot)/p(\cdot)}(B)}\Big(\tfrac{\omega(B)}{|B|^{p_B}} \tfrac{1}{\omega}\Big), \end{aligned}$$

where we used the local log-Hölder condition, the fact that $\operatorname{diam} B \leqslant 2$, and Lemma 3.4 for the equivalence. But then we can move back to a norm expression, now with the $\frac{p'(\cdot)}{p(\cdot)}$-norm. This gives us exactly a constant lower bound for the left hand side of (3.9), hence $\big\|\frac{1}{\omega}\big\|_{L^{p'(\cdot)/p(\cdot)}(B)} \approx \frac{|B|^{p_B}}{\omega(B)}$. Armed with this piece of information and Lemma 3.4 we see that the log-Hölder continuity of p also implies that

$$\big\|\tfrac{1}{\omega}\big\|_{L^{p'(\cdot)/p(\cdot)}(B)}^{\frac{1}{p_B^+-1}} \approx \big\|\tfrac{1}{\omega}\big\|_{L^{p'(\cdot)/p(\cdot)}(B)}^{\frac{1}{p_B^--1}}.$$

Hence (2.3) implies that

$$\|1\|_{L^{p'(\cdot)/p(\cdot)}(B,\omega')} = \big\|\tfrac{1}{\omega}\big\|_{L^{p'(\cdot)/p(\cdot)}(B)} \approx \Big(\varrho_{L^{p'(\cdot)/p(\cdot)}(B)}\big(\tfrac{1}{\omega}\big)\Big)^{p_B-1} = \omega'(B)^{p_B-1}$$

for balls with $\operatorname{diam} B \leqslant 2$.

Let us then look at the duality claim for small balls. So, let B be as before. Then

$$\begin{aligned}
\frac{1}{|B|^{p'_B}}\,\|\omega'\|_{L^1(B)}\,\|\tfrac{1}{\omega'}\|_{L^{p(\cdot)/p'(\cdot)}(B)} &= \frac{1}{|B|^{p'_B}}\,\omega'(B)\,\|1\|_{L^{p(\cdot)/p'(\cdot)}(B,\omega)} \\
&\approx \frac{1}{|B|^{p'_B}}\,\omega'(B)\,\omega(B)^{\frac{1}{p_B-1}} \\
&= \left[\frac{\omega(B)}{|B|}\left(\frac{\omega'(B)}{|B|}\right)^{p_B-1}\right]^{\frac{1}{p_B-1}} \\
&\approx \left[\frac{\omega(B)}{|B|^{p_B}}\,\|\tfrac{1}{\omega}\|_{L^{p'(\cdot)/p(\cdot)}(B)}\right]^{\frac{1}{p_B-1}} \leqslant \|\omega\|_{A_{p(\cdot)}}^{\frac{1}{p_B-1}}
\end{aligned} \tag{3.10}$$

where we used Corollary 3.7 for the first equivalence and the previously derived expression for the second equivalence. This shows that $\omega' \in A^{\mathrm{loc}}_{p'(\cdot)}$.

But now it follows from Lemma 3.1 that $\omega' \in A^{\mathrm{loc}}_\infty$, so in particular the measure is doubling on small balls. We proved that $\|1\|_{L^{p'(\cdot)/p(\cdot)}(B,\omega')} \approx \omega'(B)^{p_B-1}$, and, as usual, p_B can be replaced by p_∞ when $\operatorname{diam} B \in (\frac{1}{4\sqrt{n}}, 2]$. Therefore it follows from Proposition 3.6 that

$$\|1\|_{L^{p'(\cdot)/p(\cdot)}(B,\omega')} \approx \omega'(B)^{p_\infty-1}$$

for balls with $\operatorname{diam} B \geqslant 2$. Since p_∞ can here be replaced by p_B we finally obtain that

$$\frac{1}{|B|^{p_B}}\,\|\omega\|_{L^1(B)}\,\|\tfrac{1}{\omega}\|_{L^{p'(\cdot)/p(\cdot)}(B)} = \frac{\omega(B)}{|B|^{p_B}}\,\|1\|_{L^{p'(\cdot)/p(\cdot)}(B,\omega')} \approx \frac{\omega(B)}{|B|}\left(\frac{\omega'(B)}{|B|}\right)^{p_B-1}$$

for large balls, which completes the proof of the first claim. Armed with this information, we see that (3.10) holds also for large balls, hence $\omega' \in A_{p'(\cdot)}$. $\square$

Remark 3.11. One could consider taking $\sup_B \frac{\omega(B)}{|B|}\left(\frac{\omega'(B)}{|B|}\right)^{p_B-1} < \infty$ as the definition of the class $A_{p(\cdot)}$. With this definition the duality property is an immediate consequence. However, this definition would make it more difficult to show that $A_{p(\cdot)}$ is increasing in p, Lemma 3.1, which is needed to get the regularity results in Lemma 3.3–Corollary 3.7. Another possible definition would be

$$\sup_B |B|^{-1}\big\|\omega^{\frac{1}{p(\cdot)}}\big\|_{L^{p(\cdot)}(B)}\big\|\omega^{-\frac{1}{p(\cdot)}}\big\|_{L^{p'(\cdot)}(B)} < \infty, \tag{3.12}$$

which is similar to the expression considered by Kopaliani [45]. However, again the monotonicity property is missing. Note that (3.12) can also be rewritten as

$$\sup_B |B|^{-1}\|1\|_{L^{p(\cdot)}(B,\omega)}\|1\|_{L^{p'(\cdot)}(B,\omega')} < \infty,$$

It can be shown as in the previous proposition that $\omega \in A_{p(\cdot)}$ implies (3.12). On the other hand, once we have proved Theorem 1.1, the proof is almost trivial using duality:

$$\begin{aligned}\|1\|_{L^{p(\cdot)}(B,\omega)}\|1\|_{L^{p'(\cdot)}(B,\omega')} &\leqslant 2\,\|1\|_{L^{p(\cdot)}(B,\omega)} \sup_{\|g\|_{L^{p(\cdot)}(B,\omega)}\leqslant 1} \int_B g\,dx \\ &= 2\,|B| \sup_{\|g\|_{L^{p(\cdot)}(B,\omega)}\leqslant 1} \Big\|\chi_B \fint_B g\,dy\Big\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \\ &\lesssim |B| \sup_{\|g\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}\leqslant 1} \|\chi_B Mg\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \leqslant |B|.\end{aligned}$$

It is left to future investigations to consider whether the opposite implication also holds.

Let us now prove another basic property which is trivial in the constant exponent case. It is the reverse factorization result, the converse of Jones' famous factorization theorem [38].

Proposition 3.13. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $\omega_1, \omega_2 \in A_1$. Then $\omega_1\omega_2^{1-p(\cdot)} \in A_{p(\cdot)}$.*

Proof. Let us start with the L^1-part of $\|\omega_1\omega_2^{1-p(\cdot)}\|_{A_{p(\cdot)}}$. Since $\omega_2 \in A_1$, we have $\omega_2(y)^{-1} \lesssim |B|/\omega_2(B)$ for $y \in B$. Hence it follows that $\omega_2(y)^{1-p(y)} \lesssim (|B|/\omega_2(B))^{1-p_B}$ if B has radius at most 1 or satisfies $0 \notin 2B$, by log-Hölder continuity and Corollary 3.7. In this case we have $\big\|\omega_1\omega_2^{1-p(\cdot)}\big\|_{L^1(B)} \lesssim \big(\frac{|B|}{\omega_2(B)}\big)^{p_B-1}\omega_1(B)$. Suppose now that B' is a ball with radius at least 1 and $0 \in 2B'$. Let $B := 3B'$ and note that $B(0,1) \subset B$. Then we estimate

$$\begin{aligned}\big\|\omega_1\omega_2^{1-p(\cdot)}\big\|_{L^1(B)} &\lesssim \bigg\|\Big(\frac{|B|}{\omega_2(B)}\Big)^{p_B-1}\omega_2^{p_B-p(\cdot)}\omega_1\,\chi_{\{p(\cdot)\geqslant p_B\}} + \Big(\frac{|B|}{\omega_2(B)}\Big)^{p(\cdot)-1}\omega_1\,\chi_{\{p(\cdot)<p_B\}}\bigg\|_{L^1(B)} \\ &= \Big(\frac{|B|}{\omega_2(B)}\Big)^{p_B-1}\bigg\|\omega_2^{p_B-p(\cdot)}\omega_1\,\chi_{\{p(\cdot)\geqslant p_B\}} + \Big(\frac{|B|}{\omega_2(B)}\Big)^{p(\cdot)-p_B}\omega_1\,\chi_{\{p(\cdot)<p_B\}}\bigg\|_{L^1(B)}.\end{aligned}$$

Since $\omega_2 \in A_1$ and $B(0,1) \subset B$ it follows that

$$\omega_2(B(0,1)) = \int_{B(0,1)} \omega_2(x)\,dx \gtrsim \int_{B(0,1)} \fint_B \omega_2(y)\,dy\,dx = \frac{|B(0,1)|}{|B|}\omega_2(B);$$

thus we further conclude that

$$\Big(\frac{|B|}{\omega_2(B)}\Big)^{p(\cdot)-p_B}\chi_{\{p(\cdot)<p_B\}} \lesssim 1 + \Big(\frac{|B(0,1)|}{\omega_2(B(0,1))}\Big)^{p^- - p^+} = C.$$

Again using that $\omega_2 \in A_1$ we also find that

$$\omega_2(y)^{p_B-p(y)}\chi_{\{p(y)\geqslant p_B\}} \leqslant \Big(\frac{|B(y,1+|y|)|}{\omega_2(B(y,1+|y|))}\Big)^{p(y)-p_B} \lesssim \Big(\frac{(1+|y|)^n}{\omega_2(B(0,1))}\Big)^{p(y)-p_B} \leqslant C$$

by the log-Hölder decay condition. Therefore

$$\big\|\omega_1\omega_2^{1-p(\cdot)}\big\|_{L^1(B')} \leqslant \big\|\omega_1\omega_2^{1-p(\cdot)}\big\|_{L^1(B)} \lesssim \Big(\frac{|B|}{\omega_2(B)}\Big)^{p_B-1}\omega_1(B) \approx \Big(\frac{|B'|}{\omega_2(B')}\Big)^{p_{B'}-1}\omega_1(B'),$$

where we used the doubling condition of ω_1 and ω_2 in the last equivalence. We have thus shown that this inequality holds in all cases, i.e. for all balls $B' \subset \mathbb{R}^n$.

Using the conclusion of the previous paragraph and the inequality $\omega_1(y)^{-1} \leqslant |B|/\omega_1(B)$ for $y \in B$ we obtain

$$\begin{aligned}
|B|^{-p_B} \big\|\omega_1\omega_2^{1-p(\cdot)}\big\|_{L^1(B)} \big\|\omega_1^{-1}\omega_2^{p(\cdot)-1}\big\|_{L^{p'(\cdot)/p(\cdot)}(B)} \\
&\lesssim |B|^{-p_B}\left(\frac{|B|}{\omega_2(B)}\right)^{p_B-1}\omega_1(B)\left\|\frac{|B|}{\omega_1(B)}\omega_2^{p(\cdot)-1}\right\|_{L^{p'(\cdot)/p(\cdot)}(B)} \\
&= \frac{1}{\omega_2(B)^{p_B-1}}\|1\|_{L^{p'(\cdot)/p(\cdot)}(B,\omega_2)} \approx 1,
\end{aligned}$$

where we used Corollary 3.7 for the last equivalence. Therefore $\|\omega_1\omega_2^{1-p(\cdot)}\|_{A_{p(\cdot)}} < \infty$, as was to be shown. $\square$

Remark 3.14. The value of $\|\omega_1\omega_2^{1-p(\cdot)}\|_{A_{p(\cdot)}}$ depends on ω_2 also via $\omega_2(B(0,1))$, which is again a manifestation of the non-homogeneity of the variable exponent modular.

4. Self-improvement properties of the Muckenhoupt class

The referee pointed out that self-improvement of the Muckenhoupt condition is only known for $A_p(Q)$, $Q \neq \mathbb{R}^n$, when the A_p-condition is defined in terms of cubes. In our case this is not a problem, since we do not actually need quite this strong a property. We use the following lemma for connecting the $\mathcal{A}_q$ classes defined over different families of sets. Recall that $A_q^{\mathcal{B}}$ was defined in the beginning of Section 3.

Lemma 4.1. *Let $M \in \mathbb{N}$. Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are families of balls or cubes with the property that every set $B \in \mathcal{B}_1$ can be covered by M sets $B_i \in \mathcal{B}_2$, each with diameter comparable (uniformly) to that of B. If $\omega \in A_q^{\mathcal{B}_2}$ is doubling, then $\|\omega\|_{A_q^{\mathcal{B}_1}} \lesssim \|\omega\|_{A_q^{\mathcal{B}_2}}$.*

Proof. Let $B \in \mathcal{B}_1$ and let $B_i \in \mathcal{B}_2$, $i = 1,\dots,M$ be a covering with $\operatorname{diam} B_i \approx \operatorname{diam} B$, so that also $|B| \approx |B_i|$. We may assume that $B \cap B_i \neq \emptyset$ for every i. Then there exists a constant $k > 1$ such that $B \subset k\,B_i$ for every i. Since ω is doubling, $\|\omega\|_{L^1(B)} \lesssim \|\omega\|_{L^1(B_i)}$. Finally, we note the trivial estimate

$$\|\tfrac{1}{\omega}\|_{L^{q'/q}(B)} \leqslant \sum_{i=1}^{M} \|\tfrac{1}{\omega}\|_{L^{q'/q}(B_i)} \leqslant M \sup_i \|\tfrac{1}{\omega}\|_{L^{q'/q}(B_i)}.$$

Hence

$$\begin{aligned}
|B|^{-q}\|\omega\|_{L^1(B)}\|\tfrac{1}{\omega}\|_{L^{q'/q}(B)} &\lesssim \inf_{j,k}\sup_i |B_j|^{-q}\|\omega\|_{L^1(B_k)}\|\tfrac{1}{\omega}\|_{L^{q'/q}(B_i)} \\
&\leqslant \sup_i |B_i|^{-q}\|\omega\|_{L^1(B_i)}\|\tfrac{1}{\omega}\|_{L^{q'/q}(B_i)} \leqslant \|\omega\|_{A_q^{\mathcal{B}_2}}.
\end{aligned}$$

The result now follows when we take the supremum over $B \in \mathcal{B}_1$. $\square$

Corollary 4.2. *Let $\delta > 0$ and let Q be a ball or a cube. Let $\mathcal{B}_1$ be the family of all cubes in $(1+\delta)Q$, and $\mathcal{B}_2$ be the family of all balls in Q. Then $\mathcal{A}_\infty \cap \mathcal{A}_q^{\mathcal{B}_1} \subset A_q^{\mathcal{B}_2}$. The conclusion holds also if the role of balls and cubes is interchanged.*

Proof. A ball B in Q can be covered by a finite, uniformly bounded number of cubes in $(1+\delta)Q$, each with diameter comparable to B. The same holds if balls and cubes are interchanged. In both cases the result follows from Lemma 4.1. □

Now we can transfer the self-improvement property from cubes to balls:

Corollary 4.3. *Let $\delta > 0$ and let Q be a ball or a cube. If $\omega \in A_\infty(\mathbb{R}^n) \cap \mathcal{A}_q((1+\delta)Q)$, then there exists $\epsilon > 0$ such that $\omega \in A_{q-\epsilon}(Q)$.*

Proof. Let $\omega \in A_\infty(\mathbb{R}^n) \cap \mathcal{A}_q((1+\delta)Q)$. By Corollary 4.2, $\omega \in A_q^{\text{cubes}}((1+\frac{\delta}{2})Q)$. By the self-improving property of Muckenhoupt weights on cubes, there exists $\epsilon > 0$ such that $\omega \in A_{q-\epsilon}^{\text{cubes}}((1+\frac{\delta}{2})Q)$. By Corollary 4.2, again, $\omega \in A_{q-\epsilon}(Q)$. □

Corollary 4.4. *Let $\delta > 0$ and let D be a ball. Let $\mathcal{B}_1$ be the family of all cubes in $\mathbb{R}^n \setminus D$, and $\mathcal{B}_2$ be the family of all sets $B \setminus (1+\delta)D$ where B are balls with center in $\mathbb{R}^n \setminus (1+\delta)D$. Then $\mathcal{A}_\infty \cap \mathcal{A}_q^{\mathcal{B}_1} \subset A_q^{\mathcal{B}_2}$, and the embedding constant is independent of D. The role of cubes and balls can also be interchanged.*

Proof. We may assume without loss of generality that $\delta \leqslant \frac{1}{2}$. Suppose first that $\mathcal{B}_1$ are balls and $\mathcal{B}_2$ are cubes. Choose balls $B_1', \dots, B_k'$ with diameter equal to $\operatorname{diam} D$ which are externally tangent to D and cover the sphere $\partial(1+\delta)D$. Let $p_1, \dots, p_k$ on ∂D be the points of tangency of the balls $B_1', \dots, B_k'$. Note that the number k of points needed depends on δ but not on D.

Let $Q \in \mathcal{B}_2$. If $\operatorname{diam} Q < \operatorname{diam} D$, then Q can be covered exactly as in Corollary 4.2. Otherwise, let $B_1, \dots B_k$ be the balls which are externally tangent to D at the points p_j with diameter equal to $3 \operatorname{diam} Q$. Since $\operatorname{diam} Q \geqslant \operatorname{diam} D$, the balls $B_1, \dots B_k$ cover the annulus $((1+\frac{\operatorname{diam} Q}{\operatorname{diam} D})D) \setminus ((1+\delta)D)$; in particular, the balls covers Q. This family of balls satisfies the conditions of Lemma 4.1, so the claim follows in this case.

The case when $\mathcal{B}_1$ are cubes and $\mathcal{B}_2$ are balls is handled similarly. □

From this we obtain the following results using the same steps as in the proof of Corollary 4.3.

Corollary 4.5. *Let $\delta > 0$ and let D be a ball. Let $\mathcal{B}$ be the family of all sets $B \setminus (1+\delta)D$ where B are balls with center in $\mathbb{R}^n \setminus (1+\delta)D$. If $\omega \in A_\infty(\mathbb{R}^n) \cap \mathcal{A}_q(\mathbb{R}^n \setminus D)$, then there exists $\epsilon > 0$, independent of D, such that $\omega \in A_{q-\epsilon}^{\mathcal{B}}$.*

In order to use these results, we need the following generalization of Muckenhoupt's theorem, in which we use the generalized maximal operator, defined with an arbitrary family $\mathcal{B}$ of measurable sets:

$$M_{\mathcal{B}} f(z) := \sup_{\substack{B \in \mathcal{B} \\ z \in B}} \fint_B |f| \, dx.$$

Theorem 4.6 (Theorem B, [48] or Theorem 1.1, [37]). *Let $1 < q < \infty$ and let $\mathcal{B}$ be a family of open sets. Then $M_{\mathcal{B}}$ is bounded on $L^q(\mathbb{R}^n, \omega)$ if and only if $\omega \in A_q^{\mathcal{B}}$.*

5. Sufficiency of the $A_{p(\cdot)}$ condition

We start this section by proving the weighted maximal inequality for the restricted maximal operator. This result is a stepping-stone on our route to the complete maximal inequality, which is then proved.

We denote by $M_{<R}$ the restricted maximal operator which is defined as

$$M_{<R}f(y) := \sup_{r<R} \fint_{B(y,r)} |f(x)|\,dx.$$

Using Diening's trick [19] we begin with a very local version of a weighted maximal inequality.

Lemma 5.1. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$ and $\omega \in A_{p(\cdot)}$. Then there exists $r_0 \in (0,1)$ depending only on $\|\omega\|_{A_{p(\cdot)}}$ and the characteristics of p such that*

$$M_{<R}\colon L^{p(\cdot)}(2Q,\omega) \hookrightarrow L^{p(\cdot)}(Q,\omega)$$

when Q is a cube with side length at most r_0 and $R < \frac{1}{2}r_0$.

Proof. Define $c_1 := C_{\text{incl}}\|\omega\|_{A_{p(\cdot)}}$, where the constant C_{incl} is from Lemma 3.1 with exponents given $p(\cdot)$ and all $q \in [1,\infty]$ (cf. Remark 3.2). Choose $\epsilon \in (0,1)$ such that for all constants $q \in [p^-, p^+ + 1]$ and all cubes Q the inequality $\|\sigma\|_{A_q(Q)} \leqslant c_1$ implies that $\|\sigma\|_{A_{q-\epsilon}(Q)} \leqslant c_2$, where c_2 is some large constant independent of q and Q (see Corollary 4.2, below, for some further comments on the self-improving property). This choice is possible by the self-improving property of Muckenhoupt weights [61]. Next we choose $r_0 < \frac{1}{2}n^{-1/2}$ such that $p^+_{2Q} - \epsilon < p^-_{2Q}$ whenever Q has side-length at most r_0. This is possible by the uniform continuity of p. By Lemma 3.1 $\|w\|_{A_{p^+_{2Q}}(2Q)} \leqslant c_1$; hence we conclude by monotonicity that $\omega \in A_{p^+_{2Q}-\epsilon}(2Q) \subset A_{p^-_{2Q}}(2Q)$ with $\|\omega\|_{A_{p^-_{2Q}}(2Q)} \leqslant c_2$.

Let $f \in L^{p(\cdot)}(2Q,\omega)$ with $\|f\|_{L^{p(\cdot)}(2Q,\omega)} \leqslant 1$ and set $q = p/p^-_{2Q}$. Next we use a variant of Diening's trick [19]. Let $y \in Q$ and let $B := B(y,r)$ with $r < R$. We start with Hölder's inequality for a constant exponent and an elementary estimate valid for all $\beta > 0$:

$$\begin{aligned}
\bigg(\fint_B |f(x)|\,dx\bigg)^{q(x)} &\leqslant \bigg(\fint_B |f(x)|^{q^-_B}\,dx\bigg)^{\frac{q(x)}{q^-_B}} = \bigg(\fint_B \tfrac{1}{\beta}\Big[|f(x)|\,\beta^{\frac{1}{q^-_B}}\Big]^{q^-_B}\,dx\bigg)^{\frac{q(x)}{q^-_B}} \\
&\leqslant \bigg(\fint_B \tfrac{1}{\beta}\Big[|f(x)|^{q(x)}\beta^{\frac{q(x)}{q^-_B}} + 1\Big]\,dx\bigg)^{\frac{q(x)}{q^-_B}} \\
&= \bigg(\fint_B |f(x)|^{q(x)}\beta^{\frac{q(x)}{q^-_B}-1}\,dx + \tfrac{1}{\beta}\bigg)^{\frac{q(x)}{q^-_B}}.
\end{aligned}$$

Now we choose $\beta := \max\{1, \omega(Q)^{1/p^-_{2Q}}\}$. Lemma 3.3 implies that $\omega(Q) \lesssim (1+|x|)^{p+}$. As a consequence, we estimate $\beta^{q(x)/q^-_B-1} \lesssim (1+|x|)^{C(q(x)-q^-_B)} \leqslant C$, where the second inequality follows by the log-Hölder decay condition of q. Since $q(x) \geqslant q_B^-$ and $\frac{1}{\beta} \leqslant 1$ we obtain

(5.2)
$$\begin{aligned}\bigg(\fint_B |f(x)|\,dx\bigg)^{q(x)} &\lesssim \bigg(\fint_B |f(x)|^{q(x)}\,dx\bigg)^{\frac{q(x)}{q_B^-}} + \frac{1}{\beta}\\ &= |B|^{1-\frac{q(x)}{q_B^-}}\,\varrho_{L^{q(\cdot)}(B)}(f)^{\frac{q(x)-q_B^-}{q_B^-}} \fint_B |f(x)|^{q(x)}\,dx + \min\Big\{1, \omega(Q)^{-\frac{1}{p_{2Q}^-}}\Big\}.\end{aligned}$$

By the log-Hölder continuity of q, the factor $|B|^{1-q(x)/q_B^-}$ is bounded by a constant. For the modular in the second factor we use Young's inequality with exponent p_{2Q}^-:

$$\begin{aligned}\varrho_{L^{q(\cdot)}(B)}(f) = \int_B |f(x)|^{\frac{p(x)}{p_{2Q}^-}}\,dx &\leqslant \int_B |f(x)|^{p(x)}\omega(x)\,dx + \int_B \omega(x)^{-\frac{1}{p_{2Q}^- -1}}dx\\ &\leqslant 1 + \big(\|\omega\|_{A_{p_{2Q}^-}(B)}|B|^{p_{2Q}^-}\omega(B)^{-1}\big)^{\frac{1}{p_{2Q}^- -1}}.\end{aligned}$$

Here we need that $\omega \in A_{p_{2Q}^-}(B)$ and $\|f\|_{L^{p(\cdot)}(B,\omega)} \leqslant 1$. Using the log-Hölder continuity of q and Lemma 3.4 we conclude that $|B|^{q(x)-q_B^-} \leqslant c$ and $\omega(B)^{-(q(x)-q_B^-)} \leqslant c$. Hence we conclude that $\varrho_{L^{q(\cdot)}(B)}(f)^{(q(x)-q_B^-)/q_B^-} \leqslant C$. Then we take the supremum over balls $B = B(y,r)$ with $r < R$ in (5.2). This yields

$$M_{<R}f(x)^{q(x)} \lesssim M_{<R}(f^{q(\cdot)})(x) + \min\Big\{1, \omega(Q)^{-\frac{1}{p_{2Q}^-}}\Big\}$$

for $x \in Q$. Raising this to the power of p_{2Q}^- (recalling that $p = p_{2Q}^- q$) and integrating over $x \in Q$, we conclude that

$$\begin{aligned}\int_Q M_{<R}f(x)^{p(x)}\omega(x)\,dx &\lesssim \int_Q \bigg[M_{<R}(|f|^{q(\cdot)})(x) + \min\Big\{1, \omega(Q)^{-\frac{1}{p_{2Q}^-}}\Big\}\bigg]^{p_{2Q}^-}\omega(x)\,dx\\ &\approx \int_Q M_{<R}(|f|^{q(\cdot)})(x)^{p_{2Q}^-}\omega(x)\,dx + \omega(Q)\min\big\{1, \omega(Q)^{-1}\big\}\\ &\lesssim \int_{2Q} |f(x)|^{p(x)}\omega(x)\,dx + 1 \leqslant 2,\end{aligned}$$

by the boundedness of $M_{<R}\colon L^{p_{2Q}^-}(2Q,\omega) \hookrightarrow L^{p_{2Q}^-}(Q,\omega)$ which holds since $\omega \in A_{p_{2Q}^-}(2Q)$, $p_{2Q}^- \geqslant p^- > 1$, and $R < \frac{1}{2}r_0$.

Thus the proof in the case $\|f\|_{L^{p(\cdot)}(2Q,\omega)} \leqslant 1$ is complete. If $\|f\|_{L^{p(\cdot)}(2Q,\omega)} > 1$, then we reduce the claim to the previous case by considering the function $\tilde{f} := f/\|f\|_{L^{p(\cdot)}(2Q,\omega)}$ with norm equal to one. $\square$

Using the previous result and the local-to-global trick we can immediately prove the boundedness of the restricted maximal operator.

Lemma 5.3. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$, $\omega \in A_{p(\cdot)}$ and let $r_0 > 0$ be as in the previous lemma. Then*
$$M_{<R}\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n,\omega)$$

for $R < \frac{1}{2}r_0$.

Proof. Let (Q_i) be a partition of $\mathbb{R}^n$ as in Definition 2.5 into cubes with side-length r_0. Let $R < \frac{1}{2}r_0$. By Lemma 5.1 $\|M_{<R}f\|_{L^{p(\cdot)}(Q_i,\omega)} \lesssim \|f\|_{L^{p(\cdot)}(2Q_i,\omega)}$. Now we apply Theorem 2.6 and use the bounded overlap property of $(2Q_i)$:

$$\begin{aligned}\|M_{<R}f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} &\approx \big\|\|M_{<R}f\|_{L^{p(\cdot)}(Q_i,\omega)}\big\|_{l^{p_\infty}} \\ &\lesssim \big\|\|f\|_{L^{p(\cdot)}(2Q_i,\omega)}\big\|_{l^{p_\infty}} \approx \big\|\|f\|_{L^{p(\cdot)}(Q_i,\omega)}\big\|_{l^{p_\infty}} \approx \|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}. \qquad \square\end{aligned}$$

If ϕ has support in $B(0,R)$, then $f * \phi \lesssim \|\phi\|_\infty M_{<R}f$. Hence we obtain the following corollary.

Corollary 5.4. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$, $\omega \in A_{p(\cdot)}$ and let $r_0 > 0$ be as Lemma 5.1. If $\Phi f := f * \chi_{B(0,\frac{1}{2}r_0)}$, then*

$$\Phi\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n,\omega).$$

Remark 5.5. An analysis of the proofs shows that Lemmas 5.1 and 5.3 and Corollary 5.4 actually hold already with the local assumption $\omega \in A^{\mathrm{loc}}_{p(\cdot)}$ in place of $\omega \in A_{p(\cdot)}$ if we assume additionally that

$$(1+|x|)^{-\beta} \lesssim \omega(B(x,r)) \lesssim (1+|x|)^{\beta}$$

for some fixed $\beta > 0$, all $x \in \mathbb{R}^n$ and all $r \in (1,2)$.

We can also define a lower-restricted maximal operator:

$$M_{>R}f(y) := \sup_{r>R} \fint_{B(y,r)} |f(x)|\,dx.$$

Obviously, $Mf \leqslant M_{<R}f + M_{>R}f$. We already controlled the restricted maximal operator, so we now turn to the lower-restricted maximal operator.

Lemma 5.6. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$, $\omega \in A_{p(\cdot)}$ and let $R = \frac{1}{3}r_0$, with r_0 as in Lemma 5.1. Then*

$$M_{>R}\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n,\omega).$$

Proof. Let $f \in L^{p(\cdot)}(\mathbb{R}^n,\omega)$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \leqslant 1$. Using Hölder's inequality, Proposition 3.8 and Corollary 3.7 we conclude that

$$\begin{aligned}\fint_B |f(y)|\,dy &\leqslant \tfrac{2}{|B|}\,\|f\|_{L^{p(\cdot)}(B,\omega)}\|\omega^{-1}\|_{L^{p'(\cdot)}(B,\omega)} \\ &\leqslant \tfrac{2}{|B|}\,\|1\|_{L^{p'(\cdot)}(B,\omega')} \approx \frac{\omega'(B)^{\frac{1}{p'_B}}}{|B|} \lesssim \bigg(\frac{\|\omega\|_{A_{p(\cdot)}}}{\omega(B)}\bigg)^{\frac{1}{p_B}}.\end{aligned}$$

For $B = B(x,r)$ with $r > \frac{1}{6}r_0$ we have $\omega(B) \geqslant C\omega(B(0,1))(1+|x|)^{-np^+}$ by Lemma 3.3. Thus $M_{>\frac{1}{2}R}f(x) \lesssim (1+|x|)^{np^+/p^-}$ since we have assumed that $R = \frac{1}{3}r_0$.

Define $c_1 := C_{\mathrm{incl}}\|\omega\|_{A_{p(\cdot)}}$. Choose $\epsilon \in (0,1)$ such that for all constants $q \in [p^-, p^+ + 1]$ and all $\rho > 1$ the inequality $\|\sigma\|_{A_q(\mathbb{R}^n \setminus B(0,\rho))} \leqslant c_1$ implies that $\sigma \in A^{\mathcal{B}}_{q-\epsilon}$, where $\mathcal{B}$ is the family of sets of the form $B \setminus B(0,2\rho)$, where B is a ball with center in $\mathbb{R}^n \setminus B(0,2\rho)$ (cf. Corollary 4.5). Next we choose $\rho = R_0$ so large that $p^+_D - \epsilon \leqslant p_\infty$ where $D := \mathbb{R}^n \setminus B(0,R_0)$.

This is possible by the decay condition on p. Let us denote $B_k := B(0,(1+k)R_0)$ and $D_k := \mathbb{R}^n \setminus B_k$, for $k \in \mathbb{N}$. By Lemma 3.1 $\|w\|_{A_{p_D^+}(D)} \leqslant c_1$; hence we conclude that

$$\omega \in A^{\mathcal{B}}_{p_D^+ - \epsilon} \subset A^{\mathcal{B}}_{p_\infty}.$$

Suppose that g is a function with support in D_1. If B is a ball centered in D_2 with radius at least $\frac{1}{2}R$, then

$$\fint_B |g|\,dx = \frac{|B \setminus B_1|}{|B|} \fint_{B \setminus B_1} |g|\,dx \lesssim \fint_{B \setminus B_1} |g|\,dx.$$

With $\mathcal{B}$ as in the previous paragraph, we thus obtain $M_{\frac{1}{2}R}\, g \lesssim M_{\mathcal{B}} g$ in D_1. By Lemma 2.7 and the L^{p_∞}-maximal inequality (Theorem 4.6) we conclude that

$$\|M_{>\frac{1}{2}R}\, g\|_{L^{p(\cdot)}(D_2,\omega)} \approx \|M_{>\frac{1}{2}R}\, g\|_{L^{p_\infty}(D_2,\omega)} \lesssim \|M_{\mathcal{B}} g\|_{L^{p_\infty}(D_2,\omega)} \lesssim \|g\|_{L^{p_\infty}(D_1,\omega)} \tag{5.7}$$

for $g \in L^{p_\infty}(\mathbb{R}^n, \omega)$ with support in D_1.

Let finally $\Phi f := |f| * \chi_{B(0,\frac{1}{2}R)}$. It follows from

$$\fint_{B(y,r)} |f(y)|\,dy = \Big(\frac{\chi_{B(0,r)}}{|B(0,r)|} * |f|\Big)(y)$$

and $\chi_{B(0,r)} \lesssim \chi_{B(0,r)} * \chi_{B(0,R/2)} \lesssim \chi_{B(0,r+R/2)}$ for all $r > R/2$, that $M_{>R} f \approx M_{>\frac{1}{2}R} \Phi f$. Moreover, $\Phi f(x) \lesssim (1+|x|)^{np^+}$ by Lemma 3.3. Using (5.7) with $g = \Phi f\, \chi_{D_1}$, we obtain that

$$\begin{aligned}
\|M_{>R} f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} &\leqslant \|M_{>R} f\|_{L^\infty(B_2)} \|1\|_{L^{p(\cdot)}(B_2,\omega)} + \big\|M_{>\frac{1}{2}R}(\Phi f\, \chi_{D_1})\big\|_{L^{p(\cdot)}(D_2,\omega)} \\
&\quad + \big\|M_{>\frac{1}{2}R}(\Phi f\, \chi_{B_1})\big\|_{L^{p(\cdot)}(D_2,\omega)} \\
&\lesssim \|1\|_{L^{p(\cdot)}(B_2,\omega)} + \|\Phi f\|_{L^{p_\infty}(D_1,\omega)} + \|\Phi f\|_{L^\infty(B_1)} \big\|M \chi_{B_1}\big\|_{L^{p(\cdot)}(D_2,\omega)} \\
&\approx 1 + \|\Phi f\|_{L^{p(\cdot)}(D_1,\omega)} + \big\|M \chi_{B_1}\big\|_{L^{p_\infty}(D_2,\omega)},
\end{aligned}$$

where we have used Lemma 2.7 for the equivalence.

We note that $\|\Phi f\|_{L^{p(\cdot)}(D_1,\omega)} \leqslant \|f\|_{L^{p(\cdot)}(D,\omega)} \leqslant 1$ by Corollary 5.4. Let $B' \subset D_2$ be a ball of the same size as B_1. Since $M\chi_{B_1} \approx M\chi_{B'}$, we find that $\big\|M\chi_{B_1}\big\|_{L^{p_\infty}(D_2,\omega)} \approx \big\|M\chi_{B'}\big\|_{L^{p_\infty}(D_2,\omega)}$. Since the support of $\chi_{B'}$ is in D_2, the last term can be estimated as before.

This concludes the proof for the case of small norms. Applying this conclusion to the function $\tilde{f} := f/\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}$, we obtain the proof for the general case. □

Proof of sufficiency of the $A_{p(\cdot)}$-condition in Theorem 1.1. By Lemmas 5.1 and 5.6 we conclude that

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \leqslant \|M_{<R} f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} + \|M_{>R} f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \lesssim \|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}$$

for $R = \frac{1}{3} r_0$. □

6. Necessity of the $A_{p(\cdot)}$ condition

In this section we prove the other implication of Theorem 1.1, that $\omega \in A_{p(\cdot)}$ if M is bounded on $L^{p(\cdot)}(\mathbb{R}^n,\omega)$. In the constant exponent case this is a simple application of Hölder's inequality. In the variable exponent case things do not work quite so neatly.

The proof would be easy if we knew that $\omega, \omega' \in A_\infty$ and could use the results in Section 3. It seems that establishing this is no easier than showing that $\omega \in A_{p(\cdot)}$. However, there is a neat method to obtain that $\omega, \omega' \in A_\infty$ in small balls. This approach was suggested to us by the referee of the paper, and leads to a simpler proof than we originally proposed. With the local A_∞ property and the methods of Section 3 we prove the global $A_{p(\cdot)}$ result.

We say that $M\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow$ w-$L^{p(\cdot)}(\mathbb{R}^n,\omega)$ if the *weak maximal inequality*

$$\|t\,\chi_{\{Mf>t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \leqslant c\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}$$

holds for all $t>0$. Since $t\,\chi_{\{Mf>t\}} \leqslant Mf$, it is clear that this weak type inequality is implied by the boundedness $M\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n,\omega)$.

Lemma 6.1. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$. If $M\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow$ w-$L^{p(\cdot)}(\mathbb{R}^n,\omega)$ and* $\operatorname{diam} B \leqslant 2$, *then*

$$\|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{p_B}} \approx \omega(B)^{\frac{1}{p_B^-}} \approx \omega(B)^{\frac{1}{p_B^+}}.$$

Proof. The idea of the proof is similar to Lemma 3.4. Fix $x,z \in \mathbb{R}^n$ and $r,R>0$ and set $t := r^n/(|x-z|^n + r^n + R^N)$. Then $t\chi_{B(z,R)} \leqslant M\chi_{B(x,r)}$ so that

$$B(z,R) \subset \{M\chi_{B(x,r)} > t\}.$$

Hence we obtain from the weak maximal inequality that

$$t\,\|\chi_{B(z,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \leqslant \|t\chi_{\{M\chi_{B(x,r)}>t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \lesssim \|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}. \tag{6.2}$$

Of course, $\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} = \|1\|_{L^{p(\cdot)}(B(x,r),\omega)}$. Using the previous inequality twice gives

$$\frac{r^n}{1+|x|^n}\|1\|_{L^{p(\cdot)}(B(0,1),\omega)} \lesssim \|1\|_{L^{p(\cdot)}(B,\omega)} \lesssim (1+|x|^n)\,\|1\|_{L^{p(\cdot)}(B(0,1),\omega)},$$

where $B = B(x,r)$ and $r \in (0,1]$. Using the decay condition on p we conclude from this inequality that $\|1\|^{p_B^+}_{L^{p(\cdot)}(B,\omega)} \approx \|1\|^{p_B^-}_{L^{p(\cdot)}(B,\omega)}$. In view of (2.3) this implies that

$$\|1\|^{p_B^+}_{L^{p(\cdot)}(B,\omega)} \approx \|1\|^{p_B^-}_{L^{p(\cdot)}(B,\omega)} \approx \|1\|^{p_B}_{L^{p(\cdot)}(B,\omega)} \approx \varrho_{L^{p(\cdot)}(B,\omega)}(1) = \omega(B). \qquad \square$$

The proof of the following lemma was suggested by the referee.

Lemma 6.3. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$. If $M\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow$ w-$L^{p(\cdot)}(\mathbb{R}^n,\omega)$, then $\omega \in A^{\mathrm{loc}}_\infty$.*

Proof. Let $E \subset B$ be measurable for a ball B with radius at most 1. Then

$$B \subset \{M\chi_E \geqslant \tfrac{|E|}{2^n|B|}\}.$$

Hence it follows as in (6.2) that

$$\tfrac{|E|}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \lesssim \|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}.$$

Then we use that (Lemma 6.1)

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} = \|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{1/p_B^-} \approx \omega(B)^{1/p_B^+}$$

and, by (2.3),

$$\|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \leqslant \max\{\omega(E)^{\frac{1}{p_B^+}}, \omega(E)^{\frac{1}{p_B^-}}\}$$

to conclude that

$$\frac{|E|}{|B|} \lesssim \max\left\{ \left(\frac{\omega(E)}{\omega(B)}\right)^{\frac{1}{p_B^+}}, \left(\frac{\omega(E)}{\omega(B)}\right)^{\frac{1}{p_B^-}} \right\}.$$

This implies that $\omega \in A_\infty^{\mathrm{loc}}$. □

Lemma 6.4. *Let $p \in \mathcal{P}_\pm^{\log}(\mathbb{R}^n)$. If $M\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow w\text{-}L^{p(\cdot)}(\mathbb{R}^n,\omega)$, then ω has at most polynomial growth.*

Proof. We show that ω has at most polynomial growth by proving the inequality

$$\omega\big(B(0,2^{k+1}) \setminus B(0,2^k)\big) \lesssim 2^{knp_\infty}. \tag{6.5}$$

Using the weak maximal inequality we have

$$\begin{aligned} 2^{-nk}\|\chi_{B(0,2^{k+1})\setminus B(0,2^k)}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} &\lesssim 2^{-nk}\,\|\chi_{\{M\chi_{B(0,1)}>c\,2^{-nk}\}}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \\ &\lesssim \|\chi_{B(0,1)}\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)}. \end{aligned}$$

Note that the right hand side is a constant independent of k. This implies that the modular of $2^{-nk}\chi_{B(0,2^{k+1})\setminus B(0,2^k)}$ is also bounded:

$$\int_{\mathbb{R}^n} \Big(\chi_{B(0,2^{k+1})\setminus B(0,2^k)}(x)2^{-nk}\Big)^{p(x)}\omega(x)\,dx \leqslant c.$$

Using the decay condition we see that $2^{-p(x)nk} \approx 2^{-p_\infty nk}$ in $B(0,2^{k+1}) \setminus B(0,2^k)$. This implies (6.5). □

Note the two different exponents in the next result.

Corollary 6.6. *Let $p, q \in \mathcal{P}_\pm^{\log}(\mathbb{R}^n)$. If $M\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow w\text{-}L^{p(\cdot)}(\mathbb{R}^n,\omega)$, then*

$$\|1\|_{L^{q(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{q_B}}$$

for all balls $B \subset \mathbb{R}^n$.

Proof. By Lemma 6.3, $\omega \in A_\infty^{\mathrm{loc}}$. Hence it follows by Lemma 3.4 that

$$\|1\|_{L^{q(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{q_\infty}}$$

for balls with diameter between $\frac{1}{4\sqrt{n}}$ and 2. Hence the assumption of Lemma 3.6 are satisfied. Using the conclusions of that lemma, and the fact that ω has polynomial growth (Lemma 6.4), we conclude as in Corollary 3.7 that

$$\|1\|_{L^{q(\cdot)}(B,\omega)} \approx \omega(B)^{\frac{1}{q_B}}$$

for all balls, as claimed. □

The following lemma was suggested by the referee.

Lemma 6.7. *Let $p \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^n)$. If $M\colon L^{p(\cdot)}(\mathbb{R}^n,\omega) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n,\omega)$, then*

$$M\colon L^{p'(\cdot)}(\mathbb{R}^n,\omega') \hookrightarrow w\text{-}L^{p'(\cdot)}(\mathbb{R}^n,\omega').$$

Proof. For $f \in L^{p(\cdot)}(\mathbb{R}^n,\omega)$ we set $F_t := t\,\chi_{\{Mf>t\}}$. Then we can choose, by [22, Corollary 3.2.14], $g \in L^{p(\cdot)}(\mathbb{R}^n,\omega)$ with norm equal to 1 such that

$$\|t\,\chi_{\{Mf>t\}}\|_{L^{p'(\cdot)}(\mathbb{R}^n,\omega')} = \|F_t\|_{L^{p'(\cdot)}(\mathbb{R}^n,\omega')} \approx \int_{\mathbb{R}^n} F_t\, g\, dx.$$

By the Fefferman–Stein inequality and Hölder's inequality we then find that

$$\int_{\mathbb{R}^n} F_t\, g\, dx \leqslant c\int_{\mathbb{R}^n} |f|\, Mg\, dx \leqslant c\,\|f\|_{L^{p'(\cdot)}(\mathbb{R}^n,\omega')}\|g\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} = c\,\|f\|_{L^{p'(\cdot)}(\mathbb{R}^n,\omega')},$$

i.e. the weak maximal inequality holds. □

We can then conclude the proof of the main theorem.

Proof of necessity of the $A_{p(\cdot)}$-condition in Theorem 1.1. We want to use the classical test-function,

$$f := \omega^{-\frac{1}{p(\cdot)-1}}\chi_B = \omega'\chi_B,$$

even though the arguments are a bit more technical since the connection between norm and modular is not as simple as in the constant exponent case. Let B be a ball and note that $\int_B f\,dx = \omega'(B)$. Then we see that $Mf \gtrsim \frac{\omega'(B)}{|B|}$ in B. Therefore the maximal inequality implies that

$$\begin{aligned}\frac{\omega'(B)}{|B|}\|1\|_{L^{p(\cdot)}(B,\omega)} &\lesssim \|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} \\ &\lesssim \|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\omega)} = \|\omega^{-\frac{1}{p(\cdot)-1}}\|_{L^{p(\cdot)}(B,\omega)} = \|1\|_{L^{p(\cdot)}(B,\omega')}.\end{aligned} \tag{6.8}$$

By Corollary 6.6, $\|1\|_{L^{p(\cdot)}(B,\omega)} \approx \omega(B)^{1/p_B}$. By Lemma 6.7, the assumptions of Corollary 6.6 hold also for the weight ω'. Hence

$$\|1\|_{L^{p(\cdot)}(B,\omega')} \approx \omega'(B)^{\frac{1}{p_B}}.$$

Thus it follows from (6.8) that

$$\frac{\omega'(B)}{|B|}\,\omega(B)^{1/p_B} \approx \frac{\omega'(B)}{|B|}\|1\|_{L^{p(\cdot)}(B,\omega)} \lesssim \|1\|_{L^{p(\cdot)}(B,\omega')} \approx \omega'(B)^{\frac{1}{p_B}}.$$

From this we obtain that

$$\frac{\omega(B)}{|B|^{p_B}}\,\omega'(B)^{p_B-1} \lesssim 1.$$

We conclude the proof by noting that $\omega'(B)^{p_B-1} \approx \|\frac{1}{\omega}\|_{L^{p'(\cdot)/p(\cdot)}(B)}$, by Corollary 6.6, so the previous inequality is equivalent to the $A_{p(\cdot)}$ condition. □

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