A NEW PROOF OF THE BOUNDEDNESS OF MAXIMAL OPERATORS ON VARIABLE LEBESGUE SPACES

D. CRUZ-URIBE, SFO, L. DIENING, AND A. FIORENZA

ABSTRACT. We give a new proof using the classic Calderón-Zygmund decomposition that the Hardy-Littlewood maximal operator is bounded on the variable Lebesgue space $L^{p(\cdot)}$ whenever the exponent function $p(\cdot)$ satisfies log-Hölder continuity conditions. We include the case where $p(\cdot)$ assumes the value infinity. The same proof also shows that the fractional maximal operator M_α, $0<\alpha<n$, maps $L^{p(\cdot)}$ into $L^{q(\cdot)}$, where $1/p(\cdot)-1/q(\cdot)=\alpha/n$.

1. Introduction

Given a measurable function $p(\cdot):\mathbb{R}^n\to[1,\infty]$, let $\Omega_{\infty,p(\cdot)}=\{x\in\mathbb{R}^n : p(x)=\infty\}$. We define the variable Lebesgue space $L^{p(\cdot)}$ to be the set of functions such that for some $\lambda>0$,

$$\rho_{p(\cdot)}(f/\lambda)=\int_{\mathbb{R}^n\setminus\Omega_{\infty,p(\cdot)}}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx+\lambda^{-1}\|f\|_{L^\infty(\Omega_{\infty,p(\cdot)})}<\infty.$$

$L^{p(\cdot)}$ is a Banach space when equipped with the norm

$$\|f\|_{p(\cdot)}=\inf\left\{\lambda>0:\rho_{p(\cdot)}(f/\lambda)\le 1\right\}.$$

These spaces are a special case of the Musielak-Orlicz spaces (cf. Musielak [18]) and generalize the classical Lebesgue spaces: if $p(x)=p_0$, then $L^{p(\cdot)}=L^{p_0}$.

Variable Lebesgue spaces have been known since the 1930's, but have become the focus of intense investigation in the past fifteen years. (See [8, 10, 22] for further history and applications.) A central problem has been to extend the techniques of harmonic analysis to these spaces, which in turn leads naturally to the study of the

Date: March 11, 2008.

1991 *Mathematics Subject Classification.* 42B25,42B35.

Key words and phrases. variable Lebesgue space, maximal operators, Calderón-Zygmund decomposition.

The first author is partially supported by the Stewart-Dorwart faculty development fund of the Department of Mathematics of Trinity College. The first two authors would like to thank Carlo Sbordone and also the Dipartimento di Costruzioni e Metodi Matematici in Architettura, Università di Napoli, for its generous hospitality.

Hardy-Littlewood maximal operator and the closely related fractional maximal operator. The purpose of this paper is to give a new and simpler proof of the boundedness of these operators on variable Lebesgue spaces.

Before stating our main result, we first make a few key definitions. Given α, $0 \leq \alpha < n$, we define

$$M_\alpha f(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \fint_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ that contain x. (Equivalently, cubes may be replaced by balls containing x.) When $\alpha = 0$ this is the Hardy-Littlewood maximal operator and we write Mf instead of $M_0 f$. For $\alpha > 0$ this is the fractional maximal operator introduced by Muckenhoupt and Wheeden [17].

Given a function $r(\cdot) : \mathbb{R} \to [0,\infty)$, we say that $r(\cdot)$ is locally log-Hölder continuous, and write $r(\cdot) \in LH_0$, if there exists a constant C_0 such that

$$|r(x) - r(y)| \leq \frac{C_0}{-\log|x-y|}, \quad x, y \in \mathbb{R}^n, \quad |x-y| < 1/2.$$

Similarly, we say that $r(\cdot)$ is log-Hölder continuous at infinity, and write $r(\cdot) \in LH_\infty$, if there exists constants C_∞ and $r(\infty)$ such that

$$|r(x) - r(\infty)| \leq \frac{C_\infty}{\log(e+|x|)}, \qquad x \in \mathbb{R}^n.$$

We say $r(\cdot)$ is (globally) log-Hölder continuous if $r(\cdot) \in LH_0 \cap LH_\infty$ and we write $r(\cdot) \in LH$.

Remark 1.1. The LH_∞ condition is equivalent to the uniform continuity condition

$$|r(x) - r(y)| \leq \frac{C_\infty}{\log(e+|x|)}, \quad x,\, y \in \mathbb{R}^n, \quad |y| \geq |x|.$$

The LH_∞ condition was originally defined in this form in [4].

Finally, given a set $E \subset \mathbb{R}^n$, let

$$p_-(E) = \operatorname*{ess\,inf}_{x \in E} p(x), \quad p_+(E) = \operatorname*{ess\,sup}_{x \in E} p(x);$$

If $E = \mathbb{R}^n$, then we simply write p_- and p_+.

Theorem 1.2. *Given α, $0 \leq \alpha < n$, let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ be such that $1/p(\cdot) \in LH$ and $1 < p_- \leq p_+ \leq n/\alpha$. Define the exponent function $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad x \in \mathbb{R}^n,$$

where we let $1/\infty = 0$ and $1/0 = \infty$. Then

$$\|M_\alpha f\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Remark 1.3. The constant in the conclusion of Theorem 1.2 depends on the dimension n, the log-Hölder constants of $1/p(\cdot)$, p_-, and $p(\infty)$ (if this value is finite).

Remark 1.4. The assumption that $1/p(\cdot) \in LH$ implies that $1/q(\cdot) \in LH$ as well. Further, if $p_+ < \infty$, then the assumption $1/p(\cdot) \in LH$ is equivalent to assuming $p(\cdot) \in LH$, since

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| = \left|\frac{p(x)-p(y)}{p(x)p(y)}\right| \geq \left|\frac{p(x)-p(y)}{(p_+)^2}\right|.$$

Theorem 1.2 combines a number of results that have been proved by the authors and others. We first consider the case $\alpha = 0$, that is, norm inequalities for the Hardy-Littlewood maximal operator. In [6] Theorem 1.2 was proved with the stronger assumption that $p_+ < \infty$, $p(\cdot) \in LH_0$, and $p(\cdot)$ is constant outside a large ball. The more general result, but still assuming that $p_+ < \infty$, was proved in [4] and a simpler proof was given in [2]. A somewhat different version with the LH_∞ condition replaced by a weaker averaging condition at infinity was proved by Nekvinda [19]. The full result was proved in [8] (see also [9]).

In the case $\alpha > 0$ and $p_+ = n/\alpha$ this result is new. Estimates for fractional maximal operators were first considered by Kokilashvili and Samko [14]. Theorem 1.2 when $p_+ < n/\alpha$ and $p(\cdot) \in LH$ was proved in [2].

Remark 1.5. The log-Hölder condition $1/p(\cdot) \in LH$ is not necessary: see the examples due to Lerner [16] and Nekvinda [20]. In [7] a very general necessary and sufficient condition for the maximal operator to be bounded was given. However, in some sense the log-Hölder condition is close to necessary: see the example by Pick and Růžička [21] and also the example in [4]. This, combined with the relative ease with which they can be applied makes these continuity conditions useful in practice.

We can also give a new proof of a weak type inequality that extends to the endpoint case $p_- = 1$. It generalizes a result first proved in [2] in the case $p_+ < n/\alpha$.

Theorem 1.6. *Given α, $0 \leq \alpha < n$, let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ be such that $1/p(\cdot) \in LH$ and $1 = p_- \leq p_+ \leq n/\alpha$. Then*

$$\sup_{t>0} t \|\chi_{\{M_\alpha f(x) > t\}}\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}. \tag{1.1}$$

Our proofs of Theorems 1.2 and 1.6 have several features that we want to highlight. First, each proof uses the machinery of Calderón-Zygmund cubes, which are of great importance in harmonic analysis on the classical Lebesgue spaces. This machinery was not used in proving earlier versions of Theorem 1.2 (though some of it was used in [7].) We believe that these techniques will be applicable to other problems in variable Lebesgue spaces.

Second, our proofs, especially in the case $p_+ < n/\alpha$, are simpler than previous proofs. The proof in [2] for the Hardy-Littlewood maximal operator depends on the following estimate: if $\|f\|_{p(\cdot)} \leq 1$, then there exists a function $S(\cdot) \in L^1$ and $C > 0$ such that for every ball B and $x \in B$,

$$\left(\fint_B |f(y)|\,dy \right)^{p(x)} \leq C \left(\fint_B |f(y)|^{p(y)/p_-}\,dy \right)^{p_-} + S(x). \tag{1.2}$$

The proof of this inequality required considering separately the averages of $f\chi_{\{|f|\geq 1\}}$ and $f\chi_{\{|f|\leq 1\}}$, and then subdivided the argument further by considering the distance of the ball from the origin in comparison to its radius. Our proof still requires that we divide f into its large and small parts, but the Calderón-Zygmund decomposition provides the "natural" family of cubes on which to consider the averages. Furthermore, the structure of the proof makes clear the role played by the log-Hölder continuity conditions: the LH_0 condition is necessary only on the set where f is large, and the LH_∞ condition on the set where f is close to zero.

Third, our proof gives a unified treatment of the Hardy-Littlewood maximal operator and the fractional maximal operator. The proofs in [2] for the case $\alpha > 0$ required first proving that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$, and then using this fact to prove that the fractional maximal operator mapped $L^p(\cdot)$ into $L^q(\cdot)$. In the classical Lebesgue spaces, however, it is possible to give a single proof that works simultaneously for all α, $0 \leq \alpha < n$. This is well-known; the proof is sketched in Duoandikoetxea [11]. This proof uses weak type inequalities and Marcinkiewicz interpolation, and so cannot be used in the variable Lebesgue spaces. In [2] the authors conjectured a generalization of (1.2) that would yield unified proof: if $\rho_{p(\cdot)}(f) \leq 1$, then there exists constant C and $S \in L^1$ such that

$$M_\alpha f(x)^{q(x)} \leq C M_\alpha(|f(\cdot)|^{p(\cdot)/p_-})(x)^{q_-} + S(x).$$

However, this conjecture remains open.

The rest of this paper is organized as follows. In Section 2 we gather together some preliminary results about variable Lebesgue spaces and about Calderón-Zygmund cubes. For complete information about these spaces we refer the reader to the papers by Kováčik and Rákosník [15] or Fan and Zhao [12]. In Sections 3 and 4 we prove Theorem 1.2. The proof of the full result contains many technical details that obscure the overall argument, so we first prove it in the special case that $p_+ < n/\alpha$. Doing so results in some repetition, but it allows us to make clear the basic ideas of our argument and to highlight the relative simplicity of this proof compared to earlier proofs. Finally, in Section 5, we prove Theorem 1.6. Throughout the paper, C

will denote a constant that may depend on n and $p(\cdot)$ but which may otherwise change value at each appearance. In order to emphasize that we are dealing with variable exponents, we will always write $p(\cdot)$ and $q(\cdot)$ for exponent functions; p and q will denote constants. Occasionally there will be minor differences in the argument depending on whether $\alpha = 0$ or $\alpha > 0$. We will highlight these but will generally give full details only for the latter case, as the former case is usually easier.

2. Preliminary Results

The following lemmas are some key technical results needed in our proof. We have gathered them here to make our overall approach in the proofs clearer.

The first is the analogue of the monotone convergence theorem; in the more general context of Banach function spaces it is referred to as the Fatou property of the norm. (See Bennett and Sharpley [1].)

Lemma 2.1. *Given a non-negative function $f \in L^{p(\cdot)}$, suppose the sequence $\{f_N\}$ of non-negative functions increases pointwise to f almost everywhere. Then $\|f_N\|_{p(\cdot)}$ increases to $\|f\|_{p(\cdot)}$.*

Proof. We may assume $\|f\|_{p(\cdot)} > 0$ since otherwise there is nothing to prove. Fix λ, $0 < \lambda < \|f\|_{p(\cdot)}$; then by the definition of the norm,

$$\begin{aligned} 1 < \rho_{p(\cdot)}(f/\lambda) &= \int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} \left(\frac{f(x)}{\lambda}\right)^{p(\cdot)} dx + \lambda^{-1} \|f\|_{L^\infty(\Omega_{\infty,p(\cdot)})} \\ &= \lim_{N\to\infty} \int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} \left(\frac{f_N(x)}{\lambda}\right)^{p(\cdot)} dx + \lambda^{-1} \|f_N\|_{L^\infty(\Omega_{\infty,p(\cdot)})} \\ &= \lim_{N\to\infty} \rho_{p(\cdot)}(f_N/\lambda). \end{aligned}$$

Therefore, for all N sufficiently large, $\rho_{p(\cdot)}(f_N/\lambda) > 1$, so $\|f_N\|_{p(\cdot)} > \lambda$. Since we can take any such λ, the desired conclusion follows at once. □

To apply this lemma we need the following fact which is part of the "folklore" of harmonic analysis. We include its short proof.

Lemma 2.2. *Given α, $0 \le \alpha < n$, and a sequence of non-negative functions f_N increasing pointwise a.e. to a function f, then the functions $M_\alpha f_N$ increase to $M_\alpha f$ pointwise.*

Proof. It follows at once from the definition that the sequence $M_\alpha f_N$ is increasing and $M_\alpha f_N(x) \le M_\alpha f(x)$ for all x. Now fix x and K such that $K < M_\alpha f(x)$. Then there exists a cube $Q \ni x$ such that

$$K < |Q|^{\alpha/n} \fint_Q f(y)\,dx = \lim_{N\to\infty} |Q|^{\alpha/n} \fint_Q f_N(y)\,dx \le \lim_{N\to\infty} M_\alpha f_N(x).$$

The desired conclusion follows immediately. □

The next two lemmas are the only places we need to use the assumption that the exponent function is log-Hölder continuous. The first appeared in [6] with balls in place of cubes. The second is a special case of a result that appeared in [2] (see also [4, 5]). For the convenience of the reader we include their short proofs.

Lemma 2.3. *Given $r(\cdot) : \mathbb{R}^n \to [0,\infty)$ such that $r(\cdot) \in LH_0$ and $r_+ < \infty$, there exists a constant C depending on n and the LH_0 constant of $r(\cdot)$ such that given any cube Q and $x \in Q$,*

$$|Q|^{r(x)-r_+(Q)} \le C \quad \textit{and} \quad |Q|^{r_-(Q)-r(x)} \le C.$$

Proof. We prove the first inequality; the proof of the second is identical. If $\ell(Q) \ge (2\sqrt{n})^{-1}$, then

$$|Q|^{r(x)-r_+(Q)} \le (2\sqrt{n})^{n(r_+-r_-)} = C(n, r(\cdot)).$$

If $\ell(Q) < (2\sqrt{n})^{-1}$, then for all $y \in Q$, $|x-y| < 1/2$. In particular, since $r(\cdot)$ is continuous, there exists $y \in Q$ such that $r(y) = r_+(Q)$. Therefore, by the definition of LH_0,

$$\begin{aligned}|Q|^{r(x)-r_+(Q)} &\le \big(n^{-1/2}|x-y|\big)^{-n|r(x)-r(y)|} \\ &\le \exp\left(\frac{C_0(\log(n^{1/2}) - \log|x-y|)}{-\log|x-y|}\right) \le C(n, r(\cdot)).\end{aligned}$$

□

Lemma 2.4. *Let $r(\cdot) : \mathbb{R}^n \to [0,\infty)$ be such that $r(\cdot) \in LH_\infty$, and let $R(x) = (e+|x|)^{-N}$, $N > n$. Then there exists a constant C depending on n, N and the LH_∞ constant of $r(\cdot)$ such that given any set E and any function F such that $0 \le F(y) \le 1$ for $y \in E$,*

$$\int_E F(y)^{r(y)}\,dy \le C\int_E F(y)^{r(\infty)}\,dy + C\int_E R(y)^{r(\infty)}\,dy, \tag{2.1}$$

$$\int_E F(y)^{r(\infty)}\,dy \le C\int_E F(y)^{r(y)}\,dy + C\int_E R(y)^{r(\infty)}\,dy. \tag{2.2}$$

Proof. We will prove (2.1); the proof of the second inequality is essentially the same. By the LH_∞ condition,

$$R(y)^{-|r(y)-r(\infty)|} = \exp\big(N\log(e+|y|)|r(y)-r(\infty)|\big) \le \exp(NC_\infty).$$

Write the set E as $E_1 \cup E_2$, where $E_1 = \{x \in E : F(y) \le R(y)\}$ and $E_2 = \{x \in E : R(y) < F(y)\}$. Then

$$\begin{aligned}\int_{E_1} F(y)^{r(y)}\,dy &\le \int_{E_1} R(y)^{r(y)}\,dy \\ &\le \int_{E_1} R(y)^{r(\infty)} R(y)^{-|r(y)-r(\infty)|}\,dy \le \exp(NC_\infty)\int_{E_1} R(y)^{r(\infty)}\,dy.\end{aligned}$$

Similarly, since $F(y) \le 1$,

$$\begin{aligned}\int_{E_2} F(y)^{r(y)}\,dy &\le \int_{E_2} F(y)^{r(\infty)} F(y)^{-|r(y)-r(\infty)|}\,dy \\ &\le \int_{E_2} F(y)^{r(\infty)} R(y)^{-|r(y)-r(\infty)|}\,dy \le \exp(NC_\infty)\int_{E_2} F(y)^{r(\infty)}\,dy.\end{aligned}$$

□

The last two lemmas give some basic properties of cubes. The first defines the so called Calderón-Zygmund cubes. This result is well-known for $\alpha = 0$—for a proof see Duoandikoetxea [11] or García-Cuerva and Rubio de Francia [13]. The same proofs go through without significant changes for the case $\alpha > 0$. (For details, see [3].) Hereafter, given a cube Q and $r > 0$, let rQ denote the cube with the same center as Q and such that $\ell(rQ) = r\ell(Q)$.

Lemma 2.5. *Fix α, $0 \le \alpha < n$. Given a function f such that $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$, then for each $\lambda > 0$ there exists a set of pairwise disjoint dyadic cubes $\{Q_j^\lambda\}$ such that*

$$\{x \in \mathbb{R}^n : M_\alpha f(x) > 2^{2n-\alpha}\lambda\} \subset \bigcup_j 3Q_j^\lambda,$$

and

$$|Q_j^\lambda|^{\alpha/n} \fint_{Q_j^\lambda} |f(x)|\,dx > \lambda.$$

Remark 2.6. The hypothesis on f is satisfied if it is bounded and has compact support.

The final lemma is a clever application of Hölder's inequality.

Lemma 2.7. *Given α, $0 \le \alpha < n$, and p, q, such that $1 < p < n/\alpha$ and $1/p - 1/q = \alpha/n$, then for every cube Q and non-negative function f,*

$$|Q|^{\alpha/n} \fint_Q f(x)\,dx \le \left(\int_Q f(x)^p\,dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\fint_Q f(x)\,dx\right)^{p/q}.$$

Proof. When $\alpha = 0$ this reduces to an identity, so we only need to consider the case $\alpha > 0$. By Hölder's inequality with exponent $n/\alpha p > 1$ and then with exponent p,

$$\begin{aligned} |Q|^{\alpha/n} \fint_Q f(x)\,dx &= |Q|^{\alpha/n} \fint_Q f(x)^{\alpha p/n} f(x)^{1-\alpha p/n}\,dx \\ &\leq |Q|^{\alpha/n} \left(\fint_Q f(x)\,dx \right)^{\alpha p/n} \left(\fint_Q f(x)\,dx \right)^{1-\alpha p/n} \\ &\leq \left(\int_Q f(x)^p\,dx \right)^{\alpha/n} \left(\fint_Q f(x)\,dx \right)^{1-\alpha p/n} \\ &= \left(\int_Q f(x)^p\,dx \right)^{\frac{1}{p}-\frac{1}{q}} \left(\fint_Q f(x)\,dx \right)^{p/q}. \end{aligned}$$

□

Remark 2.8. As a corollary to Lemma 2.7 we have that

$$M_\alpha f(x)^q \leq \|f\|_p^{q-p} Mf(x)^p.$$

Hence, the fact that $M_\alpha : L^p \to L^q$ follows immediately from the fact that the Hardy-Littlewood maximal operator is bounded on L^p. In some sense, our proof of Theorem 1.2 is a generalization of this approach.

3. Proof of Theorem 1.2: The case $p_+ < n/\alpha$

Since $p_+ < n/\alpha$, we have that p_+, $q_+ < \infty$. Therefore, we will use our hypothesis on $p(\cdot)$ in the equivalent form that $p(\cdot)$, $q(\cdot) \in LH$.

We begin the proof by making some initial reductions. First, clearly we may assume that f is non-negative.

Second, we may assume without loss of generality that f is bounded and has compact support. For if we can prove the theorem in this case, then given any non-negative $f \in L^{p(\cdot)}$, let $f_N = \min(f, N)\chi_{\{|x| \leq N\}}$. Then f_N increases to f as N tends to infinity, and the general result follows from Lemmas 2.1 and 2.2. This assumption allows us to apply Lemma 2.5 to f.

Third, by homogeneity we may assume that $\|f\|_{p(\cdot)} = 1$. Then

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} f(x)^{p(x)}\,dx \leq 1.$$

Decompose f as $f_1 + f_2$, where $f_1 = f\chi_{\{x : f(x) > 1\}}$ and $f_2 = f\chi_{\{x : f(x) \leq 1\}}$; then $\rho_{p(\cdot)}(f_i) \leq \|f_i\|_{p(\cdot)} \leq 1$. Further, since $M_\alpha f \leq M_\alpha f_1 + M_\alpha f_2$, it will suffice to show that for

$i = 1, 2$, that $\|M_\alpha f_i\|_{q(\cdot)} \leq C(n, p(\cdot))$; since $q_+ < \infty$ it will in turn suffice to show that

$$\rho_{q(\cdot)}(M_\alpha f_i) = \int_{\mathbb{R}^n} M_\alpha f_i(x)^{q(x)}\,dx \leq C.$$

The estimate for f_1. Let $A = 2^{2n-\alpha}$, and for each $k \in \mathbb{Z}$ let

$$\Omega_k = \{x \in \mathbb{R}^n : M_\alpha f_1(x) > A^k\}.$$

Since f_1 is bounded and has compact support, by Lemma 2.7, $M_\alpha f_1 \in L^\infty$, so $M_\alpha f_1(x) < \infty$ a.e., and $\mathbb{R}^n = \bigcup_k \Omega_k \setminus \Omega_{k+1}$ (up to a set of measure 0). Further, for each k we can apply Lemma 2.5 to form the pairwise disjoint cubes $\{Q_j^k\}$ such that

$$\Omega_k \subset \bigcup_j 3Q_j^k \quad \text{and} \quad |Q_j^k|^{\alpha/n} \fint_{Q_j^k} f_1(x)\,dx > A^{k-1}.$$

Define the sets E_j^k inductively: $E_1^k = (\Omega_k \setminus \Omega_{k+1}) \cap 3Q_1^k$, $E_2^k = ((\Omega_k \setminus \Omega_{k+1}) \cap 3Q_2^k) \setminus E_1^k$, $E_3^k = ((\Omega_k \setminus \Omega_{k+1}) \cap 3Q_3^k) \setminus (E_1^k \cup E_2^k)$, etc. The sets E_j^k are pairwise disjoint for all j and k and $\Omega_k \setminus \Omega_{k+1} = \bigcup_j E_j^k$.

We now estimate as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n} M_\alpha f_1(x)^{q(x)}\,dx &= \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} M_\alpha f_1(x)^{q(x)}\,dx \\
&\leq \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} [A^{k+1}]^{q(x)}\,dx \\
&\leq A^{2q_+} 3^{q_+(n-\alpha)} \sum_{k,j} \int_{E_j^k} \left(|3Q_j^k|^{\alpha/n} \fint_{3Q_j^k} f_1(y)\,dy \right)^{q(x)} dx. \qquad (3.1)
\end{aligned}$$

To estimate the integral in the last sum, we apply Lemma 2.7 with exponents $p_{jk} = p_-(3Q_j^k)$ and $q_{jk} = q_-(3Q_j^k)$:

$$|3Q_j^k|^{\alpha/n} \fint_{3Q_j^k} f_1(y)\,dy \leq \left(\int_{3Q_j^k} f_1(y)^{p_{jk}}\,dy \right)^{\frac{1}{p_{jk}} - \frac{1}{q_{jk}}} \left(\fint_{3Q_j^k} f_1(y)\,dy \right)^{p_{jk}/q_{jk}}.$$

Since $f_1 = 0$ or $f_1 \geq 1$ pointwise,

$$\int_{3Q_j^k} f_1(y)^{p_{jk}}\,dy \leq \int_{\mathbb{R}^n} f_1(y)^{p(y)}\,dy \leq 1.$$

Therefore,

$$\sum_{k,j}\int_{E_j^k}\left(|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_1(y)\,dy\right)^{q(x)} dx \le \sum_{k,j}\int_{E_j^k}\left(\fint_{3Q_j^k} f_1(y)\,dy\right)^{p_{jk}q(x)/q_{jk}} dx$$

$$\le \sum_{k,j}\int_{E_j^k}\left(\fint_{3Q_j^k} f_1(y)^{p_{jk}/p_-}\,dy\right)^{p_-q(x)/q_{jk}} dx.$$

Since $q(\cdot) \in LH_0$ and $q_+ < \infty$, by Lemma 2.3 there exists a constant C depending on $q(\cdot)$ and n such that

$$|3Q_j^k|^{-q(x)} \le C|3Q_j^k|^{-q_{jk}}.$$

Further, arguing as before,

$$\int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy \le \int_{3Q_j^k} f_1(y)^{p(y)}\,dy \le 1.$$

Therefore, since for $x \in E_j^k \subset 3Q_j^k$, $q(x) \ge q_{jk}$,

$$\begin{aligned}
&\sum_{k,j}\int_{E_j^k}\left(\fint_{3Q_j^k} f_1(y)^{p_{jk}/p_-}\,dy\right)^{p_-q(x)/q_{jk}} dx \\
&\qquad \le C\sum_{k,j}\int_{E_j^k}|3Q_j^k|^{-p_-}\left(\int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy\right)^{q(x)p_-/q_{jk}} dx \\
&\qquad \le C\sum_{k,j}\int_{E_j^k}|3Q_j^k|^{-p_-}\left(\int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy\right)^{p_-} dx \\
&\qquad \le C\sum_{k,j}\int_{E_j^k} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx \\
&\qquad \le C\int_{\mathbb{R}^n} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx.
\end{aligned}$$

Since $p_- > 1$, M is bounded on L^{p_-}. Hence,

$$\begin{aligned}
&\le C\int_{\mathbb{R}^n} f_1(x)^{p(x)}\,dx \\
&\le C.
\end{aligned}$$

The estimate for f_2. We argue exactly as we did above for f_1, forming the sets Ω_k and E_j^k using Lemma 2.5. We thus get

$$\int_{\mathbb{R}^n} M_\alpha f_2(x)^{q(x)}\,dx \le C\sum_{k,j}\int_{E_j^k}\left(|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_2(y)\,dy\right)^{q(x)}\,dx.$$

We claim that

$$F = |3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_2(y)\,dy \le 1. \tag{3.2}$$

If $\alpha = 0$, this is immediate since $f_2 \le 1$. If $\alpha > 0$, then by Hölder's inequality and since $p(y) \le n/\alpha$,

$$F \le \left(\int_{3Q_j^k} f_2(y)^{n/\alpha}\,dy\right)^{\alpha/n} \le \left(\int_{3Q_j^k} f_2(y)^{p(y)}\,dy\right)^{\alpha/n} \le 1.$$

Therefore, by Lemma 2.4 with $R(x) = (e+|x|)^{-n}$.

$$\begin{aligned}\sum_{k,j}\int_{E_j^k}&\left(|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_2(y)\,dy\right)^{q(x)}\,dx\\ &\le C\sum_{k,j}\int_{E_j^k}\left(|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_2(y)\,dy\right)^{q(\infty)}\,dx + C\sum_{k,j}\int_{E_j^k} R(x)^{q(\infty)}\,dx.\end{aligned}$$

We can immediately estimate the second term: since $q(\infty) > 1$ and the sets E_j^k are disjoint,

$$\sum_{k,j}\int_{E_j^k} R(x)^{q(\infty)}\,dx \le \int_{\mathbb{R}^n} R(x)^{q(\infty)}\,dx \le C.$$

To estimate the first term we apply Lemma 2.7 with exponents $p(\infty)$ and $q(\infty)$:

$$|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_2(y)\,dy \le \left(\int_{3Q_j^k} f_2(y)^{p(\infty)}\right)^{\frac{1}{p(\infty)}-\frac{1}{q(\infty)}}\left(\fint_{3Q_j^k} f_2(y)\,dy\right)^{p(\infty)/q(\infty)}.$$

To estimate the first integral on the right-hand side, we again apply Lemma 2.4 with $R(x) = (e+|x|)^{-n}$:

$$\int_{3Q_j^k} f_2(y)^{p(\infty)} \le C\int_{3Q_j^k} f_2(y)^{p(y)}\,dy + C\int_{3Q_j^k} R(y)^{p(\infty)}\,dy \le C.$$

Since $p(\infty) > 1$, M is bounded on $L^{p(\infty)}$. Therefore,

$$\begin{aligned}\sum_{k,j} \int_{E_j^k} \left(|3Q_j^k|^{\alpha/n} \fint_{3Q_j^k} f_2(y)\,dy \right)^{q(\infty)} dx &\leq C \sum_{k,j} \int_{E_j^k} \left(\fint_{3Q_j^k} f_2(y)\,dy \right)^{p(\infty)} dx \\ &\leq C \sum_{k,j} \int_{E_j^k} Mf_2(x)^{p(\infty)}\,dx \\ &\leq C \int_{\mathbb{R}^n} Mf_2(x)^{p(\infty)}\,dx \\ &\leq C \int_{\mathbb{R}^n} f_2(x)^{p(\infty)}\,dx;\end{aligned}$$

since $f_2 \leq 1$ we can apply Lemma 2.4 again to conclude

$$\begin{aligned}&\leq C \int_{\mathbb{R}^n} f_2(x)^{p(x)}\,dx + C \int_{\mathbb{R}^n} R(x)^{p(\infty)}\,dx \\ &\leq C.\end{aligned}$$

4. Proof of Theorem 1.2: The general case

The proof of the general case has much the same outline as the proof when $p_+ < n/\alpha$ given in the previous section, but it is made more complicated by the technicalities needed to deal with the fact that the exponent function $q(\cdot)$ is unbounded and may in fact equal ∞ on a set of positive measure. In the proof that follows we attempt to strike a balance between brevity and completeness, and we will refer back to the proof in Section 3 for those details which remain the same.

Arguing as we did before, we may assume without loss of generality that f is non-negative, bounded, has compact support, and that $\|f\|_{p(\cdot)} = 1$. Then

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} f(x)^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_{\infty,p(\cdot)})} \leq 1.$$

Decompose f as $f_1 + f_2 + f_3$, where

$$\begin{aligned} f_1 &= f\chi_{\{x : f(x) > 1\}}, \\ f_2 &= f\chi_{\{x \in \mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)} : f(x) \leq 1\}}, \\ f_3 &= f\chi_{\{x \in \Omega_{\infty,p(\cdot)} : f(x) \leq 1\}}. \end{aligned}$$

(Note that $f_3 \neq 0$ only if $\alpha = 0$.) Then $\operatorname{supp}(f_1) \subset \mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}$ (up to a set of measure zero), and $\rho_{p(\cdot)}(f_i) \leq \|f_i\|_{p(\cdot)} \leq 1$. We will show that there exist constants $\lambda_i = \lambda_i(n, p(\cdot)) > 0$ such that

$$\rho_{q(\cdot)}(M_\alpha f_i/\lambda_i) \leq 1.$$

In each case, we will write the constant λ_i as a product of constants needed at different stages of the proof.

The estimate for f_1**.** Let $\lambda_1^{-1} = \alpha_1\beta_1\gamma_1$. Then

$$\rho_{q(\cdot)}(\alpha_1\beta_1\gamma_1 M_\alpha f_1) = \int_{\mathbb{R}^n\setminus\Omega_{\infty,q(\cdot)}} [\alpha_1\beta_1\gamma_1 M_\alpha f_1(x)]^{q(x)}\,dx + \alpha_1\beta_1\gamma_1\|M_\alpha f_1\|_{L^\infty(\Omega_{\infty,q(\cdot)})}.$$

We will show that each term on the right is bounded by 1/2. To estimate the first, let $A = 2^{2n-\alpha}$, and define

$$\Omega_k = \{x \in \mathbb{R}^n \setminus \Omega_{\infty,q(\cdot)} : M_\alpha f_1(x) > A^k\}.$$

Since $M_\alpha f_1 \in L^\infty$, $M_\alpha f_1(x) < \infty$ a.e., and so $\mathbb{R}^n \setminus \Omega_{\infty,q(\cdot)} = \bigcup_k \Omega_k \setminus \Omega_{k+1}$ (up to a set of measure 0). By Lemma 2.5 there exist pairwise disjoint cubes $\{Q_j^k\}$ such that

$$\Omega_k \subset \bigcup_j 3Q_j^k \quad \text{and} \quad |Q_j^k|^{\alpha/n} \fint_{Q_j^k} f_1(x)\,dx > A^{k-1},$$

and we can form sets E_j^k that are pairwise disjoint for all j and k and and such that $\Omega_k \setminus \Omega_{k+1} = \bigcup_j E_j^k$. Now let $\alpha_1 = A^{-2}3^{\alpha-n}$ and estimate as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus\Omega_{\infty,q(\cdot)}} [\alpha_1\beta_1\gamma_1 M_\alpha f_1(x)]^{q(x)}\,dx &= \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} [\alpha_1\beta_1\gamma_1 M_\alpha f_1(x)]^{q(x)}\,dx \\
&\le \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} [\alpha_1\beta_1\gamma_1 A^{k+1}]^{q(x)}\,dx \\
&\le \sum_{k,j} \int_{E_j^k} \left(\beta_1\gamma_1 |3Q_j^k|^{\alpha/n} \fint_{3Q_j^k} f_1(y)\,dy\right)^{q(x)} dx. \qquad (4.1)
\end{aligned}$$

To estimate the integral in the last sum, we apply Lemma 2.7 with exponents $p_{jk} = p_-(3Q_j^k)$ and $q_{jk} = q_-(3Q_j^k)$. (Since $E_j^k \subset \Omega_k \subset \mathbb{R}^n \setminus \Omega_{\infty,q(\cdot)}$, both of these exponents are finite.) This yields

$$|3Q_j^k|^{\alpha/n} \fint_{3Q_j^k} f_1(y)\,dy \le \left(\int_{3Q_j^k} f_1(y)^{p_{jk}}\,dy\right)^{\frac{1}{p_{jk}}-\frac{1}{q_{jk}}} \left(\fint_{3Q_j^k} f_1(y)\,dy\right)^{p_{jk}/q_{jk}}$$

Since $f_1 = 0$ or $f_1 \ge 1$ pointwise and $\operatorname{supp}(f_1) \subset \mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}$,

$$\int_{3Q_j^k} f_1(y)^{p_{jk}}\,dy \le \int_{\mathbb{R}^n\setminus\Omega_{\infty,p(\cdot)}} f_1(y)^{p(y)}\,dy \le \rho_{p(\cdot)}(f_1) \le 1.$$

Therefore,

$$\sum_{k,j}\int_{E_j^k}\left(\beta_1\gamma_1|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_1(y)\,dy\right)^{q(x)} dx$$
$$\leq \sum_{k,j}\int_{E_j^k}\left(\beta_1\gamma_1\left(\fint_{3Q_j^k} f_1(y)\,dy\right)^{p_{jk}/q_{jk}}\right)^{q(x)} dx$$
$$\leq \sum_{k,j}\int_{E_j^k}\left(\beta_1\gamma_1\left(\fint_{3Q_j^k} f_1(y)^{p_{jk}/p_-}\,dy\right)^{p_-/q_{jk}}\right)^{q(x)} dx.$$

Define the exponent function $r(\cdot) = 1/q(\cdot)$. Then $r(\cdot) \in LH_0$, $r_+ \leq 1$, and $r_+(3Q_j^k) = 1/q_{jk}$. Therefore, by Lemma 2.3, we can choose $\beta_1 < 1$ so that

$$\beta_1|3Q_j^k|^{-p_-/q_{jk}} \leq |3Q_j^k|^{-p_-/q(x)}.$$

Further, arguing as before,

$$\int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy \leq \int_{3Q_j^k} f_1(y)^{p(y)}\,dy \leq 1.$$

Therefore, since $x \in E_j^k \subset 3Q_j^k$, $q(x) \geq q_{jk}$, and assuming for the moment that $\gamma_1 < 1$,

$$\sum_{k,j}\int_{E_j^k}\left(\beta_1\gamma_1\left(\fint_{3Q_j^k} f_1(y)^{p_{jk}/p_-}\,dy\right)^{p_-/q_{jk}}\right)^{q(x)} dx$$
$$\leq \sum_{k,j}\int_{E_j^k}|3Q_j^k|^{-p_-}\left(\gamma_1\int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy\right)^{q(x)p_-/q_{jk}} dx$$
$$\leq \sum_{k,j}\int_{E_j^k}|3Q_j^k|^{-p_-}\left(\gamma_1\int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy\right)^{p_-} dx$$
$$\leq \sum_{k,j}\int_{E_j^k}\gamma_1^{p_-}M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx$$
$$\leq \int_{\mathbb{R}^n}\gamma_1^{p_-}M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx.$$

Since $p_- > 1$, M is bounded on L^{p_-}, so we can choose $\gamma_1 < 1$ such that

$$\int_{\mathbb{R}^n}\gamma_1^{p_-}M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx \leq \frac{1}{2}\int_{\mathbb{R}^n} f_1(y)^{p(y)}\,dy \leq \frac{1}{2}.$$

We will now show that $\alpha_1\beta_1\gamma_1\|M_\alpha f_1\|_{L^\infty(\Omega_{\infty,q(\cdot)})} \leq 1/2$. Since $\alpha_1\beta_1 \leq 1/4$, it will suffice to show (after possibly taking γ_1 smaller than the value chosen above) that

$$\gamma_1\|M_\alpha f_1\|_{L^\infty(\Omega_{\infty,q(\cdot)})} \leq 2. \tag{4.2}$$

Fix $x \in \Omega_{\infty,q(\cdot)}$. Since $\operatorname{supp}(f_1) \subset \mathbb{R}^n \setminus \Omega_{\infty,q(\cdot)}$, when computing $M_\alpha f_1(x)$ we can restrict ourselves to cubes $Q \ni x$ such that $|Q \cap \Omega \setminus \Omega_{\infty,q(\cdot)}| > 0$. In particular, there exists such a cube that satisfies

$$M_\alpha f_1(x) \leq 2|Q|^{\alpha/n} \fint_Q f_1(y)\,dy.$$

Fix r, $q_-(Q) < r < \infty$; then by the continuity of $1/q(\cdot)$ there exists $x_r \in Q \setminus \Omega_{\infty,q(\cdot)}$ such that $q(x_r) = r$. If we now repeat the argument above, beginning with the estimate of the integral in (4.1) and replacing p_- with 1, we see that for $\gamma_1 > 0$ sufficiently small (but not depending on our choice of r),

$$\left(\gamma_1|Q|^{\alpha/n} \fint_Q f_1(y)\,dy\right)^{q(x_r)} \leq \fint_Q f_1(y)^{p(y)}\,dy \leq \frac{1}{|Q|}.$$

Therefore, we have that

$$\gamma_1|Q|^{\alpha/n} \fint_Q f_1(y)\,dy \cdot |Q|^{1/r} \leq 1.$$

Since this is true for all r large, we can take the limit as $r \to \infty$ to get

$$\gamma_1 M_\alpha f_1(x) \leq 2\gamma_1|Q|^{\alpha/n} \fint_Q f_1(y)\,dy \leq 2.$$

Since this estimate holds for almost all x, we have proved inequality (4.2). Thus we have proved that $\rho_{q(\cdot)}(\alpha_1\beta_1\gamma_1 M_\alpha f_1) \leq 1$.

The estimate for f_2. Let $\lambda_2^{-1} = \alpha_2\beta_2\gamma_2\delta_2$. Then

$$\begin{aligned}\rho_{q(\cdot)}&(\alpha_2\beta_2\gamma_2\delta_2 M_\alpha f_2)\\ &= \int_{\mathbb{R}^n\setminus\Omega_{\infty,q(\cdot)}} [\alpha_2\beta_2\gamma_2\delta_2 M_\alpha f_2(x)]^{q(x)}\,dx + \alpha_2\beta_2\gamma_2\delta_2\|M_\alpha f_2\|_{L^\infty(\Omega_{\infty,q(\cdot)})}.\end{aligned}$$

We will again show that each term is bounded by $1/2$. The second is very easy to estimate. Since $M_\alpha : L^{n/\alpha} \to L^\infty$ with constant 1, and since $f_2 \leq 1$,

$$\begin{aligned}\alpha_2\beta_2\gamma_2\delta_2\|M_\alpha f_2\|_{L^\infty(\Omega_{\infty,q(\cdot)})} &\leq \alpha_2\beta_2\gamma_2\delta_2 \left(\int_{\mathbb{R}^n} f_2(y)^{n/\alpha}\,dy\right)^{\alpha/n}\\ &\leq \alpha_2\beta_2\gamma_2\delta_2 \left(\int_{\mathbb{R}^n} f_2(y)^{p(y)}\,dy\right)^{\alpha/n} \leq \alpha_2\beta_2\gamma_2\delta_2[\rho_{p(\cdot)}(f_2)]^{\alpha/n} \leq \alpha_2\beta_2\gamma_2\delta_2.\end{aligned}$$

As we will see below, $\alpha_2\beta_2\gamma_2\delta_2 \leq 1/2$.

To estimate the first term, we form the sets Ω_k, Q_j^k and E_j^k as before using Lemma 2.5. If we set $\alpha_2 = A^{-2}3^{\alpha-n}$, and argue as we did for f_1, we get

$$\int_{\mathbb{R}^n\setminus\Omega_{\infty,q(\cdot)}} [\alpha_2\beta_2\gamma_2\delta_2 M_\alpha f_2(x)]^{q(x)}\,dx \leq \sum_{k,j}\int_{E_j^k}\left(\beta_2\gamma_2\delta_2|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_2(y)\,dy\right)^{q(x)} dx.$$

At this point we consider two cases: $q(\infty)=\infty$ and $q(\infty)<\infty$. For both cases, we make use of the fact that

$$F = |3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_2(y)\,dy \leq 1;$$

this is proved exactly as we did in Section 3, inequality (3.2).

The first case is very easy. In this case, since $1/q(\cdot)\in LH_\infty$,

$$q(x) \geq C_\infty^{-1}\log(e+|x|),$$

and so we have that (since the sets E_j^k are pairwise disjoint)

$$\sum_{k,j}\int_{E_j^k}\left(\beta_2\gamma_2\delta_2|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} f_2(y)\,dy\right)^{q(x)} dx \leq \int_{\mathbb{R}^n}(\beta_2\gamma_2\delta_2)^{C_\infty^{-1}\log(e+|x|)}\,dx \leq 1/2,$$

where the last inequality holds if we fix $\beta_2\gamma_2\delta_2>0$ sufficiently close to 0.

We now consider the more difficult case when $q(\infty)<\infty$. Define $g_2(y)=f_2(y)^{p(y)}$ if $y\in\operatorname{supp}(f_2)\subset\mathbb{R}^n\setminus\Omega_{\infty,p(\cdot)}$, and set it equal to 0 elsewhere. Thus we have to estimate

$$\sum_{k,j}\int_{E_j^k}\left(\beta_2\gamma_2\delta_2|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} g_2(y)^{1/p(y)}\,dy\right)^{q(x)} dx.$$

We first estimate the integral by applying Lemma 2.7 with exponents $p(\infty)$ and $q(\infty)$:

$$|3Q_j^k|^{\alpha/n}\fint_{3Q_j^k} g_2(y)^{1/p(y)}\,dy \leq \left(\int_{3Q_j^k} g_2(y)^{p(\infty)/p(y)}\,dy\right)^{\frac{1}{p(\infty)}-\frac{1}{q(\infty)}}\left(\fint_{3Q_j^k} g_2(y)^{1/p(y)}\,dy\right)^{p(\infty)/q(\infty)}.$$

Since $g_2(y)^{p(\infty)}\leq 1$ and $1/p(\cdot)\in LH_\infty$, we can apply Lemma 2.4 to conclude

$$\int_{3Q_j^k} g_2(y)^{p(\infty)/p(y)}\,dy \leq C\int_{3Q_j^k} g_2(y)\,dy + C\int_{3Q_j^k} R(y)^{1/p(\infty)}\,dy,$$

where $R(y) = (e+|x|)^{-N}$, and N is chosen so that the second integral converges and in fact so that

$$\int_{\mathbb{R}^n} R(y)^{1/p(\infty)}\,dy \le \int_{\mathbb{R}^n} R(y)^{1/q(\infty)}\,dy \le 1.$$

(The reason for this choice will be made clear below.) Since we also have that

$$\int_{3Q_j^k} g_2(y)\,dy \le \int_{\mathbb{R}^n} f_2(y)^{p(y)}\,dy \le 1,$$

we can choose $\beta_2 > 0$ so that

$$\sum_{k,j} \int_{E_j^k} \left(\beta_2\gamma_2\delta_2 |3Q_j^k|^{\alpha/n} \fint_{3Q_j^k} g_2(y)^{1/p(y)}\,dy \right)^{q(x)} dx \\ \le \sum_{k,j} \int_{E_j^k} \gamma_2\delta_2 \left[\left(\fint_{3Q_j^k} g_2(y)^{1/p(y)}\,dy \right)^{p(\infty)} \right]^{q(x)/q(\infty)} dx.$$

Since the quantity in square brackets is less than 1, and since $1/q(\cdot) \in LH_\infty$, we can again apply Lemma 2.4 to conclude that we can choose $\gamma_2 > 0$ such that

$$\sum_{k,j} \int_{E_j^k} \gamma_2\delta_2 \left[\left(\fint_{3Q_j^k} g_2(y)^{1/p(y)}\,dy \right)^{p(\infty)} \right]^{q(x)/q(\infty)} dx \\ \le \sum_{k,j} \int_{E_j^k} \delta_2 \left(\fint_{3Q_j^k} g_2(y)^{1/p(y)}\,dy \right)^{p(\infty)} dx + \frac{1}{6} \int_{\mathbb{R}^n} R(x)^{1/q(\infty)}\,dx \\ \le \int_{\mathbb{R}^n} \delta_2 M(g_2(\cdot)^{1/p(\cdot)})(x)^{p(\infty)}\,dx + \frac{1}{6}.$$

The maximal operator is bounded on $L^{p(\infty)}$ since $p(\infty) \ge p_- > 1$, and so we can apply Lemma 2.4 a third time to conclude that

$$\int_{\mathbb{R}^n} M(g_2(\cdot)^{1/p(\cdot)})(x)^{p(\infty)}\,dx \le C \int_{\mathbb{R}^n} g_2(x)^{p(\infty)/p(x)}\,dx \\ \le C \int_{\mathbb{R}^n} g_2(x)\,dx + C \int_{\mathbb{R}^n} R(x)^{1/p(\infty)}\,dx \le C.$$

Therefore, we can choose $\delta_2 > 0$ so that

$$\int_{\mathbb{R}^n} \delta_2 M(g_2(\cdot)^{1/p(\cdot)})(x)^{p(\infty)}\,dx + \frac{1}{6} \le \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

This completes the estimate for f_2.

The estimate for f_3. Recall that by the definition of f_3 we only need this estimate when $\alpha = 0$, so $p(\cdot) = q(\cdot)$. Let $\lambda^{-1} = \alpha_3 \leq 1/2$. Then

$$\rho_{p(\cdot)}(\alpha_3 M f_3) = \int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} [\alpha_3 M f_3(x)]^{p(x)}\, dx + \alpha_3 \|M f_3\|_{L^\infty(\Omega_{\infty,p(\cdot)})}.$$

We will show that each term is less that $1/2$. To estimate the second, since M is bounded on L^∞ with constant 1,

$$\alpha_3 \|M f_3\|_{L^\infty(\Omega_{\infty,p(\cdot)})} \leq \alpha_3 \|f_3\|_{L^\infty} \leq \alpha_3 \leq \frac{1}{2}.$$

To estimate the first term, we consider two cases: $p(\infty) = \infty$ and $p(\infty) < \infty$. In the first case we argue exactly as we did before in the estimate for f_2. Since $1/p(\cdot) \in LH_\infty$, $p(x) \geq C_\infty^{-1} \log(e + |x|)$. Since $f_3 \leq 1$, $M f_3 \leq 1$. Therefore, for α_3 sufficiently close to 0,

$$\int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} [\alpha_3 M f_3(x)]^{p(x)}\, dx \leq \int_{\mathbb{R}^n} \alpha_3^{C_\infty^{-1} \log(e+|x|)}\, dx \leq \frac{1}{2}.$$

Now suppose $p(\infty) < \infty$. Then, since $M f_3 \leq 1$, by Lemma 2.4,

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} [\alpha_3 M f_3(x)]^{p(x)}\, dx &\leq \alpha_3 \int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} M f_3(x)^{p(\infty)p(x)/p(\infty)}\, dx \\
&\leq C\alpha_3 \int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} M f_3(x)^{p(\infty)}\, dx + C\alpha_3 \int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} R(x)^{1/p(\infty)}\, dx,
\end{aligned}$$

where $R(x) = (e + |x|)^{-N}$, where N so large that the last integral is less than 1. Since $p(\infty) \geq p_- > 1$, M is bounded on $L^{p(\infty)}$, and since $f_3 \leq 1$ we can again apply Lemma 2.4 (with the same function R) to conclude that

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} M f_3(x)^{p(\infty)}\, dx &\leq \int_{\mathbb{R}^n} f_3(x)^{p(\infty)}\, dx \\
&\leq C \int_{\mathbb{R}^n} f_3(x)^{p(x)}\, dx + C \int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} R(x)^{1/p(\infty)}\, dx \leq C.
\end{aligned}$$

Combining these estimates, we see that we can choose $\alpha_3 > 0$ such that

$$\int_{\mathbb{R}^n \setminus \Omega_{\infty,p(\cdot)}} [\alpha_3 M f_3(x)]^{p(x)}\, dx \leq \frac{1}{2}.$$

5. Proof of Theorem 1.6

The proof is nearly identical to the proof of Theorem 1.2 and we sketch the details. We begin by making the same reductions as before, and writing $f = f_1 + f_2 + f_3$. Then for fixed $t > 0$,

$$\{x \in \mathbb{R}^n : M_\alpha f(x) > t\} \subset \bigcup_i \{x \in \mathbb{R}^n : Mf_i(x) > t/3\} = \bigcup_i \Omega_i$$

Therefore, it will suffice to show that for each i, $t\|\chi_{\Omega_i}\|_{q(\cdot)} \leq C$, and in turn it will suffice to show that for some $\alpha_i > 0$,

$$\rho_{q(\cdot)}(\alpha_i t \chi_{\Omega_i}) = \int_{\Omega_i \setminus \Omega_{\infty,q(\cdot)}} [\alpha_i t]^{q(x)}\, dx + \alpha_i t \|\chi_{\Omega_i}\|_{L^\infty(\Omega_{\infty,q(\cdot)})} \leq 1.$$

As before, for each i we will show that each term on the right is bounded by $1/2$ for suitable choice of α_i.

The estimates for the second term are immediate. For all i,

$$\alpha_i t \|\chi_{\Omega_i}\|_{L^\infty(\Omega_{\infty,q(\cdot)})} \leq \alpha_i \|M_\alpha f_i\|_{L^\infty(\Omega_{\infty,q(\cdot)})},$$

and the bounds on the right-hand side given above did not depend on the fact that $p_- > 1$. Since the other hypotheses hold, the same proofs yield the desired estimates.

To estimate

$$\int_{\Omega_i \setminus \Omega_{\infty,q(\cdot)}} [\alpha_i t]^{q(x)}\, dx$$

we have to avoid using the Hardy-Littlewood maximal operator, since our hypotheses no longer guarantee that it is bounded. We apply Lemma 2.5 to find disjoint dyadic cubes $\{Q_j\}$ such that

$$\Omega_i \subset \bigcup_j 3Q_j \quad \text{and} \quad |Q_j|^{\alpha/n} \fint_{Q_j} f_i(y)\, dy > 2^{\alpha - 2n} \frac{t}{3}.$$

Then we can find disjoint sets E_j such that $E_j \subset 3Q_j$ and $\Omega_i \setminus \Omega_{\infty,q(\cdot)} = \bigcup_j E_j$.

Now if $i = 1$, we argue as we did in the estimate of f_1 in the previous section, replacing $3Q_j^k$ by Q_j^k and using the fact that $p_- = 1$ to get

$$\int_{\Omega_1 \setminus \Omega_{\infty,q(\cdot)}} [\alpha_1 t]^{q(x)}\, dx \leq \sum_j \int_{E_j} \gamma_1 \fint_{Q_j} f_1(y)^{p(y)}\, dy\, dx.$$

Since $E_j \subset 3Q_j$ and the cubes Q_j are disjoint, we can choose $\gamma_1 > 0$ so that

$$\sum_j \int_{E_j} \gamma_1 \fint_{Q_j} f_1(y)^{p(y)}\, dy\, dx \leq \sum_j \frac{1}{2} \int_{Q_j} f_1(y)^{p(y)}\, dy \leq \frac{1}{2} \int_{\mathbb{R}^n} f_1(y)^{p(y)}\, dy \leq \frac{1}{2}.$$

If $i = 2$, the estimate when $q(\infty) = \infty$ is exactly the same as in the estimate for f_2 above. When $q(\infty) < \infty$, then we can argue as before to get that

$$\int_{\Omega_2 \setminus \Omega_{\infty,q(\cdot)}} [\alpha_2 t]^{q(x)}\, dx \leq \sum_j \int_{E_j} \delta_2 \left(\fint_{Q_j^k} g_2(y)^{1/p(y)}\, dy \right)^{p(\infty)} dx + \frac{1}{6}$$
$$\leq \sum_j \int_{E_j} \delta_2 \fint_{Q_j^k} g_2(y)^{p(\infty)/p(y)}\, dy\, dx + \frac{1}{6} \leq \int_{\mathbb{R}^n} \delta_2 g_2(y)^{p(\infty)/p(y)}\, dy + \frac{1}{6}.$$

At this point we can repeat the end of the proof and choose $\delta_2 > 0$ such that right-hand side is bounded by $1/2$.

Finally, if $i = 3$, we may assume as before that $\alpha = 0$. In this case, since $f_3 \leq 1$, $Mf_3 \leq 1$, so Ω_3 is non-empty only if $t < 3$. If $p(\infty) = \infty$, then

$$\int_{\Omega_3 \setminus \Omega_{\infty,p(\cdot)}} [\alpha_3 t]^{p(x)}\, dx \leq \int_{\Omega_3 \setminus \Omega_{\infty,p(\cdot)}} [3\alpha_3]^{p(x)}\, dx,$$

and we can repeat the argument for the estimate of f_3 given above.

If $p(\infty) < \infty$, let $\alpha_3 = \beta_3/3$. Then arguing as before, by Lemma 2.4,

$$\int_{\Omega_3 \setminus \Omega_{\infty,p(\cdot)}} [\alpha_3 t]^{p(x)}\, dx \leq \beta_3 \int_{\Omega_3 \setminus \Omega_{\infty,p(\cdot)}} [t/3]^{p(x)}\, dx$$
$$\leq C\beta_3 \int_{\Omega_3} [t/3]^{p(\infty)}\, dx + C\beta_3 \int_{\Omega_3} R(x)^{1/p(\infty)}\, dx,$$

where the last integral is at most 1. Since $p(\infty) \geq 1$, by the weak $(p(\infty), p(\infty))$ inequality for the maximal operator and again by Lemma 2.4,

$$\int_{\Omega_3} [t/3]^{p(\infty)}\, dx \leq C \int_{\mathbb{R}^n} f_3(x)^{p(\infty)}\, dx \leq C \int_{\mathbb{R}^n} f_3(x)^{p(x)}\, dx + C \int_{\mathbb{R}^n} R(x)^{1/p(\infty)}\, dx \leq C.$$

Therefore, we can choose $\alpha_3 = \beta_3/3 > 0$ such that

$$\int_{\Omega_3 \setminus \Omega_{\infty,p(\cdot)}} [\alpha_3 t]^{p(x)}\, dx \leq \frac{1}{2}.$$

References

[1] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[2] C. Capone, D. Cruz-Uribe, and A. Fiorenza. The fractional maximal operator and fractional integrals on variable L^p spaces. *Revista Math. Iberoamericana*, To appear.

[3] D. Cruz-Uribe. New proofs of two-weight norm inequalities for the maximal operator. *Georgian Math. J.*, 7(1):33–42, 2000.

[4] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. The maximal function on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, 28(1):223–238, 2003. See errata [5].

[5] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. Corrections to: "The maximal function on variable L^p spaces" [Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 1, 223–238;]. *Ann. Acad. Sci. Fenn. Math.*, 29(1):247–249, 2004.

[6] L. Diening. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.*, 7(2):245–253, 2004.

[7] L. Diening. Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, 129(8):657–700, 2005.

[8] L. Diening. Habilitation, Universität Freiburg, 2007.

[9] L. Diening, P. Harjulehto, P. Hästo, Y. Mizuta, and T. Shimomura. Maximal functions in variable exponent spaces: limiting cases of the exponent. *Preprint*, 2007.

[10] L. Diening, P. Hästö, and A. Nekvinda. Open problems in variable exponent Lebesgue and Sobolev spaces. In *FSDONA04 Proceedings (Drabek and Rakosnik (eds.); Milovy, Czech Republic*, pages 38–58. Academy of Sciences of the Czech Republic, Prague, 2005.

[11] J. Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[12] X. Fan and D. Zhao. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 263(2):424–446, 2001.

[13] J. García-Cuerva and J.L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985.

[14] V. Kokilashvili and S. Samko. On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent. *Z. Anal. Anwendungen*, 22(4):899–910, 2003.

[15] O. Kováčik and J. Rákosník. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(116)(4):592–618, 1991.

[16] A.K. Lerner. Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces. *Math. Z.*, 251(3):509–521, 2005.

[17] B. Muckenhoupt and R. Wheeden. Weighted norm inequalities for fractional integrals. *Trans. Amer. Math. Soc.*, 192:261–274, 1974.

[18] J. Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[19] A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$. *Math. Inequal. Appl.*, 7(2):255–265, 2004.

[20] A. Nekvinda. A note on maximal operator on $\ell^{\{p_n\}}$ and $L^{p(x)}(\mathbb{R})$. *J. Funct. Spaces Appl.*, 5(1):49–88, 2007.

[21] L. Pick and M. Růžička. An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded. *Expo. Math.*, 19(4):369–371, 2001.

[22] S. Samko. On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integral Transforms Spec. Funct.*, 16(5-6):461–482, 2005.

Dept. of Mathematics, Trinity College, Hartford, CT 06106-3100, USA

E-mail address: david.cruzuribe@trincoll.edu

Section of Applied Mathematics, University of Freiburg, Eckerstr. 1, 79104 Freiburg, Germany

E-mail address: diening@mathematik.uni-freiburg.de

Dipartimento di Costruzioni e Metodi Matematici in Architettura, Università di Napoli, Via Monteoliveto, 3, I-80134 Napoli, Italy, and Istituto per le Applicazioni del Calcolo "Mauro Picone", sezione di Napoli, Consiglio Nazionale delle Ricerche, via Pietro Castellino, 111, I-80131 Napoli, Italy

E-mail address: fiorenza@unina.it