Existence of strong solutions for incompressible fluids with shear dependent viscosities

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Abstract

Certain rheological behavior of non-Newtonian fluids in engineering sciences is often modeled by a power law ansatz with $p \in (1, 2]$. In the present paper the local in time existence of strong solutions is studied. The main result includes also the degenerate case ($\delta = 0$) of the extra stress tensor and thus improves previous results of [L. Diening and M. Růžička, *J. Math. Fluid Mech.*, 7 (2005), pp. 413-450].

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1 Introduction

We study the existence of strong solutions for the system describing the motion of a homogeneous, incompressible fluid with shear dependent viscosity, which reads

$$\begin{aligned} \rho\,\mathbf{u}_t - \operatorname{div}\mathbf{S}(\mathbf{Du}) + \rho\,[\nabla\mathbf{u}]\mathbf{u} + \nabla\pi &= \rho\,\mathbf{f} && \text{in } I\times\Omega,\\ \operatorname{div}\mathbf{u} &= 0 && \text{in } I\times\Omega, \qquad (\mathrm{NS}_p)\\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega, \end{aligned}$$

where the vector field $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity, $\mathbf{S}$ is the extra stress tensor, the scalar π is the kinematic pressure, the vector $\mathbf{f} = (f_1, f_2, f_3)$ is the external body force, ρ the constant density, and $\mathbf{u}_0$ is the initial velocity. Here we used the notation $([\nabla\mathbf{u}]\mathbf{u})_i = \sum_{j=1}^{3} u_j \partial_j u_i$, $i = 1, 2, 3$, for the convective term. We divide the equation (NS_p) by the constant density ρ and relabel $\mathbf{S}/\rho$ and π/ρ again as $\mathbf{S}$ and π, respectively. Thus we consider from now on (NS_p) always with the convention that $\rho = 1$. The term $\mathbf{Du} := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$ denotes the

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symmetric part of the gradient $\nabla \mathbf{u}$. The problem (NS_p) will be considered in $\Omega = (0, 2\pi)^3 \subset \mathbb{R}^3$ and we endow the problem with space periodic boundary conditions. The latter assumption simplifies the problem, but allows us to concentrate on the difficulties which arise from the structure of the extra stress tensor. As usual $I = [0, T]$ denotes some non-vanishing time interval.

Standard examples of power-law stress tensors for $p \in (1, \infty)$ are

$$\mathbf{S}(\mathbf{D}\mathbf{u}) = \mu\,(\delta + |\mathbf{D}\mathbf{u}|^2)^{\frac{p-2}{2}}\mathbf{D}\mathbf{u} \quad \text{or} \quad \mathbf{S}(\mathbf{D}\mathbf{u}) = \mu\,(\delta + |\mathbf{D}\mathbf{u}|)^{p-2}\mathbf{D}\mathbf{u}, \tag{1.1}$$

where $\mu > 0$ and $\delta \geq 0$ are given constants. These models belong to the class of power-law ansatz to model certain non-Newtonian behavior of fluid flows, and they are frequently used in engineering literature. A classical reference (with a detailed discussion of power-law models including also early models) is the book by Bird, Armstrong, and Hassager [12]. We also refer to Málek, Rajagopal, and Růžička [39] and Málek and Rajagopal [38] for a discussion of such models. Let us mention that most real fluids that can be modeled by a constitutive law of type (1.1) are shear thinning fluids, which corresponds to a "small" shear exponent p, i.e., $p \in (1, 2]$. However there are also shear thickening fluids, which have a shear exponent $p \in [2, \infty)$. Moreover, the case $p = 3$ is very interesting also for the modeling of turbulent flows and known in applied literature as the Smagorinsky model [48]. The mathematical analysis of the problem (NS_p), (1.1) started with the work of Ladyžhenskaya [32], [33], [34]. After the papers by Nečas *et. al.* [36], [9] the problem has been studied intensively and various existence and regularity properties have been proved in the last years. The literature on this subject is very large and we focus on the papers that are mostly connected with the results we are going to prove. In particular, for the steady problem, there are several results proving existence of weak solutions [23], [17], interior regularity [1], [24] and very recently regularity up-to-the boundary for the Dirichlet problem [44], [47], [7], [8], [10]. Concerning the time-evolution Dirichlet problem in a three–dimensional domain we have recent advances on the existence of weak solutions in [49] for $p > \frac{8}{5}$ and in [22] for $p > \frac{6}{5}$. For this paper the most relevant results of (local in time) existence of strong solutions in a three–dimensional cube with space periodic boundary conditions are those in [20], for $p \in \left(\frac{7}{5}, 2\right]$. There are many other papers dealing with "strong solutions" for time-dependent problems and we refer for instance to [35], [38], [2], [3], [13], [25], [28], [29], [30], [37], [39], [43], [44], [45]. Note that in [13] the existence of local in time strong solutions for the Dirichlet problem is proved for $p \geq 1$. However, this result depends crucially on the fact that $\delta > 0$ and breaks down for $\delta = 0$.

Our aim is to prove (local in time) existence of strong solutions in the case of shear thinning fluids, i.e., in the case $p \in (1, 2]$ and to extend the results in [20] to the degenerate case $\delta = 0$. Our interest in the existence of regular solutions in the time evolution problem is also motivated by the fact that error estimates needed for the analysis of numerical methods require improved smoothness. In this respect weak solutions are not enough to obtain suitable estimates. We note that results of existence proved here are employed in [11] to improve error estimates for Euler schemes previously studied in [41], [18], [19], [14].

We also note that the stability of the numerical results for asymptotically small δ in (1.1) is a problem of certain relevance. This was the hint to try to understand whether the degenerate case $\delta = 0$ (which corresponds to a p-Laplacian, but with a divergence-free constraint and the pressure) can be treated in the same way. From the physical point of view the fact that the viscosity can grow without limits is debatable, but from the pure mathematical point of view, it is interesting that the limit case can be covered by an approximation technique.

In particular, in our main result Theorem 5.1 we focus on the "stability" of existence results in terms of $\delta \to 0^+$. Our main task is showing local existence of strong solutions, independently of the value of $\delta \in (0, \delta_0]$. As by product, we shall also show that the (degenerate) limit problem has locally a smooth solution, which shares several good properties of smoothness with the solution of the non-degenerate problem. The main tools are precise a priori estimates, the notion of shifted N-functions, and a suitable approximation procedure to treat the degenerate problem.

Outline of the paper. The paper is organized as follows: In the section 2 we fix the notation, we introduce our assumptions on the extra stress tensor, and we recall basic properties of related Orlicz functions. In section 3 we collect some features of the extra stress tensor and related quantities which naturally occur in the investigation of the problem (NS_p). These results are valid for all $p \in (1, \infty)$. Then, in section 4 we restrict ourselves to the case $p \in (1, 2]$ and prove several estimates (specific of the shear thinning case) necessary for the main theorem. In section 5 we prove the main result, namely the existence of local in time strong solutions for the problem (NS_p) for $p \in \left(\frac{7}{5}, 2\right]$ and $\delta \geq 0$ (cf. Theorem 5.1). Thus we extend previous results to the degenerate case. Finally, in section 6 we study steady problems.

2 Notations and assumptions on the extra stress tensor S

Let us first introduce the notation which will be used in the sequel. We shall use the customary Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$ and we do not distinguish between scalar, vector, or tensor function spaces. We shall denote by $\|\,.\,\|_p$ the norm in $L^p(\Omega)$ and by $\|\,.\,\|_{k,p}$ the norm in $W^{k,p}(\Omega)$. In this paper we are considering the space periodic case*, i.e., $\Omega = (0, 2\pi)^d$, $d \geq 2$, and each function f we consider will satisfy $f(x + 2\pi\, e_i) = f(x)$, $i = 1, \ldots, d$, where $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$. Often we will also require that the functions have vanishing mean value, i.e., $\int_\Omega f(x)\, dx = 0$. This is a standard request in order to have Poincaré's inequality. We define $\mathcal{V}$ as the space of vector-valued functions on Ω that are smooth, divergence-free, and space periodic with

*However, all results in sections 2, 3 and 4 also hold for sufficiently smooth domains $\Omega \subset \mathbb{R}^d$.

zero mean value and set

$$W^{1,p}_{\mathrm{div}}(\Omega) := \big\{\text{closure of } \mathcal{V} \text{ in } W^{1,p}(\Omega)\big\}.$$

Since we deal with a time dependent problem, we shall make use of the spaces $L^p(I;X), 1 \le p \le \infty$, where $(X, \|\,.\,\|_X)$ is a Banach space. The subscript "$_t$" denotes differentiation with respect to time. We write $f \simeq g$ if there exist positive constants c_0 and c_1 such that

$$c_0 f \le g \le c_1 f.$$

Let us now discuss the structure of the extra stress tensor $\mathbf{S}$ and motivate our assumptions for it. Due to the principle of objectivity the extra stress tensor $\mathbf{S}$ depends on the velocity gradient $\nabla \mathbf{u}$ only through its symmetric part $\mathbf{D}\mathbf{u} := \frac{1}{2}\big(\nabla \mathbf{u} + \nabla \mathbf{u}^\top\big)$. Therefore we assume that the extra stress tensor $\mathbf{S}\colon \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, where $\mathbb{R}^{d\times d}_{\mathrm{sym}} := \big\{\mathbf{A} \in \mathbb{R}^{d\times d} \,\big|\, \mathbf{A} = \mathbf{A}^\top\big\}$ satisfies $\mathbf{S}(\mathbf{A}) = \mathbf{S}\big(\mathbf{A}^{\mathrm{sym}}\big)$ and $\mathbf{S}(\mathbf{0}) = \mathbf{0}$, where $\mathbf{A}^{\mathrm{sym}} := \frac{1}{2}\big(\mathbf{A} + \mathbf{A}^\top\big)$.

Often $\mathbf{S}$ is derived from a potential, i.e., there exists a convex function $\Phi\colon \mathbb{R}^{\ge 0} \to \mathbb{R}^{\ge 0}$ which belongs to $C^1(\mathbb{R}^{\ge 0}) \cap C^2(\mathbb{R}^{>0})$ and which satisfies $\Phi(0) = \Phi'(0) = 0$, such that for all $\mathbf{A} \in \mathbb{R}^{d\times d} \setminus \{\mathbf{0}\}$ and $i, j = 1, \dots, d$ it holds that†

$$S_{ij}(\mathbf{A}) = \partial_{ij}\big(\Phi(|\mathbf{A}^{\mathrm{sym}}|)\big) = \Phi'\big(|\mathbf{A}^{\mathrm{sym}}|\big)\frac{A^{\mathrm{sym}}_{ij}}{|\mathbf{A}^{\mathrm{sym}}|}\,. \tag{2.1}$$

This assumption is too restrictive and we are able to cover a wider class of stress tensors, as we shall see in the next subsection.

2.1 On N-functions and shear dependent fluids

In this section we recall some basic properties of N-functions and state some results which will be useful in the sequel. In particular, this abstract approach turns out to be very fruitful to treat problems with shear dependent viscosity in ad hoc function spaces, see e.g., recent results in [15], [21], [16]. In addition, note that the introduction of quasi-norms in the study of degenerate problems dates back to [4], [5].

In many cases relevant classes of stress tensors are those derived from a potential Φ with *p-structure*, or more precisely with (p,δ)*-structure*. This means that there exist $p \in (1,\infty)$, $\delta \in [0,\infty)$, and constants $\nu_0, \nu_1 > 0$ such that for all $t \in \mathbb{R}^{\ge 0}$ holds

$$\nu_0(\delta + t)^{p-2} \le \Phi''(t) \le \nu_1(\delta + t)^{p-2}\,. \tag{2.2}$$

From (2.2) and [27, Lemma 8.3] (cf. [14, Lemma 6.2], [46, Section 6]) one easily deduces that uniformly in $t \ge 0$

$$\Phi'(t) \simeq \Phi''(t)\,t\,, \tag{2.3a}$$

$$\Phi(t) \simeq \Phi'(t)\,t\,, \tag{2.3b}$$

† For functions $g\colon \mathbb{R}^{d\times d} \to \mathbb{R}$ we use the notation $\partial_{kl} g(\mathbf{A}) := \frac{\partial g(\mathbf{A})}{\partial A_{kl}}$.

where the constants in (2.3) depend only on ν_0, ν_1, and p. Note, that if $\mathbf{S}$ is derived from a potential we have for all $\mathbf{A} \in \mathbb{R}^{d\times d}\setminus\{\mathbf{0}\}$ and all $i,j,k,l = 1,\ldots,d$

$$\partial_{kl} S_{ij}(\mathbf{A}) = \frac{\Phi'(|\mathbf{A}^{\mathrm{sym}}|)}{|\mathbf{A}^{\mathrm{sym}}|}\left(\delta^{\mathrm{sym}}_{ij,kl} - \frac{A^{\mathrm{sym}}_{ij}A^{\mathrm{sym}}_{kl}}{|\mathbf{A}^{\mathrm{sym}}|^2}\right) + \Phi''(|\mathbf{A}^{\mathrm{sym}}|)\,\frac{A^{\mathrm{sym}}_{ij}A^{\mathrm{sym}}_{kl}}{|\mathbf{A}^{\mathrm{sym}}|^2}\,,$$

where $\delta^{\mathrm{sym}}_{ij,kl} := \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Using this one can conclude as in [15, Lemma 6.3], [46, Lemma 6.7, Section 8] that there are constants $\nu_2, \nu_3 > 0$, which depend only on ν_0, ν_1 and p, such that for all $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{d\times d}$ with $\mathbf{A}^{\mathrm{sym}} \neq \mathbf{0}$ and $i,j,k,l = 1,\ldots,d$ hold

$$\begin{aligned}\sum_{i,j,k,l=1}^{d} \partial_{kl} S_{ij}(\mathbf{A}) C_{ij} C_{kl} &\geq \nu_2 \big(\delta + |\mathbf{A}^{\mathrm{sym}}|\big)^{p-2} |\mathbf{C}^{\mathrm{sym}}|^2\,,\\ \big|\partial_{kl} S_{ij}(\mathbf{A})\big| &\leq \nu_3 \big(\delta + |\mathbf{A}^{\mathrm{sym}}|\big)^{p-2}.\end{aligned} \tag{2.4}$$

These two relations concerning growth and coercivity will be the main abstract hypotheses we shall need on $\mathbf{S}$, see Assumption 1.

Closely related to the extra stress tensor $\mathbf{S}$ with p-structure is the function $\mathbf{F}\colon \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ defined through

$$\mathbf{F}(\mathbf{A}) := \big(\delta + |\mathbf{A}^{\mathrm{sym}}|\big)^{\frac{p-2}{2}} \mathbf{A}^{\mathrm{sym}}\,, \tag{2.5}$$

where $\delta \geq 0$ is the same as in (2.2) and (2.4). If the dependence on δ is of relevance we write $\mathbf{F}^{\delta}(\mathbf{A})$. Moreover, there is a close relation to Orlicz spaces and N-functions (cf. [31], [40], [42], [46] for a detailed description.)

Remark 2.6. If not otherwise stated we will use the convention that in formulas relating the quantities $\mathbf{S}$ and $\mathbf{F}$ the value of δ is the same in each of the quantities and it is suppressed for shortage of notation.

Definition 2.7 (N-function)**.** *A function $\phi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is called an* N-function *(where N stands for "nice") if ϕ is continuous, convex, strictly positive for $t > 0$, and such that*

$$\lim_{t\to 0^+} \frac{\phi(t)}{t} = 0 \qquad\qquad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty.$$

Note that ϕ being convex has a right-derivative ϕ' which is right-continuous. The *complementary function* ϕ^* defined by

$$\phi^*(t) := \int_0^t (\phi')^{-1}(s)\,ds := \int_0^t \sup\{u \in \mathbb{R}^{\geq 0} | \phi'(u) \leq s\}\,ds,$$

is again an N-function. We have the following versions of Young's inequality.

Lemma 2.8 (Young's type inequalities)**.** *For all $t, u \geq 0$ there holds*

$$tu \leq \phi(t) + \phi^*(u).$$

In addition, the following inequality for derivatives is valid: for each $\delta > 0$ there exists $c_\delta > 0$, which only depends on ν_0, ν_1, and p, such that for all $t, u \geq 0$ there holds

$$t\,\phi'(u) + \phi'(t)\,u \leq \delta\,\phi(t) + c_\delta\,\phi(u). \tag{2.9}$$

Proof. The first inequality derives immediately from the equivalent definition of complementary function

$$\phi^*(u) := \sup_{t\geq 0}\big(ut - \phi(t)\big).$$

The proof of (2.9) (cf. [15]) follows by the Young's inequality and by observing that, in addition to (2.3b), one also has

$$\phi^*(\phi'(t)) \simeq \phi(t) \qquad \text{uniformly in } t \geq 0.$$

The above relation can be derived immediately from

$$\phi\left(\frac{\phi^*(t)}{t}\right) \leq \phi^*(t) \leq \phi\left(\frac{2\phi^*(t)}{t}\right), \qquad \forall\, t > 0,$$

see also [46, Lemma 5.1]. □

In this abstract setting one may also consider an important subclass of N-functions, those satisfying the Δ_2-condition.

Definition 2.10 (Δ_2-condition)**.** *A function $\phi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ satisfies the Δ_2-condition if*

$$\phi(t) \leq \phi(2t) \leq K\phi(t) \qquad \forall\, t \geq 0, \tag{2.11}$$

for some constant $K \geq 2$. The Δ_2-constant of ϕ is the smallest constant K having this property.

In the sequel we shall also consider functions satisfying the Δ_2-condition. Moreover, we shall also assume that the complementary function satisfies the Δ_2-condition, with constant K_*. Standard (relevant) examples are the functions $\phi(t) = t^p$, $\phi(t) = (\delta + t)^{p-2}t^2$, and $\phi(t) = \int_0^t (\delta + s)^{p-2} s\, ds$.

Remark 2.12. It is easy to show that inequalities (2.3) hold true with constants depending only on the Δ_2-constant of ϕ.

Definition 2.13 (Shifted N-functions)**.** *Let ϕ be an N-function. We define the family of* shifted N-functions $\{\phi_a\}_{a\geq 0}$ *by*

$$\phi_a'(t) := \phi'(a + t)\frac{t}{a + t}. \tag{2.14}$$

One can show that for all $s, t \geq 0$ with $s + t > 0$ holds

$$\begin{aligned}\phi_s(|s-t|) &\simeq \phi_s'(|s-t|)|s-t| \\ &\simeq \phi''(s+t)|s-t|^2 \\ &\simeq (\delta+s+t)^{p-2}|s-t|^2\,,\end{aligned} \tag{2.15}$$

where the constants depend only on ν_0, ν_1, and p (cf. [15, Lemma 6.6], [46, Lemma 6.3]).

We report some results and inequalities on shifted N-functions, which we shall need later. A very complete account of inequalities for these functions is given in [46], to which we shall constantly refer for all results on N-functions.

Lemma 2.16. *Let $\phi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be an N-function satisfying the Δ_2-condition with constant K and let $K' \in [K, K^2]$ denote the Δ_2-condition constant of ϕ' (which satisfies the Δ_2-condition due to [46, Lemma 5.2]). Then, for all $\mathbf{P}$, $\mathbf{Q} \in \mathbb{R}^{N\times n}$ the following inequalities hold true:*

$$\phi'_{|\mathbf{P}|}(t) \leq 2K'\phi'_{|\mathbf{Q}|}(t) + \phi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) \qquad \forall\, t \geq 0, \tag{2.17a}$$

$$\phi'_{|\mathbf{P}|}(t) \leq 2K'\phi'_{|\mathbf{Q}|}(t) + 2K'\phi'_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|) \qquad \forall\, t \geq 0. \tag{2.17b}$$

Moreover, if we assume that the complementary function ϕ^* satisfy the Δ_2-condition with constant K_*, then

$$\frac{1}{2K_*'K'}(\phi^*)'_{\phi'(a)}(u) \leq ((\phi_a)^*)'(u) \leq 2K_*(\phi^*)'_{\phi'(a)}(u) \quad \forall\, a, u \geq 0. \tag{2.18}$$

Proof. For the proofs see Lemmas 5.9-5.13 and Corollary 5.14 in [46]. □

From the above lemma we derive immediately a fundamental inequality, which will be used several times in the sequel.

Corollary 2.19. *The following relation*

$$\delta^{\frac{p}{2}} + t^{\frac{p}{2}} \simeq (\delta+t)^{\frac{p-2}{2}}t + \delta^{\frac{p}{2}} \tag{2.20}$$

holds for all $\delta, t \geq 0$ with constants depending only on p (and not on δ).

As claimed, one improvement with respect to previous results is that here it is not necessary that $\mathbf{S}$ is derived from a potential. It is sufficient that $\mathbf{S}$ *is a stress tensor with p-structure* or more precisely (p, δ)-structure. This means that $\mathbf{S}$ satisfies (2.4). In order to clearly formulate the results we introduce the function

$$\varphi(t) := \frac{1}{p}\,t^p, \tag{2.21}$$

and the corresponding shifted functions φ_δ, where $\delta \geq 0$ is the same constant as in (2.4). Note that the $\{\varphi_\delta\}_{\delta\geq 0}$ belong to $C^1(\mathbb{R}^{\geq 0}) \cap C^2(\mathbb{R}^{>0})$ and are N-functions satisfying the Δ_2-condition with Δ_2-constants independent of $\delta \geq 0$.

Moreover, we have $\varphi_\delta'(t) = (\delta + t)^{p-2} t$ and $\min\{1, p-1\}\,(\delta+t)^{p-2} \le \varphi_\delta''(t) \le \max\{1, p-1\}(\delta+t)^{p-2}$.

Now we can precisely formulate our assumption on the extra stress tensor with (p,δ)-structure.

Assumption 1 (extra stress tensor)**.** *We assume that the extra stress tensor* $\mathbf{S}\colon \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *belongs to* $C^1(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \cap C^2(\mathbb{R}^{d\times d}\setminus\{\mathbf{0}\}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ *and satisfies* $\mathbf{S}(\mathbf{A}) = \mathbf{S}\big(\mathbf{A}^{\mathrm{sym}}\big)$ *and* $\mathbf{S}(\mathbf{0}) = \mathbf{0}$*. Moreover, we assume that* $\mathbf{S}$ *has* (p,δ)*-structure, i.e., there exist* $p \in (1,\infty)$*,* $\delta \in [0,\infty)$*, and constants* $C_0, C_1 > 0$ *such that*

$$\sum_{i,j,k,l=1}^{d} \partial_{kl} S_{ij}(\mathbf{A}) C_{ij} C_{kl} \ge C_0 \big(\delta + |\mathbf{A}^{\mathrm{sym}}|\big)^{p-2} |\mathbf{C}^{\mathrm{sym}}|^2, \tag{2.22a}$$

$$\big|\partial_{kl} S_{ij}(\mathbf{A})\big| \le C_1 \big(\delta + |\mathbf{A}^{\mathrm{sym}}|\big)^{p-2} \tag{2.22b}$$

is satisfied for all $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{d\times d}$ *with* $\mathbf{A}^{\mathrm{sym}} \ne \mathbf{0}$ *and all* $i,j,k,l = 1,\dots,d$.

In terms of φ_δ (where φ has been defined in (2.21)), inequalities (2.22) defining the (p,δ)-structure can be written equivalently as

$$\sum_{i,j,k,l=1}^{d} \partial_{kl} S_{ij}(\mathbf{A}) C_{ij} C_{kl} \ge C_0\, \varphi_\delta''\big(|\mathbf{A}^{\mathrm{sym}}|\big) |\mathbf{C}^{\mathrm{sym}}|^2\,, \tag{2.23a}$$

$$\big|\partial_{kl} S_{ij}(\mathbf{A})\big| \le \frac{C_1}{p-1}\, \varphi_\delta''\big(|\mathbf{A}^{\mathrm{sym}}|\big)\,. \tag{2.23b}$$

Moreover, even if the stress tensor does not derive from a potential we can still introduce $\mathbf{F}$ (cf. (2.5)). For that we observe that if ϕ is an N-function, then we set

$$\psi'(t) := \sqrt{\phi'(t)\,t} \qquad t \ge 0,$$

and we define the *associated N-function* by

$$\psi(t) := \int_0^t \psi'(s)\,ds.$$

The properties of the N-function ψ are treated in detail in [15] and in [46, Section 6]. For a given function ϕ we denote by $\mathbf{F}$ the operator with N-potential ψ, i.e., $\mathbf{F}(\mathbf{0}) := \mathbf{0}$ and for all $\mathbf{A} \in \mathbb{R}^{d\times d}\backslash\{\mathbf{0}\}$

$$\mathbf{F}(\mathbf{A}) := \psi'(|\mathbf{A}^{\mathrm{sym}}|)\, \frac{\mathbf{A}^{\mathrm{sym}}}{|\mathbf{A}^{\mathrm{sym}}|}. \tag{2.24}$$

This is the abstract setting for the definition of $\mathbf{F}$. In order to get $\mathbf{F}$ (or more precisely $\mathbf{F}^\delta$) defined in (2.5) one has to use $\phi(t) = \varphi_\delta(t)$ in the above construction. The main result is that if $\mathbf{S}$ is a stress tensor with (p,δ)-structure then

$$\big|\mathbf{F}^\delta(\mathbf{A}) - \mathbf{F}^\delta(\mathbf{B})\big|^2 \simeq (\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}),$$

and this defines a quantity which is relevant in the study of partial regularity of fluids with shear dependent viscosity, cf. [15]. In the next sections we collect several results related to stress tensors with p-structure, by using (when necessary) the formalism of N-functions. This abstract setting will be particularly useful in section 3.1 to derive suitable estimates and continuity properties of approximate stress tensors.

3 Properties of the extra stress tensor S

In this section we collect general properties of the extra stress tensor $\mathbf{S}$ (with p-structure) and related quantities that naturally occur in the analysis of the system (NS_p). All results in this section hold for all $p \in (1,\infty)$ hence they are not specific of the shear thinning case. In addition, no restriction on the space dimension is requested in this section. The results of this section are rather standard. What is relevant is that we carefully checked that all constants appearing in the various inequalities turn out to be independent of $\delta \in (0,\infty)$. This will allow us to obtain uniform (in δ) estimates on solutions to (NS_p).

Let us start with the following crucial lemma, which shows the equivalence of several quantities which are useful in the analysis of the system (NS_p).

Lemma 3.1. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in (1,\infty)$ *and* $\delta \in [0,\infty)$*, let* $\mathbf{F}$ *be defined by* (2.5)*, and let* φ *be defined in* (2.21)*. Then for all* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ *there holds*

$$\begin{aligned}(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B}))\cdot(\mathbf{A}-\mathbf{B}) &\simeq |\mathbf{A}^{\mathrm{sym}}-\mathbf{B}^{\mathrm{sym}}|^2(\delta+|\mathbf{B}^{\mathrm{sym}}|+|\mathbf{A}^{\mathrm{sym}}|)^{p-2}\\ &\simeq \varphi_{|\mathbf{A}^{\mathrm{sym}}|}(|\mathbf{A}^{\mathrm{sym}}-\mathbf{B}^{\mathrm{sym}}|) \\ &\simeq |\mathbf{F}(\mathbf{A})-\mathbf{F}(\mathbf{B})|^2\,,\end{aligned} \tag{3.2}$$

$$|\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B})| \simeq |\mathbf{A}^{\mathrm{sym}}-\mathbf{B}^{\mathrm{sym}}|(\delta+|\mathbf{B}^{\mathrm{sym}}|+|\mathbf{A}^{\mathrm{sym}}|)^{p-2}, \tag{3.3}$$

where the constants depend only on C_0, C_1*, and* p*. In particular, the constants are independent of* $\delta \geq 0$*.*

Proof. For the proof see [14, Lemma 2.1], [15, Lemma 2.3], [46, Lemma 6.16, Section 6]. $\square$

Remark 3.4. Since in the following we will insert into $\mathbf{S}$, $\mathbf{F}$, φ_δ, and ψ_δ, $\delta \geq 0$, only symmetric tensors, we can drop in the above formulas the superscript " $^{\mathrm{sym}}$" and restrict the admitted tensors to symmetric ones.

The following lemma is a version of Young's inequality and will be used frequently in the sequel.

Lemma 3.5. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in (1,\infty)$ *and* $\delta \in [0,\infty)$*, and let* $\mathbf{F}$ *be defined by* (2.5)*. Then for each* $\varepsilon > 0$ *there exists* $c_\varepsilon(p) > 0$*, such that* *for all* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *there holds*

$$\begin{aligned}&\big(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B})\big)\cdot\big(\mathbf{A}-\mathbf{C}\big)\\ &\quad\le \varepsilon\,\big(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B})\big)\cdot\big(\mathbf{A}-\mathbf{B}\big)+c_\varepsilon\,\big(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{C})\big)\cdot\big(\mathbf{A}-\mathbf{C}\big)\end{aligned}$$

and

$$\big(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B})\big)\cdot\big(\mathbf{A}-\mathbf{C}\big)\le\varepsilon\,\big|\mathbf{F}(\mathbf{A})-\mathbf{F}(\mathbf{B})\big|^2+c_\varepsilon\,\big|\mathbf{F}(\mathbf{A})-\mathbf{F}(\mathbf{C})\big|^2.$$

Proof. Using (2.9), (2.15), (3.2), and (3.3) the result follows (cf. [6, Lemma 2.2]). □

Especially, for $\mathbf{v},\mathbf{w}_1,\mathbf{w}_2\in W^{1,p}(\Omega)$ we easily deduce from Lemma 3.5 the following useful inequality.

$$\begin{aligned}&\int_\Omega(\mathbf{S}(\mathbf{D}\mathbf{v})-\mathbf{S}(\mathbf{D}\mathbf{w}_1))\cdot(\mathbf{D}\mathbf{v}-\mathbf{D}\mathbf{w}_2)\,dx\\ &\qquad\le\varepsilon\,\|\mathbf{F}(\mathbf{D}\mathbf{v})-\mathbf{F}(\mathbf{D}\mathbf{w}_1)\|_2^2+c_\varepsilon\,\|\mathbf{F}(\mathbf{D}\mathbf{v})-\mathbf{F}(\mathbf{D}\mathbf{w}_2)\|_2^2.\end{aligned}\tag{3.6}$$

We recall the definition of two quantities that will be used extensively in the sequel (and that are common in the literature concerning (NS_p)). The terms $\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u})\|_2$ and $\|(\mathbf{F}(\mathbf{D}\mathbf{u}))_t\|_2$ are related to those coming from testing the term $-\operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{u})$ with $-\Delta\mathbf{u}$ and $\mathbf{u}_{tt}$, respectively. They are defined for $\delta>0$ through

$$\begin{aligned}\mathcal{I}(\mathbf{u})(t)&:=\int_\Omega(\delta+|\mathbf{D}\mathbf{u}(t)|)^{p-2}|\nabla\mathbf{D}\mathbf{u}(t)|^2\,dx\,,\\ \mathcal{J}(\mathbf{u})(t)&:=\int_\Omega(\delta+|\mathbf{D}\mathbf{u}(t)|)^{p-2}|\mathbf{D}\mathbf{u}_t(t)|^2\,dx\,.\end{aligned}\tag{3.7}$$

Let us first prove that the integrands of $\mathcal{I}(\mathbf{u})(t)$ and $\mathcal{J}(\mathbf{u})(t)$ are equivalent to $|\nabla\mathbf{F}(\mathbf{D}\mathbf{u})|^2$ and $\big|\big(\mathbf{F}(\mathbf{D}\mathbf{u})\big)_t\big|^2$, respectively.

Lemma 3.8. *Let $\delta\in(0,\infty)$ and let $\mathbf{F}$ be defined by (2.5). Then, for all sufficiently smooth $\mathbf{u}$ defined on $I\times\Omega$ there holds a.e.*

$$\begin{aligned}C_2\,(\delta+|\mathbf{D}\mathbf{u}|)^{p-2}|\nabla\mathbf{D}\mathbf{u}|^2&\le|\nabla\mathbf{F}(\mathbf{D}\mathbf{u})|^2\le C_3\,(\delta+|\mathbf{D}\mathbf{u}|)^{p-2}|\nabla\mathbf{D}\mathbf{u}|^2\,,\\ C_2\,(\delta+|\mathbf{D}\mathbf{u}|)^{p-2}|\mathbf{D}\mathbf{u}_t|^2&\le\big|\big(\mathbf{F}(\mathbf{D}\mathbf{u})\big)_t\big|^2\le C_3\,(\delta+|\mathbf{D}\mathbf{u}|)^{p-2}|\mathbf{D}\mathbf{u}_t|^2\,,\end{aligned}$$

where $C_2=\min\{1,\frac{p^2}{4}\}$*, and* $C_3=\max\{1,\frac{p^2}{4}\}$*.*

Proof. We show the first inequality, the other follows analogously. On the set $\{\mathbf{D}\mathbf{u}=\mathbf{0}\}$ we have $\nabla\mathbf{D}\mathbf{u}=\mathbf{0}$ almost everywhere, so $(\delta+|\mathbf{D}\mathbf{u}|)^{p-2}|\nabla\mathbf{D}\mathbf{u}|^2=0$ on $\{\mathbf{D}\mathbf{u}=\mathbf{0}\}$. Since $\{\mathbf{D}\mathbf{u}=\mathbf{0}\}=\{\mathbf{F}(\mathbf{D}\mathbf{u})=\mathbf{0}\}$, also $\nabla\mathbf{F}(\mathbf{D}\mathbf{u})=\mathbf{0}$ almost everywhere in $\{\mathbf{D}\mathbf{u}=\mathbf{0}\}$. This proves the inequality on the set $\{\mathbf{D}\mathbf{u}=\mathbf{0}\}$. Therefore, we can assume in the following that $|\mathbf{D}\mathbf{u}|>0$. We easily calculate

$$\begin{aligned}\partial_iF_{mn}\big(\mathbf{D}\mathbf{u}\big)&=\frac{p-2}{2}\big(\delta+|\mathbf{D}\mathbf{u}|\big)^{\frac{p-4}{2}}D_{mn}\mathbf{u}\,\partial_i|\mathbf{D}\mathbf{u}|+\big(\delta+|\mathbf{D}\mathbf{u}|\big)^{\frac{p-2}{2}}\partial_iD_{mn}\mathbf{u}\\ &=:\mathbf{A}+\mathbf{B}.\end{aligned}$$

Consequently we get

$$|\nabla \mathbf{F}(\mathbf{Du})|^2 = |\mathbf{A}|^2 + 2\mathbf{A}\cdot\mathbf{B} + |\mathbf{B}|^2.$$

We observe that

$$\begin{aligned} |\mathbf{B}|^2 &= \big(\delta + |\mathbf{Du}|\big)^{p-2}\big|\nabla \mathbf{Du}\big|^2, & (3.9)\\ 2\,\mathbf{A}\cdot\mathbf{B} &= (p-2)\big(\delta+|\mathbf{Du}|\big)^{p-3}|\mathbf{Du}|\big|\nabla|\mathbf{Du}|\big|^2, & (3.10)\\ |\mathbf{A}|^2 &= \left(\frac{p-2}{2}\right)^2\big(\delta+|\mathbf{Du}|\big)^{p-4}|\mathbf{Du}|^2\big|\nabla|\mathbf{Du}|\big|^2 \le \left(\frac{p-2}{2}\right)^2|\mathbf{B}|^2. & (3.11)\end{aligned}$$

Let us begin with the case $p \ge 2$. Then $2\,\mathbf{A}\cdot\mathbf{B} \ge 0$ by (3.10) and consequently

$$|\nabla \mathbf{F}(\mathbf{Du})|^2 = |\mathbf{A}|^2 + 2\mathbf{A}\cdot\mathbf{B} + |\mathbf{B}|^2 \ge |\mathbf{B}|^2.$$

To prove the upper bound we observe that $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|$ and by (3.11) it follows that

$$\begin{aligned}|\nabla \mathbf{F}(\mathbf{Du})|^2 &= |\mathbf{A}|^2 + 2\,|\mathbf{A}||\mathbf{B}| + |\mathbf{B}|^2\\ &\le \left(\left(\frac{p-2}{2}\right)^2 + (p-2) + 1\right)|\mathbf{B}|^2 = \frac{p^2}{4}|\mathbf{B}|^2.\end{aligned}$$

Let us consider the case $p \in (1,2)$. From (3.11) we get $|\mathbf{A}| \le \frac{2-p}{2}|\mathbf{B}| \le \frac{2+p}{2}|\mathbf{B}|$. This implies

$$\begin{aligned}|\nabla \mathbf{F}(\mathbf{Du})|^2 &= |\mathbf{A}|^2 - 2|\mathbf{A}||\mathbf{B}| + |\mathbf{B}|^2\\ &= \left(|\mathbf{A}| - \frac{p+2}{2}|\mathbf{B}|\right)\left(|\mathbf{A}| - \frac{2-p}{2}|\mathbf{B}|\right) + \frac{p^2}{4}|\mathbf{B}|^2\\ &\ge \frac{p^2}{4}|\mathbf{B}|^2.\end{aligned}$$

From (3.11) we get $|\mathbf{A}| \le \frac{2-p}{2}|\mathbf{B}| \le 2|\mathbf{B}|$ and

$$\begin{aligned}|\nabla \mathbf{F}(\mathbf{Du})|^2 &= |\mathbf{A}|^2 - 2\,|\mathbf{A}||\mathbf{B}| + |\mathbf{B}|^2\\ &= |\mathbf{A}|\big(|\mathbf{A}| - 2|\mathbf{B}|\big) + |\mathbf{B}|^2\\ &\le |\mathbf{B}|^2.\end{aligned}$$

This ends the proof. $\square$

Corollary 3.12. *Let $\mathcal{I}(\mathbf{u})(t)$ and $\mathcal{J}(\mathbf{u})(t)$ be defined in* (3.7) *with $\delta \in (0,\infty)$ and let $\mathbf{F}$ be defined by* (2.5)*. Then, for all sufficiently smooth functions $\mathbf{u}$ and almost all times $t \in I$ there holds*

$$\begin{aligned}\|\nabla \mathbf{F}(\mathbf{Du}(t))\|_2^2 &\simeq \mathcal{I}(\mathbf{u})(t)\,,\\ \big\|\big(\mathbf{F}(\mathbf{Du}(t))\big)_t\big\|_2^2 &\simeq \mathcal{J}(\mathbf{u})(t)\,,\end{aligned}$$

with constants depending only on p.

3.1 Approximation of degenerate stress tensors by non-degenerate ones

In this subsection we construct an approximation for degenerate stress tensors by non-degenerate ones. This enables us to use estimates that are uniform with respect to $\delta > 0$ for the treatment of the problem (NS_p) with $\delta = 0$.

Thus we assume that $\mathbf{S}$ satisfies Assumption 1 with $\delta = 0$ and $p \in (1, \infty)$. For $\kappa > 0$ we define the tensor valued function $\mathbf{S}^\kappa \colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ through

$$\begin{aligned} \mathbf{S}^\kappa(\mathbf{A}) : &= \big(\eta_\kappa * \mathbf{S}\big)(\mathbf{A}) - \big(\eta_\kappa * \mathbf{S}\big)(\mathbf{0}) \\ &= \int_{\mathbb{R}^{3\times 3}} \big(\mathbf{S}(\mathbf{A}-\mathbf{B}) - \mathbf{S}(-\mathbf{B})\big)\, \eta_\kappa(\mathbf{B})\, d\mathbf{B}\,, \end{aligned} \tag{3.13}$$

where $\eta \in C_0^\infty(\mathbb{R}^{3\times 3})$ with $c\,\chi_{B_{1/2}(0)} \le \eta \le C\,\chi_{B_1(0)}$, $\operatorname{supp}\eta \subset B_1(0)$, and $\int_{\mathbb{R}^9} \eta(\mathbf{B})\, d\mathbf{B} = 1$ is a standard mollification kernel and $\eta_\kappa(\mathbf{B}) := \kappa^{-9}\, \eta(\mathbf{B}/\kappa)$. One easily verifies that $\mathbf{S}^\kappa(\mathbf{A}) = \mathbf{S}^\kappa(\mathbf{A}^{\mathrm{sym}})$, $\mathbf{S}^\kappa(\mathbf{0}) = \mathbf{0}$, and that for all $i,j,k,l = 1,2,3$ it holds

$$\partial_{kl} S^\kappa_{ij}(\mathbf{A}) = (\eta_\kappa * \partial_{kl} S_{ij})(\mathbf{A})\,.$$

Since $\mathbf{S}$ satisfies Assumption 1 we obtain, for all $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{3\times 3}$ with $\mathbf{A}^{\mathrm{sym}} \neq \mathbf{0}$, and all $i,j,k,l = 1,2,3$

$$\sum_{i,j,k,l=1}^{3} \partial_{kl} S^\kappa_{ij}(\mathbf{A}) C_{ij} C_{kl} \ge C_0 \left(\eta_\kappa * \varphi''(|\cdot|)\right)\big(|\mathbf{A}^{\mathrm{sym}}|\big)\, |\mathbf{C}^{\mathrm{sym}}|^2,$$

$$\big|\partial_{kl} S^\kappa_{ij}(\mathbf{A})\big| \le \frac{C_1}{p-1} \left(\eta_\kappa * \varphi''(|\cdot|)\right)\big(|\mathbf{A}^{\mathrm{sym}}|\big),$$

where $\varphi(t) = \frac{1}{p} t^p$.

In order to show that $\mathbf{S}^\kappa$ satisfies Assumption 1 with $\delta = \kappa$ it is thus sufficient to show the following result.

Lemma 3.14. *Let η_κ, φ'', and p be as above. Then, we have for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}$*

$$\int_{B_\kappa(\mathbf{0})} \eta_\kappa(\mathbf{B}) \varphi''(|\mathbf{A}-\mathbf{B}|)\, d\mathbf{B} \simeq \varphi''(\kappa + |\mathbf{A}|)\,,$$

with constants depending only on p.

Proof. If $|\mathbf{B}| \le \kappa$, then for $|\mathbf{A}| \ge 2\,\kappa$ we have $\frac{1}{4}\big(|\mathbf{A}| + \kappa\big) \le \big||\mathbf{A}| - |\mathbf{B}|\big| \le |\mathbf{A}-\mathbf{B}| \le |\mathbf{A}| + \kappa$ and consequently we get $\varphi''(|\mathbf{A}-\mathbf{B}|) \simeq \varphi''(\kappa + |\mathbf{A}|)$. Using $\int \eta_\kappa(\mathbf{B})\, d\mathbf{B} = 1$ we thus get for $|\mathbf{A}| \ge 2\kappa$ that

$$\int_{B_\kappa(\mathbf{0})} \eta_\kappa(\mathbf{B}) \varphi''(|\mathbf{A}-\mathbf{B}|)\, d\mathbf{B} \simeq \varphi''(\kappa + |\mathbf{A}|)\,.$$

For $|\mathbf{A}| \le 2\,\kappa$ we have $\kappa \simeq |\mathbf{A}|+\kappa$ and consequently $\varphi''(\kappa) \simeq \varphi''(\kappa+|\mathbf{A}|)$. Using the properties of η_κ and $|\mathbf{A}| \le 2\,\kappa$ we get

$$\begin{aligned}
\int_{B_\kappa(\mathbf{0})} \eta_\kappa(\mathbf{B})\varphi''(|\mathbf{A}-\mathbf{B}|)\,d\mathbf{B} &= \int_{B_\kappa(\mathbf{A})} \eta_\kappa(\mathbf{A}-\mathbf{B})\varphi''(|\mathbf{B}|)\,d\mathbf{B} \\
&\le \frac{C}{\kappa^9}\int_{B_{3\kappa}(\mathbf{0})} \varphi''(|\mathbf{B}|)\,d\mathbf{B} \\
&\simeq \varphi''(3\kappa) \simeq \varphi''(\kappa+|\mathbf{A}|)\,.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\int_{B_\kappa(\mathbf{0})} \eta_\kappa(\mathbf{B})\varphi''(|\mathbf{A}-\mathbf{B}|)\,d\mathbf{B} &= \int_{B_\kappa(\mathbf{A})} \eta_\kappa(\mathbf{A}-\mathbf{B})\varphi''(|\mathbf{B}|)\,d\mathbf{B} \\
&\ge \frac{c}{\kappa^9}\int_{B_{\kappa/2}(\mathbf{A})} \varphi''(|\mathbf{B}|)\,d\mathbf{B} \\
&\ge \frac{c}{\kappa^9}\int_{B_{\kappa/2}(\mathbf{A})\setminus B_{\kappa/4}(\mathbf{0})} \varphi''(|\mathbf{B}|)\,d\mathbf{B} \\
&\simeq \varphi''(\kappa/2) \simeq \varphi''(\kappa+|\mathbf{A}|)\,.
\end{aligned}$$

This finishes the proof. □

Using Lemma 3.14 and the equivalence $\varphi''(\kappa+t) \simeq \varphi''_\kappa(t)$ with constants depending only on p we thus proved the following result.

Theorem 3.15. *If $\mathbf{S}$ satisfies Assumption 1 with $p \in (1,\infty)$ and $\delta = 0$, then $\mathbf{S}^\kappa$ defined in (3.13) satisfies Assumption 1 with the same p and $\delta = \kappa > 0$, i.e., for all $\mathbf{A},\mathbf{C} \in \mathbb{R}^{3\times 3}$ with $\mathbf{A}^{\mathrm{sym}} \neq \mathbf{0}$, and all $i,j,k,l = 1,2,3$ holds*

$$\sum_{i,j,k,l=1}^{3} \partial_{kl} S^\kappa_{ij}(\mathbf{A}) C_{ij} C_{kl} \ge \widetilde{C}_0\, \varphi''_\kappa\big(|\mathbf{A}^{\mathrm{sym}}|\big)|\mathbf{C}^{\mathrm{sym}}|^2,$$
$$\big|\partial_{kl} S^\kappa_{ij}(\mathbf{A})\big| \le \widetilde{C}_1\, \varphi''_\kappa\big(|\mathbf{A}^{\mathrm{sym}}|\big),$$

with $\widetilde{C}_0$ and $\widetilde{C}_1$ depending only on p, C_0, and C_1. In particular they are independent of κ.

If we denote $\mathbf{S}^0 := \mathbf{S}$, then we get from the properties of the mollifier that $\mathbf{S}^\kappa$ converges locally-uniformly to $\mathbf{S}^0$ for $\kappa \to 0^+$. This means that for any $R > 0$

$$\lim_{\kappa\to 0^+} \mathbf{S}^\kappa(\mathbf{A}) = \mathbf{S}^0(\mathbf{A}) \quad \text{uniformly for } |\mathbf{A}| \le R.$$

From the properties of mollifiers and N-functions, we can deduce the following results, which are crucial in order to pass to the limit as the approximation parameter κ goes to zero.

Lemma 3.16. *Let $\mathbf{S}$ satisfy Assumption 1 with $p \in (1,\infty)$ and $\delta = 0$. Moreover, let $\mathbf{S}^\kappa$ be defined in (3.13). Then, the mapping $(\kappa,\mathbf{A}) \mapsto \mathbf{S}^\kappa(\mathbf{A})$ is continuous on $\mathbb{R}^{\ge 0} \times \mathbb{R}^{d\times d}$.*

Proof. Let $(\kappa_n, \mathbf{A}_n)$ converge to $(\kappa_0, \mathbf{A}_0)$ as $n \to \infty$ We have

$$\left|\mathbf{S}^{\kappa_n}(\mathbf{A}_n) - \mathbf{S}^{\kappa_0}(\mathbf{A}_0)\right| \le \left|\mathbf{S}^{\kappa_n}(\mathbf{A}_n) - \mathbf{S}^{\kappa_n}(\mathbf{A}_0)\right| + \left|\mathbf{S}^{\kappa_n}(\mathbf{A}_0) - \mathbf{S}^{\kappa_0}(\mathbf{A}_0)\right|.$$

If $\kappa_0 = 0$ then the second term converges to 0 since $\mathbf{S}^{\kappa_n}$ converges locally uniformly to $\mathbf{S}$ by the discussion above. If $\kappa_0 > 0$ we have that η_{κ_n} converges locally uniformly to η_{κ_0} and is bounded. Thus, also in this case the second term converges to 0. To treat the first term, we need some formulas on the shift change for N-functions. For the first term we have due to (3.3), (2.15), and (2.17b)

$$\begin{aligned}\left|\mathbf{S}^{\kappa_n}(\mathbf{A}_n) - \mathbf{S}^{\kappa_n}(\mathbf{A}_0)\right| &\le c\,|\mathbf{A}_n - \mathbf{A}_0|\big(\kappa_n + |\mathbf{A}_n| + |\mathbf{A}_0|\big)^{p-2} \\ &\le c\,\varphi'_{\kappa_n}(|\mathbf{A}_n - \mathbf{A}_0|) \\ &\le c\,\varphi'_{\kappa_0}(|\mathbf{A}_n - \mathbf{A}_0|) + c\,\varphi'_{\kappa_0}(|\kappa_n - \kappa_0|).\end{aligned}$$

From the continuity properties of the shifted N-functions it follows that the right-hand-side converges to 0. □

In the case that the extra stress tensor $\mathbf{S}$ is derived from a potential Φ, i.e., (2.1), (2.2) hold with $\delta = 0$ and $p \in (1, \infty)$, one can use another approximation. In the situation of a stress tensor derived from a potential we define $\mathbf{S}^\kappa(\mathbf{0}) := \mathbf{0}$ and for all $\mathbf{A} \in \mathbb{R}^{3\times 3} \setminus \{\mathbf{0}\}$ we set

$$\mathbf{S}^\kappa(\mathbf{A}) := \Phi'_\kappa(|\mathbf{A}^{\mathrm{sym}}|)\frac{\mathbf{A}^{\mathrm{sym}}}{|\mathbf{A}^{\mathrm{sym}}|}, \tag{3.17}$$

where Φ'_κ is the shifted N-function (cf. (2.14)) corresponding to the N-function Φ. Of course we set again $\mathbf{S}^0 := \mathbf{S}$. This approximation (which is very natural) suggested to us the more general approximation (3.13) necessary to treat a wider class of stress tensors with p-structure. In addition, results here will be used to detect relevant properties of the function $\mathbf{F}$, which is also derived from a potential (cf. (2.24)).

Lemma 3.18. *Let the potential Φ satisfy* (2.2) *with $\delta = 0$ and let $\mathbf{S}^\kappa$ be defined in* (3.17)*. Then the mappings $(\kappa, t) \mapsto \Phi'_\kappa(t)$ and $(\kappa, \mathbf{A}) \mapsto \mathbf{S}^\kappa(\mathbf{A})$ are continuous on $\mathbb{R}^{\ge 0} \times \mathbb{R}^{\ge 0}$ and $\mathbb{R}^{\ge 0} \times \mathbb{R}^{d\times d}$, respectively.*

Proof. Let $(\kappa_n, t_n) \to (\kappa_0, t_0)$. If $\kappa_0 + t_0 > 0$ then by the definition of $\Phi'_\kappa(t)$ it follows that $\Phi'_{\kappa_n}(t_n) \to \Phi'_{\kappa_0}(t_0)$. If $\kappa_0 = t_0 = 0$ then by (2.17b)

$$\Phi'_{\kappa_n}(t_n) \le c\left(\Phi'_0(t_n) + \Phi'_0(|t_n - \kappa_n|)\right)$$

and since $\Phi'_0(0) = \Phi'(0) = 0$ we get

$$|\Phi'_{\kappa_n}(t_n) - \Phi'_0(0)| \le c\left(\Phi'_0(t_n) + \Phi'_0(|t_n - \kappa_n|)\right) \to 0.$$

Thus the assertion for $\Phi'_\kappa(t)$ is proved.

Let now $(\kappa_n, \mathbf{A}_n)$ converge to $(\kappa_0, \mathbf{A}_0)$. If $|\mathbf{A}_0| > 0$, due to the definition of $\mathbf{S}^\kappa$ and the continuity of $\Phi'_\kappa(t)$ we have that $\mathbf{S}^{\kappa_n}(\mathbf{A}_n) \to \mathbf{S}^{\kappa_0}(\mathbf{A}_0)$. For $\mathbf{A}_0 = \mathbf{0}$ we have $\mathbf{S}^\kappa(\mathbf{A}_0) = \mathbf{0}$ and

$$|\mathbf{S}^{\kappa_n}(\mathbf{A}_n)| \le \Phi'_{\kappa_n}(|\mathbf{A}_n^{\mathrm{sym}}|) \to 0\,, \tag{3.19}$$

which shows $\mathbf{S}^{\kappa_n}(\mathbf{A}_n) \to \mathbf{0}$. Thus also the assertion for $\mathbf{S}^\kappa(\mathbf{A})$ is proved. $\square$

Next, we derive some information on the function $\mathbf{F}$. First, we observe that $\mathbf{F}$ in (2.5) for $\delta = 0$ and $p \in (1, \infty)$ is derived from the potential $\psi(t) := \frac{2}{p+2} t^{\frac{p+2}{2}}$, i.e.,

$$\mathbf{F}^0(\mathbf{A}) := \mathbf{F}(\mathbf{A}) = \psi'(|\mathbf{A}^{\mathrm{sym}}|)\frac{\mathbf{A}^{\mathrm{sym}}}{|\mathbf{A}^{\mathrm{sym}}|}, \tag{3.20}$$

and this happens also if the stress tensor $\mathbf{S}$ in (NS$_p$) does not derive from a potential.

Consequently we set

$$\mathbf{F}^\kappa(\mathbf{A}) := \psi'_\kappa(|\mathbf{A}^{\mathrm{sym}}|)\frac{\mathbf{A}^{\mathrm{sym}}}{|\mathbf{A}^{\mathrm{sym}}|}\,. \tag{3.21}$$

Now Lemma 3.18 with p replaced by $\frac{p+2}{2} \in (3/2, \infty)$ implies the following result.

Corollary 3.22. *Let $p \in (1, \infty)$ and let the potential ψ and $\mathbf{F}^\kappa$ be defined in* (3.20) *and* (3.21)*, respectively. Then, the mappings $(\kappa, t) \mapsto \psi'_\kappa(t)$ and $(\kappa, \mathbf{A}) \mapsto \mathbf{F}^\kappa(\mathbf{A})$ are continuous on $\mathbb{R}^{\ge 0} \times \mathbb{R}^{\ge 0}$ and $\mathbb{R}^{\ge 0} \times \mathbb{R}^{d\times d}$, respectively.*

In order to pass to the limit and to identify appropriate limits with solutions of the degenerate equations, we shall use also continuity in the approximation of the inverse of $\mathbf{F}$. We have the following lemma.

Lemma 3.23. *Let $p \in (1, \infty)$ and let the potential ψ and $\mathbf{F}^\kappa$ be defined in* (3.20) *and* (3.21)*, respectively. Then, the mapping $(\kappa, \mathbf{A}) \mapsto \big(\mathbf{F}^\kappa\big)^{-1}(\mathbf{A})$ is continuous on $\mathbb{R}^{\ge 0} \times \mathbb{R}^{d\times d}$.*

Proof. Note that $\big(\mathbf{F}^\kappa\big)^{-1}(\mathbf{A}) = \big(\psi^*_\kappa\big)'(|\mathbf{A}^{\mathrm{sym}}|)\frac{\mathbf{A}^{\mathrm{sym}}}{|\mathbf{A}^{\mathrm{sym}}|}$, where $\big(\psi^*_\kappa\big)'(t) = \big(\psi'_\kappa\big)^{-1}(t)$. From (2.18) it follows that $\big(\psi^*_\kappa\big)'(t) \simeq \big(\kappa^{\frac{p}{2}} + t\big)^{\frac{2-p}{p}}\, t$. This implies that the assumptions of Lemma 3.16 are satisfied with p replaced by $\frac{p+2}{p} \in (1, 3)$, and the assertion follows. $\square$

4 Some estimates specific of the case $\mathbf{p} \in (\mathbf{1}, \mathbf{2}]$

In preparation for the proof of the main result in the section 5 we prove in this section the necessary estimates and relations needed to extend the existence result for strong solutions proved in [20] to the degenerate case $\delta = 0$. In particular, we prove several results that are confined to the case $1 < p \le 2$. We proceed as in [20] and show the necessary changes to the various lemmas needed to get suitable a priori estimates. In particular, we shall replace the quantity

$$(\widetilde{\mathbf{D}}\mathbf{u})^p = (1 + |\mathbf{D}\mathbf{u}|^2)^{p/2}$$

used in [20] by the more natural (vectorial) quantity $\mathbf{F}(\mathbf{D}\mathbf{u})$ and in view of Corollary 3.12 we will also replace the quantities $\mathcal{I}(\mathbf{u})$ and $\mathcal{J}(\mathbf{u})$ used in the previous papers by $\|\nabla \mathbf{F}(\mathbf{D}\mathbf{u})\|_2^2$ and $\|(\mathbf{F}(\mathbf{D}\mathbf{u}))_t\|_2^2$, respectively. Moreover, we show that all constants in the estimates are independent of $\delta > 0$.

Throughout this section we shall assume that $d = 3$ and $1 < p \leq 2$. We also recall the following result, taken from [20, Lemma 8].

Lemma 4.1. *Let* $\mathbf{S}$ *satisfy Assumption 1 with* $p \in (1, 2]$ *and* $\delta \in [0, \infty)$*, and let* $\mathbf{F}$ *be defined by* (2.5)*. Then, for sufficiently smooth* $\mathbf{u}$*,* $\mathbf{v}$ *and* $q \in [1, 2]$ *holds*

$$\|\mathbf{D}(\mathbf{u} - \mathbf{v})\|_q^2 \leq c\, \|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\mathbf{v})\|_2^2 \, \big\|(\delta + |\mathbf{D}\mathbf{u}| + |\mathbf{D}\mathbf{v}|)^{2-p}\big\|_{\frac{q}{2-q}} \,,$$

where the constant c *depends only on* C_0*,* C_1*, and* p*. Moreover,* $\frac{q}{2-q} = \infty$ *for* $q = 2$*. For* $p \in (1, 2]$*,* $\delta \in [0, \infty)$*,* $r \in [1, \infty]$*, and* $\delta + \|\mathbf{D}\mathbf{u}\|_r + \|\mathbf{D}\mathbf{v}\|_r > 0$ *we can formulate this result also as follows*

$$\begin{aligned} &\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{u}) - \mathbf{S}(\mathbf{D}\mathbf{v})) \cdot \mathbf{D}(\mathbf{u} - \mathbf{v})\, dx \\ &\qquad \geq c\, \|\mathbf{D}\mathbf{u} - \mathbf{D}\mathbf{v}\|^2_{\frac{2r}{2-p+r}} (\delta + \|\mathbf{D}\mathbf{u}\|_r + \|\mathbf{D}\mathbf{u} - \mathbf{D}\mathbf{v}\|_r)^{p-2}. \end{aligned}$$

Proof. From (3.2) it follows that $|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\mathbf{v})|^q (\delta + |\mathbf{D}\mathbf{u}| + |\mathbf{D}\mathbf{v}|)^{\frac{(2-p)q}{2}} \simeq |\mathbf{D}\mathbf{u} - \mathbf{D}\mathbf{v}|^q$. Integrating this and applying Hölder's inequality gives the assertion (cf. also [20, Lemma 8]). For the second inequality we also use that for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}$ holds $|\mathbf{A}| + |\mathbf{A} - \mathbf{B}| \simeq |\mathbf{A}| + |\mathbf{B}|$. $\square$

Lemma 4.2. *Let* $\delta \in (0, \infty)$*. Then for all sufficiently smooth* $\mathbf{v}$ *and* $\mathbf{w}$ *and for all* $q \in [1, 2]$ *and* $p \in (1, 2]$ *and for almost all* $t \in I$ *there holds*

$$\begin{aligned} \|\mathbf{D}\mathbf{v}\|_q^2 &\leq \int_\Omega (\delta + |\mathbf{D}\mathbf{w}|)^{p-2} |\mathbf{D}\mathbf{v}|^2 \, dx \, \big\|\big(\delta + |\mathbf{D}\mathbf{w}|\big)^{2-p}\big\|_{\frac{q}{2-q}}, \\ &\leq c \int_\Omega (\delta + |\mathbf{D}\mathbf{w}|)^{p-2} |\mathbf{D}\mathbf{v}|^2 \, dx \, \Big\|\big(\delta^{\frac{p}{2}} + |\mathbf{F}(\mathbf{D}\mathbf{w})|\big)^{\frac{2-p}{p}}\Big\|^2_{\frac{2q}{2-q}}, \end{aligned}$$

where $q/(2-q) = \infty$ *if* $q = 2$ *and where the constant* c *depends only on* p*.*

Proof. The first inequality follows directly from Hölder's inequality (cf. [20, Lemma 7]). Using the relation (2.20) and the definition of $\mathbf{F}$ the second inequality follows. $\square$

Remark 4.3. The reason that we exclude the case $\delta = 0$ in the above lemma is that, even for sufficiently smooth $\mathbf{w} \neq \mathbf{v}$, we do not know a priori whether or not the right-hand-side is finite.

We prove now two lemmas, which show which information on second derivatives of $\mathbf{u}$ can be extracted from first derivatives of $\mathbf{F}(\mathbf{Du})$.

Lemma 4.4. *Let $p \in (1,2]$ and $\delta \in (0,\infty)$. Then for all sufficiently smooth function $\mathbf{u}$ there holds*

$$\|\nabla^2 \mathbf{u}\|_p^p \le c\left(\|\nabla \mathbf{F}(\mathbf{Du})\|_2^2 + \big\|\delta + |\mathbf{Du}|\big\|_p^p \right),$$

$$c\,\|\nabla \mathbf{u}_t\|_p^p \le c\,\|\mathbf{Du}_t\|_p^p \le c\left(\big\|\big(\mathbf{F}(\mathbf{Du})\big)_t\big\|_2^2 + \big\|\delta + |\mathbf{Du}|\big\|_p^p \right)$$

with constants c depending only on p and Ω.

Proof. The proof of this lemma is given in [20, Lemma 6] in the case $\delta = 1$. We show now that the same bounds hold, without any dependence on δ. We use the inequality

$$a^p \le a^2 b^{p-2} + b^p,$$

which holds for all $0 \le a$, $0 < b$, and $p \in [1,2]$. The previous inequality implies

$$|\nabla^2 \mathbf{u}|^p \le (\delta + |\mathbf{Du}|)^{p-2} |\nabla^2 \mathbf{u}|^2 + (\delta + |\mathbf{Du}|)^p.$$

Observing $|\nabla^2 \mathbf{u}| \le 3\,|\nabla \mathbf{Du}|$ and using Corollary 3.12 the first assertion follows immediately. For the second assertion we proceed analogously and use Korn's inequality. $\square$

Lemma 4.5. *Let $p \in (1,2]$ and $\delta \in (0,\infty)$. Then, for all sufficiently smooth functions $\mathbf{u}$ with vanishing mean value over Ω there holds for $s \in [1,\infty)$*

$$\begin{aligned}
\|\mathbf{u}\|^p_{W^{2,\frac{3p}{p+1}}(\Omega)} &\le c\left(\|\nabla \mathbf{F}(\mathbf{Du})\|_2^2 + \delta^p \right), \\
\|\nabla \mathbf{u}\|^2_{\frac{6s}{6-3p+s}} + \|\nabla^2 \mathbf{u}\|^2_{\frac{2s}{2-p+s}} &\le c\, \|\nabla \mathbf{F}(\mathbf{Du})\|_2^2 \left(\delta + \|\nabla \mathbf{u}\|_s\right)^{2-p}, \\
\|\mathbf{u}_t\|^p_{W^{1,\frac{3p}{p+1}}(\Omega)} &\le c\, \|(\mathbf{F}(\mathbf{Du}))_t\|_2^p \left(\|\nabla \mathbf{F}(\mathbf{Du})\|_2^2 + \delta^p \right)^{\frac{2-p}{2}} \\
&\le c\left(\|\nabla \mathbf{F}(\mathbf{Du})\|_2^2 + \|(\mathbf{F}(\mathbf{Du}))_t\|_2^2 + \delta^p \right), \\
\|\mathbf{u}_t\|^2_{\frac{6s}{6-3p+s}} + \|\nabla \mathbf{u}_t\|^2_{\frac{2s}{2-p+s}} &\le c\, \|(\mathbf{F}(\mathbf{Du}))_t\|_2^2 \left(\delta + \|\nabla \mathbf{u}\|_s\right)^{2-p},
\end{aligned}$$

with constants depending only on Ω, p, and s and independent of $\delta > 0$.

Proof. Using Lemma 4.2 and Corollary 3.12 one can adapt the proof of [20, Lemma 10] and [19, Lemma 4.2] easily to obtain the results. $\square$

The above lemmas are enough in order to prove existence of strong solutions for small times or small data in the case $5/3 < p \le 2$, since essentially in this case testing with $-\Delta \mathbf{u}$ is needed and the convective term can be handled by assuming that the time interval is small (cf. [35]).

In order to have existence of solutions also for smaller values of p it is necessary to test the equations simultaneously with $-\Delta\mathbf{u}$ and "$\mathbf{u}_{tt}$."[‡] The next lemma shows how to extract information from the additional term $\frac{d}{dt}\|\mathbf{F}(\mathbf{Du})\|_q^q$ on the left-hand-side.

Lemma 4.6. *Let $p \in (1,2]$, $\delta \in (0,\infty)$, $\mathbf{F}$ be defined by (2.5), and $q \in [1,\infty)$. Then, for sufficiently smooth functions $\mathbf{u}$ there holds*

$$\begin{aligned}\frac{d}{dt}\|\mathbf{F}(\mathbf{Du})\|_q^q &\le c\ \|(\mathbf{F}(\mathbf{Du}))_t\|_2 \left\|\mathbf{F}(\mathbf{Du})\right\|_{2(q-1)}^{q-1}\\ &\le \varepsilon\ \|(\mathbf{F}(\mathbf{Du}))_t\|_2^2 + c_\varepsilon \left\|\mathbf{F}(\mathbf{Du})\right\|_{2(q-1)}^{2(q-1)},\end{aligned}$$

where $\left\|\mathbf{F}(\mathbf{Du})\right\|_{2(q-1)}^{2(q-1)} := \int_\Omega |\mathbf{F}(\mathbf{Du})|^{2(q-1)}\,dx$ even if $2(q-1) < 1$, with constants $c = c(p,q)$ and $c_\varepsilon = c(\varepsilon,p,q)$ for all $\varepsilon > 0$.

Proof. This follows by direct computation of the time derivative of $|\mathbf{F}(\mathbf{Du}(t))|^q$, the definition of $\mathbf{F}(\mathbf{Du})$, Hölder's inequality, and Corollary 3.12 (for more details cf. [20, Lemma 11]). The last estimate follows from Young's inequality. □

This lemma can be used to produce the time derivative of $\|\mathbf{F}(\mathbf{Du})\|_q^q$ on the left-hand-side, provided that we add a multiple of $\left\|\mathbf{F}(\mathbf{Du})\right\|_{2(q-1)}^{2(q-1)}$ on the right-hand-side. Moreover, the information coming from $\|(\mathbf{F}(\mathbf{Du}))_t\|_2^2$ is stronger than that coming from $-\int_\Omega \mathbf{u}_t \cdot \Delta\mathbf{u}\,dx = \frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2$. Thus we leave this term in its original form and move it to the right-hand-side, where it must be estimated. This has the advantage that the equation (NS$_p$) tested with $-\Delta\mathbf{u}$ can be raised to some power. The following lemmas show how the terms coming from the convective term, the additional terms $\left\|\mathbf{F}(\mathbf{Du})\right\|_{2(q-1)}^{2(q-1)}$ and $-\int_\Omega \mathbf{u}_t \cdot \Delta\mathbf{u}\,dx$ can be estimated.

Lemma 4.7. *Let $p \in (1,2]$, $\delta \in (0,\infty)$, and $q \in \left(\frac{9-3p}{p},\infty\right)$. Then, there exists a constant $R_1 = R_1(p)$ such that for all $\varepsilon > 0$ and all sufficiently smooth functions $\mathbf{u}$ there is a constant $c_\varepsilon = c(\varepsilon,p,\Omega)$ such that there holds*

$$\|\nabla\mathbf{u}\|_3^3 \le c_\varepsilon \left(\left\|\mathbf{F}(\mathbf{Du})\right\|_q^{R_1} + \delta^3\right) + \varepsilon\left(\|\mathbf{F}(\mathbf{Du})\|_2^2 + \|\nabla\mathbf{F}(\mathbf{Du})\|_2^2\right).$$

Proof. It suffices to consider the case $q < 6/p$, since if $q \ge 6/p$ we can get directly the above estimate, without using interpolation. Using Korn's inequality we get $\|\nabla\mathbf{u}\|_3^3 \le c\,\|\mathbf{Du}\|_3^3$. From relation (2.20) we get $\|\nabla\mathbf{u}\|_3^3 \le c\,(\|\mathbf{F}(\mathbf{Du})\|_{6/p}^{6/p} + \delta^3)$, with a constant independent of δ. Thus, we interpolate $L^{6/p}(\Omega)$ between $L^q(\Omega)$ and $L^6(\Omega)$ to obtain

$$\|\mathbf{F}(\mathbf{Du})\|_{6/p}^{6/p} \le \|\mathbf{F}(\mathbf{Du})\|_q^{\frac{6(1-\theta)}{p}} \|\mathbf{F}(\mathbf{Du})\|_6^{\frac{6\theta}{p}},$$

[‡]This means to take the derivative of the equations with respect to t, then multiply by $\mathbf{u}_t$, and perform suitable integrations by parts.

with $\theta = \frac{pq-6}{q-6}$. If $\frac{6\theta}{p} < 2$ we can use Young's inequality and the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ to obtain the assertion of the lemma. This condition is equivalent to the requirement $\frac{9-3p}{p} < q < 6$. $\square$

Lemma 4.8. *Let $p \in (1,2]$, $\delta \in (0,\infty)$, and $q \in \big(\frac{9-3p}{p},\infty\big)$. Then, there exist constants $R_2 = R_2(p)$, $R_3 = R_3(p)$, and $R_4 = R_4(p)$ such that for all $\varepsilon > 0$ and all sufficiently smooth functions $\mathbf{u}$ there is a constant $c_\varepsilon = c(\varepsilon, p, \Omega)$ such that there holds*

$$\left|\int_\Omega [\nabla \mathbf{u}_t]\mathbf{u} \cdot \mathbf{u}_t \, dx\right| \le \varepsilon \, \|(\mathbf{F}(\mathbf{Du}))_t\|_2^2 + c_\varepsilon \Big(\|\mathbf{u}_t\|_2^{R_2} + \big\|\mathbf{F}(\mathbf{Du})\big\|_q^{R_3} + \delta^{R_4}\Big).$$

Proof. Korn's and Hölder's inequalities imply for $q > \frac{2}{p}$

$$\left|\int_\Omega [\nabla \mathbf{u}]\mathbf{u}_t \cdot \mathbf{u}_t \, dx\right| \le c\,\|\mathbf{u}_t\|^2_{\frac{2pq}{pq-2}} \|\mathbf{Du}\|_{\frac{pq}{2}}$$

Next, by using (2.20) and the definition of $\mathbf{F}(\mathbf{Du})$ we get

$$\begin{aligned}\left|\int_\Omega [\nabla \mathbf{u}]\mathbf{u}_t \cdot \mathbf{u}_t \, dx\right| &\le c\,\|\mathbf{u}_t\|^2_{\frac{2pq}{pq-2}} \big(\big\|\mathbf{F}(\mathbf{Du})\big\|_q^{\frac{2}{p}} + \delta\big), \\ &\le c\,\|\mathbf{u}_t\|_2^{2(1-\theta)} \|\mathbf{u}_t\|^{2\theta}_{\frac{6qp}{12-6p+pq}} \big(\big\|\mathbf{F}(\mathbf{Du})\big\|_q^{\frac{2}{p}} + \delta\big),\end{aligned}$$

with a constant c independent of δ. Note that the interpolation $L^{\frac{2pq}{pq-2}}(\Omega) = [L^2(\Omega), L^{\frac{6qp}{12-6p+pq}}(\Omega)]_\theta$, is possible since $q > \frac{9-3p}{p}$. Next, Lemma 4.2 (replace q there by $\frac{2pq}{4-2p+pq}$ and $\mathbf{v}$ by $\mathbf{u}_t$), Corollary 3.12, the definition of $\mathbf{F}(\mathbf{Du})$, and (2.20) imply

$$\|\mathbf{Du}_t\|_{\frac{2pq}{4-2p+pq}} \le c\, \|(\mathbf{F}(\mathbf{Du}))_t\|_2 \, \big\|\delta^{\frac{p}{2}} + |\mathbf{F}(\mathbf{Du})|\big\|_q^{\frac{2(2-p)}{p}}.$$

Note, that $\frac{2pq}{4-2p+pq} \in (1,2]$ for $q > \frac{9-3p}{p}$ and $p \in [1,2]$. Korn's inequality and the embedding $W^{1,\frac{2pq}{4-2p+pq}}(\Omega) \hookrightarrow L^{\frac{6qp}{12-6p+pq}}(\Omega)$ thus imply

$$\begin{aligned}&\left|\int_\Omega [\nabla \mathbf{u}]\mathbf{u}_t \cdot \mathbf{u}_t \, dx\right| \\ &\quad\le c\,\|\mathbf{u}_t\|_2^{2(1-\theta)} \Big(\|(\mathbf{F}(\mathbf{Du}))_t\|_2 \, \big\|\delta^{\frac{p}{2}} + |\mathbf{F}(\mathbf{Du})|\big\|_q^{\frac{2(2-p)}{p}}\Big)^{2\theta} \big(\|\mathbf{F}(\mathbf{Du})\|_q^{\frac{2}{p}} + \delta\big) \\ &\quad\le \varepsilon\, \|(\mathbf{F}(\mathbf{Du}))_t\|_2^2 + c_\varepsilon \Big(\|\mathbf{u}_t\|_2^{R_2} + \big\|\mathbf{F}(\mathbf{Du})\big\|_q^{R_3} + \delta^{R_4}\Big),\end{aligned}$$

which proves the assertion. Note, that the q here is obtained from the q in [20, Lemma 13] by multiplying with $\frac{2}{p}$. $\square$

Proceeding similarly one can modify also Lemmas 14 and 16 in [20].

Lemma 4.9. *Let Let $p \in (1,2]$, $\delta \in (0,\infty)$. Then, there exists a constant $c = c(p,\Omega)$ such that for all sufficiently smooth functions $\mathbf{u}$ there holds*

$$\left|\int_\Omega \mathbf{u}_t \cdot \Delta\mathbf{u}\,dx\right| \le c\,\|\mathbf{u}_t\|_2^{\frac{4(p-1)}{3p-2}}\ \|(\mathbf{F}(\mathbf{Du}))_t\|_2^{\frac{2-p}{3p-2}}\left(\|\nabla\mathbf{F}(\mathbf{Du})\|_2^2 + \delta^p\right)^{\frac{p+2}{2(3p-2)}}.$$

Proof. Cf. [20, Lemma 14] and Corollary 3.12. □

Lemma 4.10. *Let $p \in (1,2]$, $\delta \in (0,\infty)$, $r \in [1,\infty)$, and $2 < q < \min\{4, \frac{6(r+1)}{3+r}\}$. Then, there exists a constant $R_5 = R_5(p) > 1$ such that for all $\varepsilon > 0$ there is a constant $c_\varepsilon = c(\varepsilon,p,r,\Omega)$ such that for all sufficiently smooth functions $\mathbf{u}$ there holds*

$$\|\mathbf{F}(\mathbf{Du})\|_{2(q-1)}^{2(q-1)} \le c_\varepsilon \|\mathbf{F}(\mathbf{Du})\|_q^{R_5} + \varepsilon\left(\|\mathbf{F}(\mathbf{Du})\|_2^{2r} + \|\nabla\mathbf{F}(\mathbf{Du})\|_2^{2r}\right).$$

Proof. For $2 < q < 4$ we can interpolate $L^{2(q-1)}(\Omega) = [L^q(\Omega), L^6(\Omega)]_\theta$, with $\theta = \frac{3(q-2)}{(q-1)(6-q)}$. Using Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ we thus obtain

$$\|\mathbf{F}(\mathbf{Du})\|_{2(q-1)}^{2(q-1)} \le c\,\|\mathbf{F}(\mathbf{Du})\|_q^{2(q-1)(1-\theta)}\|\mathbf{F}(\mathbf{Du})\|_{W^{1,2}(\Omega)}^{2(q-1)\theta}.$$

If $2(q-1)\theta < 2r$ we can use Young's inequality to get the assertion. This condition is equivalent to $q < \frac{6(r+1)}{3+r}$ (cf. [20, Lemma 16]). □

Using the above results one can show for each $\delta > 0$ (cf. [20] and see also section 5) suitable a priori estimates which can be used to prove the local in time existence of strong solutions of the problem (NS$_p$), via the Galerkin method. Since all previous estimates are independent of $\delta > 0$, we shall be able treat the case $\delta = 0$ by a limiting procedure. However, we shall need the results of subsection 3.1 in order to justify also the limit in the extra stress tensor $\mathbf{S}$ and related quantities.

5 Main Theorem

Now we have prepared everything to formulate and prove the main result of the paper.

Theorem 5.1. *Let $\mathbf{S}$ satisfy Assumption 1 with $p \in \left(\frac{7}{5}, 2\right]$ and $\delta \in [0,\delta_0]$ where $\delta_0 > 0$. Assume that $\mathbf{f} \in L^\infty(I; W^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega))$, where $I = [0,T]$, and $\mathbf{u}_0 \in W^{2,2}_{\operatorname{div}}(\Omega)$, $\operatorname{div}\mathbf{S}(\mathbf{Du}_0) \in L^2(\Omega)$. Then there exists a time $T' = T'(\delta_0, p, C_0, \mathbf{f}, \mathbf{u}_0, T, \Omega)$, with $0 < T' \le T$, such that the system (NS$_p$) has a strong solution $\mathbf{u} \in L^p(I'; W^{1,p}_{\operatorname{div}}(\Omega))$, $I' = [0,T']$, satisfying for a.e. $t \in I'$ and for all $\boldsymbol{\varphi} \in W^{1,p}_{\operatorname{div}}(\Omega)$*

$$\int_\Omega \mathbf{u}_t(t)\cdot\boldsymbol{\varphi} + \mathbf{S}(\mathbf{Du}(t))\cdot\mathbf{D}\boldsymbol{\varphi} + [\nabla\mathbf{u}(t)]\mathbf{u}(t)\cdot\boldsymbol{\varphi}\,dx = \int_\Omega \mathbf{f}(t)\cdot\boldsymbol{\varphi}\,dx, \qquad (5.2)$$

and

$$\|\mathbf{u}_t\|_{L^\infty(I';L^2(\Omega))}+\|\mathbf{F}(\mathbf{Du})\|_{W^{1,2}(I'\times\Omega)} + \|\mathbf{F}(\mathbf{Du})\|_{L^{2\frac{5p-6}{2-p}}(I';W^{1,2}(\Omega))} \le c\,, \tag{5.3}$$

with a constant $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$*. In particular, we have*

$$\mathbf{u} \in L^{\frac{p(5p-6)}{2-p}}(I'; W^{2,\frac{3p}{p+1}}(\Omega)) \cap C(I'; W^{1,s}(\Omega)) \qquad 1 \le s < 6(p-1) \tag{5.4a}$$

$$\mathbf{u}_t \in L^\infty(I'; L^2(\Omega)) \cap L^{\frac{p(5p-6)}{(3p-2)(p-1)}}(I'; W^{1,\frac{3p}{p+1}}(\Omega))\,, \tag{5.4b}$$

with norms of $\mathbf{u}$ *and* $\mathbf{u}_t$ *bounded by constants* $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega, s)$ *and* $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$*, respectively. Due to* (5.4a) *and* $p > \frac{7}{5}$ *we have* $\mathbf{u} \in C(I, W^{1,\frac{12}{5}}(\Omega))$*. This solution is unique within* $C(I; W^{1,\frac{12}{5}}(\Omega))$*.*

Remark 5.5. Moreover, for $\delta > 0$ there exists a pressure π satisfying

$$\nabla\pi \in L^{\frac{2(5p-6)}{2-p}}(I'; L^2(\Omega)) \tag{5.6}$$

and the second time derivative satisfies

$$\mathbf{u}_{tt} \in L^2(I'; \big(W^{1,2}_{\mathrm{div}}(\Omega)\big)^*)\,, \tag{5.7}$$

with both norms bounded by a constant $c = c(\delta, p, C_0, C_1, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$, which may explode as $\delta \to 0^+$.

Proof of Theorem 5.1. We split the proof into two parts: first we treat the non-degenerate case $\delta > 0$ and then the degenerate one $\delta = 0$.

The case $\delta > 0$*:* In the non-degenerate case the proof follows exactly along the line of the corresponding statements in [20] (cf. [45]), if one replaces the lemmas there with corresponding lemmas here. We sketch the proof here in order to obtain a precise dependence of the constants with respect to δ.

The proof is obtained by an application of the Galerkin method. Let $\boldsymbol{\omega}^k$, with $k \in \mathbb{N}$, be the eigenfunctions of the Stokes operator and let λ^k be the corresponding eigenvalues. Note that $\int_\Omega \boldsymbol{\omega}^k\,dx = 0$. We denote $X_N := \mathrm{span}\{\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^N\}$ and we define the projection $P^N\mathbf{u} = \sum_{r=1}^N \int_\Omega \mathbf{u}\cdot\boldsymbol{\omega}^r\,dx\;\;\boldsymbol{\omega}^r$. Note that $P^N : W^{s,2} \to (X_N, \|\cdot\|_{s,2})$ are uniformly continuous for all $0 \le s \le 3$. As usual, we seek the Galerkin approximation $\mathbf{u}^N(t,x) = \sum_{r=1}^N c_r^N(t)\,\boldsymbol{\omega}^r(x)$, $N \in \mathbb{N}$, as the solutions of the Galerkin system (for all $1 \le r \le N$, $t \in I$)

$$\int_\Omega \mathbf{u}_t^N(t)\cdot\boldsymbol{\omega}^r + \mathbf{S}(\mathbf{Du}^N(t))\cdot\mathbf{D}\boldsymbol{\omega}^r + [\nabla\mathbf{u}^N(t)]\mathbf{u}^N(t)\cdot\boldsymbol{\omega}^r\,dx = \int_\Omega \mathbf{f}(t)\cdot\boldsymbol{\omega}^r\,dx,$$
$$\mathbf{u}^N(0) = P^N\mathbf{u}_0. \tag{5.8}$$

Existence of the function $\mathbf{u}^N$ follows from standard theory for systems of ordinary differential equations. Moreover, the Galerkin approximate functions $\mathbf{u}^N$ are sufficiently smooth due to our assumptions on the data. We can consequently use $\mathbf{u}^N$ as a test function in (5.8) to obtain the "energy" inequality

$$\|\mathbf{u}^N\|^2_{L^\infty(I;L^2(\Omega))} + \|\mathbf{u}^N\|^p_{L^p(I;W^{1,p}(\Omega))} \le c(\|\mathbf{f}\|, \|\mathbf{u}_0\|) + c(\Omega, T)\,\delta^p, \tag{5.9}$$

where we used (3.2) (with $\mathbf{B} = \mathbf{0}$), $\delta^p + t^p \simeq (\delta + t)^{p-2}t^2 + \delta^p$ (cf. [46, Remark 6.15]) and Korn's inequality. By following a well-established technique (cf. [35], [36], [20]) for (NS_p) with space periodic boundary conditions, we use $-\Delta\mathbf{u}^N$ as a test function in (5.8). Moreover, we move the term $-\int \mathbf{u}_t^N \cdot \Delta\mathbf{u}^N dx$ to the right-hand-side to get

$$\frac{C_0}{C_3} \left\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\right\|_2^2 \le \|\nabla\mathbf{u}^N\|_3^3 + \left|\int_\Omega \nabla\mathbf{f} \cdot \nabla\mathbf{u}^N \, dx\right| + \left|\int_\Omega \mathbf{u}_t^N \cdot \Delta\mathbf{u}^N \, dx\right|,$$

where we used (2.22a) and Corollary 3.12. Using the assumptions on $\mathbf{f}$, inequality (5.9), Lemma 4.7, Lemma 4.9, and Young's inequality we can estimate the right-hand-side to obtain for $q > \frac{9-3p}{p}$ and for a.e. $t \in I$

$$\begin{aligned}
&\left\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_2^2 \\
&\quad\le c_\varepsilon \Big(\|\nabla\mathbf{u}^N(t)\|_2\|\nabla\mathbf{f}(t)\|_2 + \max\{\delta^p, \delta^3\} + \left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_q^{R_1} \\
&\qquad\qquad + \|\mathbf{u}_t^N(t)\|_2^{\frac{8(p-1)}{5p-6}} \left\|\big(\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\big)_t\right\|_2^{2\frac{2-p}{5p-6}} + \left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_2^2\Big) \\
&\qquad + \varepsilon\left\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_2^2,
\end{aligned} \tag{5.10}$$

where $c_\varepsilon = c_\varepsilon(p, \Omega)$. Raising this inequality to the power $r \in [1, \frac{5p-6}{2-p})$ we obtain with Young's inequality and absorbing the last term on the right-hand-side in the left-hand-side

$$\begin{aligned}
&\left\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_2^{2r} \\
&\quad\le c_\varepsilon \Big(\|\nabla\mathbf{f}(t)\|_2^{R_6} + \max\{\delta^{pr}, \delta^{3r}\} + \left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_q^{R_7} + \|\mathbf{u}_t^N(t)\|_2^{R_8} \\
&\qquad\qquad + \|\nabla\mathbf{u}^N(t)\|_2^{R_9} + \left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_2^{2r}\Big) + \varepsilon\left\|\big(\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\big)_t\right\|_2^2,
\end{aligned} \tag{5.11}$$

where $R_6 = R_6(p)$, $R_7 = R_7(p)$, $R_8 = R_8(p)$, $R_9 = R_9(p)$, and $c_\varepsilon = c_\varepsilon(p, r, \Omega)$.

We take now the time derivative of (5.8) and, by using $\mathbf{u}_t^N$ as a test function, we arrive at the following equality:

$$\tfrac{1}{2}d_t\|\mathbf{u}_t^N\|_2^2 + \int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{u}^N))_t \cdot \mathbf{D}\mathbf{u}_t^N + \big([\nabla\mathbf{u}^N]\mathbf{u}^N\big)_t \cdot \mathbf{u}_t^N \, dx = \int_\Omega \mathbf{f}_t \cdot \mathbf{u}_t^N \, dx.$$

Using (2.22a), Lemma 4.8, Corollary 3.12 and the assumptions on $\mathbf{f}$ we get for a.e. $t \in I$ and $q > \frac{9-3p}{p}$

$$\begin{aligned}
&\frac{d}{dt}\|\mathbf{u}_t^N(t)\|_2^2 + \frac{2\,C_0}{C_3}\left\|\big(\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\big)_t\right\|_2^2 \\
&\quad\le c_\varepsilon \Big(\|\mathbf{f}_t(t)\|_2^2 + \|\mathbf{u}_t^N(t)\|_2^{\max\{2,R_2\}} + \left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_q^{R_3} + \delta^{R_4}\Big) \\
&\qquad + \varepsilon\left\|\big(\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\big)_t\right\|_2^2,
\end{aligned}$$

where $c_\varepsilon = c(\varepsilon, p, \Omega)$. Adding $\frac{d}{dt}\left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N)\right\|_q^q$ to both sides and using Lemmas 4.6 and 4.10 we thus obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\mathbf{u}_t^N(t)\|_2^2 + \frac{d}{dt}\left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_q^q + \frac{2\,C_0}{C_3}\left\|\left(\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right)_t\right\|_2^2 \\
&\le 2\,\varepsilon\left\|\left(\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right)_t\right\|_2^2 + \varepsilon\left\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_2^{2r} \\
&\quad + c\left(\|\mathbf{f}_t(t)\|_2^2 + \|\mathbf{u}_t^N(t)\|_2^{\max\{2,R_2\}} + \left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_q^{\max\{R_3,R_5\}} + \delta^{R_4}\right),
\end{aligned}
\tag{5.12}
$$

Summing up (5.11) and (5.12), and choosing ε sufficiently small we arrive, for a.e. $t \in I$, to the following

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\mathbf{u}_t^N(t)\|_2^2 + \left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_q^q\right) + \left\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_2^{2r} + \left\|\left(\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right)_t\right\|_2^2 \\
&\le c\left(\max\{\delta^{pr}, \delta^{3r}, \delta^{R_4}\} + \|\nabla\mathbf{u}^N(t)\|_2^{R_9} + \left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_q^{\max\{2,R_3,R_5,R_7\}} + \right. \\
&\quad \left. + \|\mathbf{u}_t^N(t)\|_2^{\max\{2,R_2,R_8\}} + \|\nabla\mathbf{f}(t)\|_2^{R_6} + \|\mathbf{f}_t(t)\|_2^2\right),
\end{aligned}
$$

as long as

$$
\max\left\{2, \tfrac{9-3p}{p}\right\} < q < \min\left\{4, \tfrac{6(r+1)}{3+r}\right\} \quad \text{and} \quad 1 \le r < \tfrac{5p-6}{2-p}\,,
$$

with a constant $c = c(p, r, C_0, \Omega)$. In order to control the term $\|\nabla\mathbf{u}^N\|_2$ on the right-hand-side we need to have also $q > 4/p$ since by recalling (2.20) $\left\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(t))\right\|_q + \delta^{p/2} \simeq \|\nabla\mathbf{u}^N\|_{pq/2}^{p/2} + \delta^{p/2}$.

The restrictions on q are then equivalent to

$$
\max\left\{\tfrac{4}{p}, \tfrac{9-3p}{p}\right\} < q < \min\left\{4, \tfrac{12(p-1)}{p}\right\}.
$$

One easily checks that we can find such q and r as long as $p > 7/5$. Moreover, from the assumptions on the data, the fact that $\mathbf{u}^N(0) = P^N\mathbf{u}_0$ is a solution of the Galerkin system (5.8) at time $t = 0$, and the properties of the projection P^N follows that

$$
\|\mathbf{F}(\mathbf{D}\mathbf{u}^N(0))\|_q + \|\mathbf{u}_t^N(0)\|_2 \le c(\mathbf{f}, \mathbf{u}_0).
$$

Thus, we can apply the local Gronwall lemma (cf. [20, Lemma 24]) which yields that there exists a time $T' = T'(\delta_0, p, r, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$ such that on the interval $I' := [0, T']$ we have in particular

$$
\|\mathbf{u}_t^N\|_{L^\infty(I';L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_{W^{1,2}(I'\times\Omega)} + \|\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_{L^{2r}(I';W^{1,2}(\Omega))} \le c\,,
$$

with a constant $c = c(\delta_0, p, r, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$ and $1 \le r < \frac{5p-6}{2-p}$. Using this estimate and (5.10) one can get rid of the dependence on r. Thus we finally obtain

$$
\|\mathbf{u}_t^N\|_{L^\infty(I';L^2(\Omega))} + \|\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_{W^{1,2}(I'\times\Omega)} + \|\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_{L^{2\frac{5p-6}{2-p}}(I';W^{1,2}(\Omega))} \le c\,,
\tag{5.13}
$$

with a constant $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$. From (5.13) one can pass to the limit as $N \to \infty$ and show the existence of a strong solution $\mathbf{u}$ of (NS_p) defined through

$$\mathbf{u} := \lim_{N\to\infty} \mathbf{u}^N.$$

This can be done with standard compactness results and De Giorgi's semi-continuity theorem (see e.g, [26]). By using interpolation theory one can show that the limit function $\mathbf{u}$ has the regularity properties stated in the theorem (cf. [20, Section 6] for full-details). Note that

$$\int_\Omega |\nabla(\mathbf{S}(\mathbf{Du}))|^2\,dx \le C_1 \int_\Omega (\delta + \mathbf{Du})^{2(p-2)} |\nabla \mathbf{Du}|^2\,dx \le C_1\,\delta^{p-2}\,\|\nabla \mathbf{F}(\mathbf{Du})\|_2^2\,,$$
$$\int_\Omega |(\mathbf{S}(\mathbf{Du}))_t|^2\,dx \le C_1 \int_\Omega (\delta + \mathbf{Du})^{2(p-2)} |\mathbf{Du}_t|^2\,dx \le C_1\,\delta^{p-2}\,\|(\mathbf{F}(\mathbf{Du}))_t\|_2^2\,,$$

which together with (5.3) implies (by comparison) the statements for $\mathbf{u}_{tt}$ and π in Remark 5.5. Uniqueness follows in a standard way (compare [20]) from regularity in (5.4). This finishes the proof for $\delta > 0$.

The case $\delta = 0$: In the degenerate case we cannot use directly the same tools, but we approximate the (now degenerate) system (NS_p) by the non-degenerate one

$$\begin{aligned} \mathbf{u}_t^\kappa - \operatorname{div} \mathbf{S}^\kappa(\mathbf{Du}^\kappa) + [\nabla \mathbf{u}^\kappa]\mathbf{u}^\kappa + \nabla \pi^\kappa &= \mathbf{f} \quad \text{in } I \times \Omega,\\ \operatorname{div} \mathbf{u}^\kappa &= 0 \quad \text{in } I \times \Omega,\\ \mathbf{u}^\kappa(0) &= \mathbf{u}_0 \quad \text{in } \Omega, \end{aligned} \tag{NS_p^κ}$$

where $\mathbf{S}^\kappa$ (for $0 < \kappa < 1$) is defined in (3.13). Since $\mathbf{S}^\kappa$ satisfies Assumption 1 with $\delta = 0$ replaced now by $\kappa > 0$, by using the above theory (and by recalling the independence of δ of results in section 4) we obtain that there exists a unique strong solution $\mathbf{u}^\kappa$ satisfying (5.3) uniformly in κ, i.e., for $\kappa \in (0, 1]$ we have

$$\|\mathbf{u}_t^\kappa\|_{L^\infty(I';L^2(\Omega))} + \big\|\mathbf{F}^\kappa(\mathbf{Du}^\kappa)\big\|_{W^{1,2}(I'\times\Omega)} + \big\|\mathbf{F}^\kappa(\mathbf{Du}^\kappa)\big\|_{L^{2\frac{5p-6}{2-p}}(I';W^{1,2}(\Omega))} \le c, \tag{5.14}$$

where $c = c(p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$. For each "smooth enough" function $\mathbf{v}$ we write (cf. (3.21))

$$\mathbf{F}^\kappa(\mathbf{Dv}) := \big(\kappa + |\mathbf{Dv}|\big)^{\frac{p-2}{2}} \mathbf{Dv},$$

to be able to follow exactly the dependence on κ. The limiting process $\kappa \to 0^+$ in all terms in (NS_p^κ) except the extra stress tensor $\mathbf{S}^\kappa$ is clear. Using Lemma 3.16 and Lemma 3.23 one can identify the limit

$$\lim_{n\to\infty} \int_{I'} \int_\Omega \mathbf{S}^{\kappa_n}(\mathbf{Du}^{\kappa_n}) \cdot \mathbf{Dw}\,dx\,dt \tag{5.15}$$

for smooth $\mathbf{w}$ and $\kappa_n \to 0^+$ as $n \to \infty$. Indeed, from (5.14) it follows that $\mathbf{F}^{\kappa_n}(\mathbf{Du}^{\kappa_n})$ is bounded in $W^{1,2}(I' \times \Omega)$. Thus, there exist $\mathbf{Q} \in W^{1,2}(I' \times \Omega)$

and a subsequence (labeled again $\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n})$) such that

$$
\begin{aligned}
\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n}) &\to \mathbf{Q} \qquad && \text{a.e. in } I' \times \Omega,\\
\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n}) &\rightharpoonup \mathbf{Q} && \text{in } W^{1,2}(I' \times \Omega)\\
\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n}) &\to \mathbf{Q} && \text{in } L^2(I' \times \Omega)
\end{aligned}
$$

We set $\mathbf{P} := (\mathbf{F}^0)^{-1}(\mathbf{Q})$. From Lemma 3.23 it follows that

$$
\mathbf{D}\mathbf{u}^{\kappa_n} = (\mathbf{F}^{\kappa_n})^{-1}(\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n})) \to (\mathbf{F}^0)^{-1}(\mathbf{Q}) = \mathbf{P} \qquad \text{a.e. in } I' \times \Omega.
$$

From (5.14) we obtain that $\mathbf{D}\mathbf{u}^{\kappa_n}$ is bounded in $L^p(I'; W^{1,p}(\Omega))$ and thus there exists a subsequence, labeled again $\mathbf{D}\mathbf{u}^{\kappa_n}$, which converges weakly to $\mathbf{D}\mathbf{u}$ in $L^p(I'; W^{1,p}(\Omega))$. Since weak and a.e. limit coincide we obtain that

$$
\mathbf{D}\mathbf{u}^{\kappa_n} \to \mathbf{D}\mathbf{u} = \mathbf{P} \qquad \text{a.e. in } I' \times \Omega. \tag{5.16}
$$

Lemma 3.16 and Corollary 3.22 imply now that

$$
\begin{aligned}
\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n}) &\to \mathbf{F}(\mathbf{D}\mathbf{u}) \qquad && \text{a.e. in } I' \times \Omega\,,\\
\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n}) &\to \mathbf{F}(\mathbf{D}\mathbf{u}) && \text{in } L^2(I' \times \Omega)\,,\\
\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n}) &\rightharpoonup \mathbf{F}(\mathbf{D}\mathbf{u}) && \text{in } W^{1,2}(I' \times \Omega)\,,\\
\mathbf{S}^{\kappa_n}(\mathbf{D}\mathbf{u}^{\kappa_n}) &\to \mathbf{S}(\mathbf{D}\mathbf{u}) && \text{a.e. in } I' \times \Omega\,.
\end{aligned}
$$

The limit in (5.15) is now easily identified by Vitali's convergence theorem, due to the growth condition of $\mathbf{S}^{\kappa_n}$ and (5.14). We define the solution of the degenerate problem as

$$
\mathbf{u} := \lim_{n\to\infty} \mathbf{u}^{\kappa_n}.
$$

The lower semicontinuity of the norm implies that

$$
\|\mathbf{u}_t\|_{L^\infty(I';L^2(\Omega))} + \big\|\mathbf{F}(\mathbf{D}\mathbf{u})\big\|_{W^{1,2}(I'\times\Omega)} + \big\|\mathbf{F}(\mathbf{D}\mathbf{u})\big\|_{L^{2\frac{5p-6}{2-p}}(I';W^{1,2}(\Omega))} \le c, \tag{5.17}
$$

with a constant $c = c(p, C_0, \|\mathbf{f}\|, \|\mathbf{u}_0\|, T, \Omega)$. From these results, the estimates (5.4) can be derived in the same way as for $\delta > 0$. These estimates, together with (5.16), imply also that

$$
\begin{aligned}
\mathbf{D}\mathbf{u}^{\kappa_n} &\to \mathbf{D}\mathbf{u} \qquad && \text{in } L^1(I' \times \Omega),\\
\nabla^2\mathbf{u}^{\kappa_n} &\rightharpoonup \nabla^2\mathbf{u} && \text{in } L^1(I' \times \Omega),\\
\nabla\mathbf{u}_t^{\kappa_n} &\rightharpoonup \nabla\mathbf{u}_t && \text{in } L^1(I' \times \Omega).
\end{aligned} \tag{5.18}
$$

□

6 Steady problems

The same technique introduced in the previous section can be also used to prove related results for steady space periodic problems. From the a priori estimates which can be derived by using $\mathbf{u}$ and $-\Delta\mathbf{u}$ as test function one can easily prove existence results for strong solutions. In the degenerate case one can use the same approximation $\{\mathbf{S}^\kappa\}_{\kappa>0}$ as above.

We consider two steady problems: the p-Navier-Stokes and a quasi-p-Oseen problem. The first system is the natural one for fluids with shear dependent viscosity, while the study of the second one is motivated by the fact that it is needed in the error analysis of time discretization problems, we performed in [11]. The first result we prove is the following.

Theorem 6.1. *Let us consider the steady problem*

$$\begin{aligned} -\operatorname{div}\mathbf{S}(\mathbf{Du}) + [\nabla\mathbf{u}]\mathbf{u} + \nabla\pi &= \mathbf{f} \qquad && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, \end{aligned} \tag{6.2}$$

with space periodic boundary conditions. Let the extra stress tensor $\mathbf{S}$ *satisfy Assumption 1 with* $p \in \left(\frac{9}{5}, 2\right]$ *and* $\delta \in [0, \delta_0]$ *for some* $\delta_0 > 0$ *and assume that* $\mathbf{f} \in W^{1,2}(\Omega)$*. Then, the system* (6.2) *has a strong solution* $\mathbf{u} \in W^{1,p}_{\operatorname{div}}(\Omega)$*, satisfying for all* $\boldsymbol{\varphi} \in W^{1,p}_{\operatorname{div}}(\Omega)$

$$\int_\Omega \mathbf{S}(\mathbf{Du}) \cdot \mathbf{D}\boldsymbol{\varphi} + [\nabla\mathbf{u}]\mathbf{u} \cdot \boldsymbol{\varphi}\, dx = \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\, dx,$$

and

$$\|\mathbf{F}(\mathbf{Du})\|_{W^{1,2}(\Omega)} \le c,$$

with a constant $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \Omega)$*. In particular, we have*

$$\mathbf{u} \in W^{2,\frac{3p}{p+1}}(\Omega),$$

with norm bounded by a constant $c = c(\delta_0, p, C_0, \|\mathbf{f}\|, \Omega)$*.*

Moreover, for $\delta > 0$ *there exists a pressure* π *satisfying*

$$\nabla\pi \in L^2(\Omega)$$

with norm bounded by a constant $c = c(\delta, p, C_0, C_1, \|\mathbf{f}\|, \Omega)$*, which may explode as* $\delta \to 0^+$*.*

Proof. We only sketch the proof of this result, since it can be obtained by following exactly the same arguments of the previous one. First, we observe that weak solutions have gradients in $L^p(\Omega)$. In fact, by employing a Galerkin approximation and using $\mathbf{u}^N$ as test function we get (independently of the value of $\delta \ge 0$)

$$\|\mathbf{Du}^N\|_p^p \le c\left(\|\mathbf{f}\|_2^{p'} + \delta^p\right).$$

To prove existence of strong solutions, for $\delta > 0$, we consider again the Galerkin system, and after using $-\Delta\mathbf{u}^N$ as test function and performing suitable integration by parts, we estimate the integral coming from the convection term as follows (by definition of $\mathbf{F}$ and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$)

$$\begin{aligned}
\|\nabla \mathbf{u}^N\|_3^3 &\leq c\,\|\mathbf{D}\mathbf{u}^N\|_p^{3\theta}\|\mathbf{D}\mathbf{u}^N\|_{3p}^{3(1-\theta)} \\
&\leq c\,\|\mathbf{D}\mathbf{u}^N\|_p^{3\theta}\big(\|\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_6^{\frac{6(1-\theta)}{p}} + \delta^{3(1-\theta)}\big) \\
&\leq c\,\|\mathbf{D}\mathbf{u}^N\|_p^{3\theta}\big(\|\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_2^{\frac{6(1-\theta)}{p}} + \|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_2^{\frac{6(1-\theta)}{p}} + \delta^{3(1-\theta)}\big),
\end{aligned}$$

where $\theta = (p-1)/2$. By using Corollary 3.12, the equivalence $\|\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_2 + \delta^{p/2} \simeq \|\mathbf{D}\mathbf{u}^N\|_p^{\frac{p}{2}} + \delta^{p/2}$, and also that $\|\mathbf{D}\mathbf{u}^N\|_p$ is bounded uniformly in terms of the data, we consequently get the following a priori estimate

$$\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_2^2 \leq c\,\big\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\big\|_2^{\frac{6(1-\theta)}{p}}$$

with $c = (p, C_0, C_1, \|\mathbf{f}\|, \Omega, \delta_0)$, and, in order to absorb $\big\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\big\|_2$ in the left-hand-side, we need

$$\frac{6(1-\theta)}{p} < 2 \Longleftrightarrow p > \frac{9}{5}.$$

Passing to the limit for $N \to \infty$ is done with standard compactness tools.

In the degenerate case $\delta = 0$ we use the approximation technique of the previous section and we prove the existence of $\mathbf{u}^\kappa$ (with bounds independent of κ) for the approximate problem with $\mathbf{S}$ replaced by $\mathbf{S}^\kappa$. The limiting process $\kappa \to 0^+$ follows exactly as in the previous section. $\square$

Since the problem is steady we cannot use $\mathbf{u}_{tt}$ as a test function. This results in a narrower range of admissible p compared to the time evolution problem.

We consider now an Oseen-like problem and have the following result.

Theorem 6.3. *Let us consider the steady problem*

$$\begin{aligned}
\mathbf{u} - \operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{u}) + [\nabla\mathbf{u}]\mathbf{v} + \nabla\pi &= \mathbf{f} \qquad &&\text{in } \Omega, \\
\operatorname{div}\mathbf{u} &= 0 &&\text{in } \Omega,
\end{aligned} \tag{6.4}$$

with space periodic boundary conditions. Let the extra stress tensor $\mathbf{S}$ *satisfy Assumption 1 with* $p \in \left(\frac{7}{5}, 2\right]$ *and* $\delta \in [0, \delta_0]$ *for some* $\delta_0 > 0$. *Assume that* $\mathbf{f} \in W^{1,2}(\Omega)$ *and that* $\mathbf{v} \in W^{1,3p}_{\operatorname{div}}(\Omega)$ *are given. Then, the system (6.4) has a strong solution* $\mathbf{u} \in W^{1,p}_{\operatorname{div}}(\Omega)$, *satisfying for all* $\boldsymbol{\varphi} \in W^{1,p}_{\operatorname{div}}(\Omega)$

$$\int_\Omega \mathbf{u}\cdot\boldsymbol{\varphi} + \mathbf{S}(\mathbf{D}\mathbf{u})\cdot\mathbf{D}\boldsymbol{\varphi} + [\nabla\mathbf{u}]\mathbf{v}\cdot\boldsymbol{\varphi}\,dx = \int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}\,dx,$$

and

$$\|\nabla\mathbf{u}\|_2^2 + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{W^{1,2}(\Omega)}^2 \leq c,$$

with a constant $c = c(\delta_0, p, C_0, \|\mathbf{v}\|, \|\mathbf{f}\|, \Omega)$. *In particular, we have*

$$\mathbf{u} \in W^{2,\frac{3p}{p+1}}(\Omega),$$

with norm bounded by a constant $c = c(\delta_0, p, C_0, \|\mathbf{v}\|, \|\mathbf{f}\|, \Omega)$. *This solution is unique within the class* $W^{1,p}_{\mathrm{div}}(\Omega)$ *for* $p > \frac{3}{2}$ *and within the class* $W^{1,3p}_{\mathrm{div}}(\Omega)$ *for* $p > \frac{7}{5}$.

Moreover, for $\delta > 0$ *there exists a pressure* π *satisfying*

$$\nabla\pi \in L^2(\Omega)$$

with norm bounded by a constant $c = c(\delta, p, C_0, C_1, \|\mathbf{v}\|, \|\mathbf{f}\|, \Omega)$*, which may explode as* $\delta \to 0$.

Proof. Again we give the only basic steps of the proof, since details are similar to the previous results. We show the a priori estimate for the Galerkin solutions $\mathbf{u}^N$. First we get (independently of the value of $\delta \geq 0$) the estimate obtained by testing with $\mathbf{u}^N$

$$\|\mathbf{u}^N\|_2^2 + \|\mathbf{D}\mathbf{u}^N\|_p^p \leq c\left(\|\mathbf{f}\|_2^{p'} + \delta^p\right).$$

For $\delta > 0$, to prove the existence of strong solution we use $-\Delta\mathbf{u}^N$ as test function and perform integrations by parts. By using Corollary 3.12 we get

$$\begin{aligned}\|\nabla\mathbf{u}^N\|_2^2 + \left\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\right\|_2^2 &\leq c\left(\|\mathbf{f}\|_{W^{1,2}}^2 + \int_\Omega |\nabla\mathbf{v}|\,|\nabla\mathbf{u}^N|^2\,dx\right)\\ &\leq c\left(\|\mathbf{f}\|_{W^{1,2}}^2 + \|\nabla\mathbf{v}\|_{3p}\|\nabla\mathbf{u}^N\|_{\frac{6p}{3p-1}}^2\right).\end{aligned}$$

With Korn's inequality, interpolation $L^{\frac{6p}{3p-1}} = [L^p, L^{\frac{3p}{3-p}}]_\theta$ (which is possible for $p > \frac{7}{5}$), and a Sobolev embedding, we obtain

$$\begin{aligned}\|\nabla\mathbf{v}\|_{3p}\|\nabla\mathbf{u}^N\|_{\frac{6p}{3p-1}}^2 &\leq c\,\|\mathbf{D}\mathbf{v}\|_{3p}\|\mathbf{D}\mathbf{u}^N\|_p^{\frac{5p-7}{p}}\,\|\mathbf{D}\mathbf{u}^N\|_{\frac{3p}{3-p}}^{\frac{7-3p}{p}}\\ &\leq c_\varepsilon\,\|\mathbf{D}\mathbf{v}\|_{3p}^{\frac{2p}{5p-7}}\,\|\mathbf{D}\mathbf{u}^N\|_p^2 + \varepsilon\,\|\mathbf{D}\mathbf{u}^N\|_{\frac{3p}{3-p}}^2\\ &\leq c_\varepsilon\,\|\mathbf{D}\mathbf{v}\|_{3p}^{\frac{2p}{5p-7}}\,\|\mathbf{D}\mathbf{u}^N\|_p^2 + \varepsilon\,c\,\|\nabla\mathbf{D}\mathbf{u}^N\|_p^2.\end{aligned}$$

Finally, by using Lemma 4.5 (first and second inequality with $s = p$), by recalling the uniform bound for $\|\mathbf{D}\mathbf{u}^N\|_p$, by using Korn's inequality, $\delta \in (0, \delta_0]$, and $p \in (1, 2]$, we get

$$\begin{aligned}&\|\nabla\mathbf{v}\|_{3p}\|\nabla\mathbf{u}^N\|_{\frac{6p}{3p-1}}^2\\ &\leq\; c_\varepsilon\,\|\mathbf{D}\mathbf{v}\|_{3p}^{\frac{2}{5p-7}}\,\|\mathbf{D}\mathbf{u}^N\|_p^2 + \varepsilon\,c\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_2^2\left(\delta + \|\nabla\mathbf{u}^N\|_p\right)^{2-p}\\ &\leq c_\varepsilon\,\|\mathbf{D}\mathbf{v}\|_{3p}^{\frac{2}{5p-7}} + \varepsilon\,c\,\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\|_2^2,\end{aligned}$$

with $c = c(\delta, p, C_0, C_1, \|\mathbf{v}\|, \|\mathbf{f}\|, \Omega)$. By choosing ε small enough we can absorb the last term in the left-hand-side to get

$$\|\nabla\mathbf{u}^N\|_2^2 + \left\|\nabla\mathbf{F}(\mathbf{D}\mathbf{u}^N)\right\|_2^2 \leq c.$$

This last estimate shows the a priori bound which can be used (passing to the limit as in the previous theorem) to show the existence of the strong solution $\mathbf{u}$ to problem (6.4). Moreover, the fact that the quantity $\|\nabla \mathbf{u}\|_2^2 + \|\nabla \mathbf{F}(\mathbf{D}\mathbf{u})\|_2^2$ is bounded uniformly for $\delta \in (0, \delta_0]$ can be employed, by the same approximation technique, to show existence also in the degenerate case. Details can be easily fixed by using the machinery of the previous section.

The uniqueness result is easily obtained by testing the difference of the equations by the difference of the solutions. The bounds for p are due to the justification of the computations. $\square$

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References

[1] E. Acerbi, G. Mingione, and G. A. Seregin. Regularity results for parabolic systems related to a class of non-Newtonian fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(1):25–60, 2004.

[2] H. Amann. Stability of the rest state of a viscous incompressible fluid. *Arch. Rat. Mech. Anal.*, 126:231–242, 1994.

[3] H.O. Bae. Existence and regularity of solutions of non-Newtonian flow. *Quart. Appl. Math.*, 58(2):379–400, 2000.

[4] W. Bao and J.W. Barrett. A priori and a posteriori error bounds for a nonconforming linear finite element approximation of a non-Newtonian flow. *RAIRO Modél. Math. Anal. Numér.*, 32:843–858, 1998.

[5] J. Barrett and W. B. Liu. Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow. *Numer. Math.*, 68(4):437–456, 1994.

[6] J. Barrett and W. B. Liu. Finite element approximation of the parabolic p-Laplacian. *SIAM J. Numer. Anal.*, 31:413–428, 1994.

[7] H. Beirão da Veiga. Concerning the Ladyzhenskaya-Smagorinsky turbulence model of the Navier-Stokes equations. *C. R. Math. Acad. Sci. Paris*, 345(5):249–252, 2007.

[8] H. Beirão da Veiga. Navier-Stokes equations with shear thinning viscosity. regularity up to the boundary. *J. Math. Fluid Mech.*, 2008.

[9] H. Bellout, F. Bloom, and J. Nečas. Young measure-valued solutions for non-Newtonian incompressible fluids. *Comm. PDE*, 19:1763–1803, 1994.

[10] L. C. Berselli. On the $W^{2,q}$-regularity of incompressible fluids with shear-dependent viscosities: The shear-thinning case. *J. Math. Fluid Mech.*, 2008.

[11] L. C. Berselli, L. Diening, and M. Růžička. Optimal estimates for a semi implicit euler scheme for incompressible fluids with shear dependent viscosities. Technical Report 7, University of Pisa, Department of Applied Mathematics, 2008.

[12] R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamic of Polymer Liquids*. John Wiley, 1987. 2nd edition.

[13] D. Bothe and J. Prüss. L^p-theory for a class of Non–Newtonian fluids. *SIAM J. Math. Anal.*, 39(2):379–421, 2007.

[14] L. Diening, C. Ebmeyer, and M. Růžička. Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure. *SIAM J. Numer. Anal.*, 45:457–472, 2007.

[15] L. Diening and F. Ettwein. Fractional estimates for non–differentiable elliptic systems with general growth. *Forum Mathematicum*, (3):523–556, 2008.

[16] L. Diening and Ch. Kreuzer. Linear convergence of an adaptive finite element method for the p-Laplacian equation. *SIAM J. Numer. Anal.*, 46:614–638, 2008.

[17] L. Diening, J. Málek, and M. Steinhauer. On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. *ESAIM: Control, Optimisation and Calculus of Variations*, pages –, 2008. DOI: 10.1051/cocv:2007049.

[18] L. Diening, A. Prohl, and M. Růžička. On time-discretizations for generalized newtonian fluids. In M. Sh. Birman, S. Hildebrandt, V. Solonnikov, and N.N. Uraltseva, editors, *Nonlinear Problems in Mathematical Physics and Related Topics II*, pages 89–118. Kluwer/Plenum, New York, 2002. In Honour of Professor O.A. Ladyzhenskaya.

[19] L. Diening, A. Prohl, and M. Růžička. Semi–implicit euler scheme for generalized Newtonian fluids. *SIAM J. Numer. Anal.*, 42:1172–1190, 2006.

[20] L. Diening and M. Růžička. Strong solutions for generalized Newtonian fluids. *J. Math. Fluid Mech.*, 7:413–450, 2005.

[21] L. Diening and M. Růžička. Error estimates for interpolation operators in Orlicz–Sobolev spaces. *Num. Math.*, 107:107–129, 2007. DOI 10.1007/s00211-007-0079-9.

[22] L. Diening, M. Růžička, and J. Wolf. Existence of weak solutions for unsteady motions of generalized newtonian fluids. Technical Report 2, University of Freiburg, Department of Mathematics and Physics, 2008.

[23] J. Frehse, J. Málek, and M. Steinhauer. On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. *SIAM J. Math. Anal.*, 34(5):1064–1083 (electronic), 2003.

[24] M. Fuchs and G. Seregin. *Variational methods for problems from plasticity theory and for generalized Newtonian fluids*, volume 1749 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[25] M. Fuchs and G. Seregin. A global nonlinear evolution problem for generalized Newtonian fluids: local initial regularity of the strong solution. *Comput. Math. Appl.*, 53(3-4):509–520, 2007.

[26] M. Giaquinta, G. Modica, and J. Souček. *Cartesian currents in the calculus of variations. II*, volume 38 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer, Berlin, 1998.

[27] E. Giusti. *Metodi Diretti nel Calcolo delle Variazioni.* Unione Matematica Italiana, Bologna, 1994.

[28] P. Kaplický. Regularity of flows of a non-Newtonian fluid subject to Dirichlet boundary conditions. *Z. Anal. Anwendungen*, 24(3):467–486, 2005.

[29] P. Kaplický. Time regularity of flows of non-Newtonian fluids. *IASME Trans.*, 2(7):1232–1236, 2005.

[30] P. Kaplický, J. Málek, and J. Stará. Global-in-time Hölder continuity of the velocity gradients for fluids with shear-dependent viscosities. *Nonlinear Differ. Equ. Appl. (NoDEA)*, 9:175–195, 2002.

[31] M. A. Krasnosel′skiĭ and J. B. Rutickiĭ. *Convex functions and Orlicz spaces.* Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961.

[32] O. A. Ladyzhenskaya. New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. *Proc. Stek. Inst. Math.*, 102:95–118, 1967.

[33] O. A. Ladyzhenskaya. On some modifications of the Navier-Stokes equations for large gradients of velocity. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)*, 7:126–154, 1968.

[34] O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York, 1969. 2nd edition.

[35] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computations.* Chapman & Hall, London, 1996.

[36] J. Málek, J. Nečas, and M. Růžička. On the non-Newtonian incompressible fluids. *Math. Models Methods Appl. Sci.*, 3:35–63, 1993.

[37] J. Málek, J. Nečas, and M. Růžička. On weak solutions to a class of non–Newtonian incompressible fluids in bounded three-dimensional domains. the case $p \geq 2$. *Adv. Diff. Equ.*, 6:257–302, 2001.

[38] J. Málek and K. R. Rajagopal. Mathematical issues concerning the Navier-Stokes equations and some of its generalizations. In *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pages 371–459. Elsevier/North-Holland, Amsterdam, 2005.

[39] J. Málek, K. R. Rajagopal, and M. Růžička. Existence and regularity of solutions and the stability of the rest state for fluids with shear dependent viscosity. *Math. Models Methods Appl. Sci.*, 5:789–812, 1995.

[40] J. Musielak. *Orlicz Spaces and Modular Spaces*. Springer, Berlin, 1983.

[41] A. Prohl and M. Růžička. On fully implicit space-time discretization for motions of incompressible fluids with shear dependent viscosities: The case $p \leq 2$. *SIAM J. Num. Anal.*, 39:214–249, 2001.

[42] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1991.

[43] M. Růžička. Flow of shear dependent electrorheological fluids: unsteady space periodic case. In A. Sequeira, editor, *Applied nonlinear analysis*, pages 485–504. Kluwer/Plenum, New York, 1999.

[44] M. Růžička. *Electrorheological Fluids: Modeling and Mathematical Theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer, 2000.

[45] M. Růžička. Modeling, mathematical and numerical analysis of electrorheological fluids. *Appl. Math.*, 49:565–609, 2004.

[46] M. Růžička and L. Diening. Non–Newtonian fluids and function spaces. In *Proceedings of NAFSA 2006, Prague*, volume 8, pages 95–144, 2007.

[47] T. N. Shilkin. Regularity up to the boundary of solutions to boundary-value problems of the theory of generalized Newtonian liquids. *J. Math. Sci. (New York)*, 92(6):4386–4403, 1998. Some questions of mathematical physics and function theory.

[48] J. S. Smagorinsky. General circulation experiments with the primitive equations. *Mon. Weather Review*, 91:99–164, 1963.

[49] J. Wolf. Existence of weak solutions to the equations of nonstationary motion of non–Newtonian fluids with shear–dependent viscosity. *J. Math. Fluid Mech.*, 9:104–138, 2007.