1.1 Ultraviolet divergences due to the photon field (Version 160815)

In this section we will take a look at the so-called "ultraviolet" problem of the photons. In order to focus solely on this problem we consider a very simple toy model of QED. When cutting off ultraviolet frequencies above some cut-off parameter in the interaction, the corresponding equation of motion can be treated in mathematical rigorous terms. Thanks to its simplicity, it can be solved explicitly, and hence, allows to observe the manifestation of the ultraviolet divergences when the cut-off parameter is send to infinity.

We will make two observations:

1. There is a ultraviolet problem due to the self-interaction of the electrons, which renders the corresponding Hamiltonian ill-defined.
2. There is another mathematical problem due to the representation of solutions which after removal of the cut-off do not lie anymore in standard Fock space.

The first problem is inherited from classical electrodynamics of point-charges and is therefore of conceptual nature and deeply anchored in the way we introduce electrodynamic interaction. The second problem is rather man-made and can be circumvented by an adaption of our Fock space representation.

We start to study the difficulties related to photon field in a toy model.

from $H_{QED} \longrightarrow H_{Toy}$

Simplifications:

- A fixed number of electrons at fixed positions
- No pair creation
- No spin degrees of freedom for both the electron and the photon
- Photon may have mass $\mu \geq 0$

$$H_{QED} = \int d^3x\, \overline{\psi}(0,\underline{x})\left(-i\,\underline{\gamma}\cdot\nabla_{\underline{x}} + m\right)\psi(0,\underline{x}) + \sum_{\lambda}\int d^3k\, \overset{\omega_k = |\underline{k}|}{\omega_k}\, a^*_{\underline{k},\lambda}\, a_{\underline{k},\lambda}$$

$$+\, e\int d^3x\, \overline{\psi}(0,\underline{x})\, \not{A}(0,\underline{x})\, \psi(0,\underline{x})$$

$$\xrightarrow{\text{simplifications}} H_{TOY} = \underset{\substack{\text{number of}\\ \text{nucleons}}}{N}\cdot \overset{\text{rest mass}}{m} + \underbrace{\int d^3k\, \overset{\omega_k=\sqrt{\mu^2+k^2}}{\omega_k}\, a_k^+ a_k}_{=H_f} + \overset{\text{coupling const.}}{g}\sum_{k=1}^{N}\varphi(0,x_j)$$

goal: make sense of corr. Schrödinger eq.

$$\varphi(0,x_j) = \int d^3k\, \gamma_k\left(a_k e^{ikx_j} + a_k^+ e^{-ikx_j}\right), \qquad \gamma_k = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}$$

HW: What is the corresponding classical Hamiltonian?

HW: What determines the form of γ_k? $[\varphi(x),\varphi(y)] = \Delta_{\text{Pauli-Jordan}}(x-y)$!

- In which sense are the creation and annihilation operators defined? On which space do they act?

Fock space

DEF: $\mathcal{F}_b = \bigoplus_{n=0}^{\infty} \mathcal{H}_b^n$, $\mathcal{H}_b^n = L^2(\mathbb{R}^3,\mathbb{C})^{\odot n} = L^2(\mathbb{R}^3,\mathbb{C})\odot \dots \odot L^2(\mathbb{R}^3,\mathbb{C})$

$\Psi \in \mathcal{F}_b$: $\Psi = (\psi_n)_{n\in\mathbb{N}_0}$ for $\psi_0 \in \mathbb{C}$, $\psi_1 \in L^2(\mathbb{R}^3,\mathbb{C})$, $\psi_2 \in L^2(\mathbb{R}^3,\mathbb{C})^{\odot 2}$, ...

scalar product $\quad \langle \Psi, \Phi\rangle = \sum_{n=0}^{\infty} \langle \psi_n, \phi_n\rangle_{\mathcal{H}_b^n}$

induced norm $\quad \|\Psi\|^2_{\mathcal{F}_b} = \sum_{n=0}^{\infty}\int d^3x_1 \dots \int d^3x_n\, |\psi_n(x_1,\dots,x_n)|^2$

THM [Simon/Reed I]: $\mathcal{F}_b$ is a seperable complex Hilbert space, i.e., a complete inner product space with a countable basis.

HW

the creation/annihilation operators are given by

omitted

$$(a^*(f)\Psi)_n(x_1,\dots,x_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(x_j)\,\Psi_{n-1}(x_1,\dots,\hat{x}_j,\dots,x_n)$$

$$(a(f)\Psi)_n(x_1,\dots,x_n) = \sqrt{n+1}\int d^3x_{n+1}\, f^*(x_{n+1})\,\Psi_{n+1}(x_1,\dots,x_n,x_{n+1})$$

$a(f)1 = 0$ we usually write $\Omega = 1$

a_k, a_k^+

clearly, for $\Psi \in \mathcal{F}_b$: $a^*(f)\Psi \in \mathcal{F}_b \implies f \in \mathcal{H}$

therefore, for all $f, g \in \mathcal{H}$:

$$a^*(f): \mathcal{H}^{\otimes n} \longrightarrow \mathcal{H}^{\otimes n+1}$$
$$a(f): \mathcal{H}^{\otimes n} \longrightarrow \mathcal{H}^{\otimes (n-1)}$$

restriction to the n-particle sector

$$[a(f), a^*(g)] = \langle f, g\rangle_{\mathcal{H}}$$
$$[a^\#(f), a^\#(g)] = 0$$

Back to our toy model for which we would like to define the corresponding equation of motion:

$$i\partial_t \Psi_t = H\Psi_t \tag{2.1}$$

for which we would have to show that H is self-adjoint on $\mathcal{F}_b$.

H_g self-adj.

THM [Simon/Reed II]: H_g is self-adj. on $\mathcal{D}(H_g) = \{\Psi \in \mathcal{F}_b \mid H_g\Psi \in \mathcal{F}_b\}$.

HW

but the interaction Hamiltonian is not even defined in the sense of above

but H_{Toy} not even well-def.

- $\int d^3k\, \gamma_k\, a_k^*\, e^{-ikx} = a^*(\gamma_k e^{-ikx})$

but $\|\gamma_k e^{-ikx}\|_{\mathcal{H}} = \|\underset{|k|\to\infty}{\mathcal{O}}(\tfrac{1}{|k|})\|_{\mathcal{H}} = \infty$!

$\Rightarrow \varphi(x)$ ill-defined !

Introduction of UV cut-off

This is the first occurrence of the so-called ultraviolet divergence.
Mathematical remedy:

$\gamma_k \longrightarrow \gamma_k^\Lambda = \dfrac{g_\Lambda(k)}{\sqrt{2\omega_k}}$ s.t. i) for $\Lambda < \infty$: γ_k^Λ Schwartz function

ii) $g^\Lambda(k) \xrightarrow[\Lambda\to\infty]{} \dfrac{1}{(2\pi)^{3/2}}$

HW: Position space interpretation?

<u>THM:</u> Let $\Lambda < \infty$ and $g < 1$, then $H^\Lambda = Nm + H_f + g\sum_{j=1}^{N} \varphi_\Lambda(0,x)$ is self-adjoint on $D(H_f)$ where $\varphi_\Lambda(x) = g\int d^3k\, \gamma_k^\Lambda \left(a_k e^{+ikx} + a_k^+ e^{-ikx}\right)$.

H^Λ self-adj.

<u>Proof:</u> Application of KLMN theorem [Reed/Simon I]:

Let A be a pos. self-adj. operator, $\beta(\Phi,\Psi)$ a symmetric quadratic form def. $\forall\, \Psi,\Phi \in D(A^{1/2})$ s.t. $\exists\, a<1,\ b<\infty$ with

$$|\beta(\Psi,\Psi)| \le a\langle\Psi, A\Psi\rangle + b\langle\Psi,\Psi\rangle \quad \forall\, \Psi \in D(A^{1/2}) \qquad \boxed{3.1}$$

Then $\exists!$ self-adjoint operator C with $D(C) \subseteq D(A^{1/2})$ and

$$\langle\Psi, C\Psi\rangle = \langle\Psi, A\Psi\rangle + \beta(\Psi,\Psi).$$

- Our setting $A = H_f$, $\beta(\Psi,\Psi) = \langle\Psi, \sum_{j=1}^{N} g\,\varphi(x_j)\Psi\rangle \quad \forall\, \Psi \in D(H_f^{1/2})$

$$|\langle\Psi, \sum_{j=1}^{N} g\,\varphi(x_j)\Psi\rangle| \le g\sum_{j=1}^{N} |\langle\Psi, \int d^3k\, \gamma_k^\Lambda a_k e^{ikx}\Psi\rangle| + g\sum_{j=1}^{N} |\langle \int d^3k\, \gamma_k^\Lambda a_k e^{ikx}\Psi, \Psi\rangle|$$

HW: $|\langle\Psi, \int d^3k\, f(k) a_k \Psi\rangle| \le \|\Psi\|_{\mathcal{F}} \left\|\frac{f}{\sqrt{\omega}}\right\|_{\mathfrak{h}} \|H_f^{1/2}\Psi\|_{\mathcal{F}} \le \left\|\frac{f}{\sqrt{\omega}}\right\|_{\mathfrak{h}}^2 \langle\Psi,\Psi\rangle + \langle\Psi, H_f\Psi\rangle$

$$\le g\left[\langle\Psi, H_f\Psi\rangle + N^2 \left\|\frac{\gamma_k^\Lambda}{\sqrt{\omega}}\right\|_{\mathfrak{h}}^2 \langle\Psi,\Psi\rangle\right]$$

HW: Can this be done for operators (Kato's theorem) instead of forms? Advantage/Disadv.?

Which allows to fulfill $\boxed{3.1}$ for $g < 1$. $\square$

by Stone's theorem:

time evolution operators generated by H_g, H^Λ

COR: $\exists!$ unitary group $U^\Lambda(t)$, $t\in\mathbb{R}$, generated by H^Λ.

Likewise, $\exists!$ unitary group $U_0(t)$, $t\in\mathbb{R}$, generated by H_g.

HW: Connection btw. self-adj. and the initial value problem

Hence, $\forall\ \Psi_0 \in D(H^\Lambda)$ [2.1] is fulfilled for $\Psi_t = U^\Lambda(t)\Psi_0$.

By introducing the cut-off γ_Λ we established a well-posed initial value problem for this toy model.

Let us understand why the removal of the cut-off causes problems:

generated fields

In the interaction picture: $i\partial_t \tilde{\Psi}_t = H_I(t)\,\Psi_t$

$$\text{for } H_I^\Lambda(t) = U_0(s)^* H_I^\Lambda U_0(s) = g\sum_{j=1}^N \int d^3k\, \gamma_k^\Lambda \left(a_k e^{ikx_j - i\omega_k s} + a_k^+ e^{-ikx_j + i\omega_k s}\right)$$

For test states $\Phi \in \mathcal{C}_c^\infty$:

$$U_0^\Lambda(s)^* a_k\, U_0^\Lambda(s) = e^{-i\omega_k s} a_k$$

- $$\lim_{t_0\to-\infty} \underbrace{\langle \Phi, U_\Lambda(0,t_0)\Omega\rangle}_{} = ?$$

HW: Why convergent?

$$= \langle \Phi, \sum_{n=0}^\infty \frac{(-i)^n}{n!}\int_{t_0}^0 dt_1 \cdots \int_{t_0}^0 dt_n\, T\, H_I^\Lambda(t_1)\cdots H_I^\Lambda(t_n)\,\Omega\rangle$$

T: time ordered

$$= \sum_{n=0}^\infty \frac{(-i)^n}{n!}\langle \Phi, T\left(\int_{t_0}^0 ds\ g\sum_{j=1}^N \int d^3k\, \gamma_k^\Lambda\left(a_k e^{ikx_j - i\omega_k s} + a_k^+ e^{-ikx_j + i\omega_k}\right)\right)^n \Omega\rangle$$

$$= \langle \Phi,\Omega\rangle - i\underbrace{\langle \Phi, \int_{t_0}^0 ds\ g\sum_{j=1}^N \int d^3k\, \gamma_k^\Lambda\, a_k^+ e^{-ikx_j + i\omega_k s}\ \Omega\rangle}_{} + \mathrm{Rest}(g^2)$$

$$= -ig\sum_{j=1}^N \sum_{m=0}^\infty \int dk_1\ldots d^3k_m\, \phi^*(k_1\ldots k_m)\int_{t_0}^0 ds\, \langle \Omega, a_{k_1}\cdots a_{k_m} \int d^3k\, \gamma_k^\Lambda a_k^+ e^{-ikx_j + i\omega_k s}\ \Omega\rangle$$

$$= -ig\sum_{j=1}^N \int dk\, \phi^*(k)\int_{t_0}^0 ds\, \gamma_k^\Lambda e^{-ikx_j + i\omega_k s} = -ig\sum_{j=1}^N \int d^3k\, \phi^*(k)\,\frac{\gamma_k^\Lambda}{i\omega_k}\, e^{-ikx_j}\left(1 - e^{+i\omega_k t_0}\right)$$

$$\xrightarrow[t_0\to\infty]{} -g\sum_{j=1}^{N}\int d^3k\, \phi^*(k)\,\frac{\gamma_k^\Lambda}{\omega_k}\,e^{-ikx_j} \qquad \text{using a stationary phase argument}$$

By induction, one can give a closed form for all orders:

$$\langle \Phi, U_\Lambda(0,t_0)\Omega\rangle = \langle \Phi, \underbrace{\sum_{n=0}^{\infty}\frac{(-g)^n}{n!}\Big(\sum_{j=1}^{N}\int d^3k\,\frac{\gamma_k^\Lambda}{\omega_k}\big(a_k^+ e^{-ikx_j} - a_k e^{+ikx_j}\big)\Big)^n}\,\Omega\rangle$$

$$\mathrm{Dress}_\Lambda = \exp\Big(-g\sum_{j=1}^{N}\int d^3k\,\frac{\gamma_k^\Lambda}{\omega_k}\big(a_k^+ e^{-ikx_j} - a_k e^{+ikx_j}\big)\Big)$$

One can check that $a_k\,\mathrm{Dress}_\Lambda\Omega = \underbrace{-g\sum_{j=1}^{N}\int d^3k\,\frac{\gamma_k^\Lambda}{\omega_k}\,e^{-ikx_j}}_{=f(k)}\ \mathrm{Dress}_\Lambda\Omega$

and hence

$$H_\Lambda \mathrm{Dress}_\Lambda\Omega = \Big[N\cdot m + \int d^3k\,\omega_k a_k^+ a_k + \sum_{j=1}^{N} g\int d^3k\,\gamma_k^\Lambda\big(a_k e^{ikx_j} + a_k^+ e^{-ikx_j}\big)\Big]\mathrm{Dress}_\Lambda\Omega$$

$$= \Big[N\cdot m + \int d^3k\,\omega_k a_k^+ f(k) + \sum_{j=1}^{N} g\int d^3k\,\gamma_k^\Lambda\big(f(k)e^{ikx_j} + a_k^+ e^{-ikx_j}\big)\Big]\mathrm{Dress}_\Lambda\Omega$$

$$= \Big(N\cdot m - \sum_{j,i=1}^{N} g^2 \underbrace{\int d^3k\,\frac{(\gamma_k^\Lambda)^2}{\omega_k}\,e^{ik(x_j-x_i)}}\Big)\mathrm{Dress}_\Lambda\Omega \qquad V_{Yukawa}(x) = \frac{e^{-\mu|x|}}{|x|}$$

$$= \frac{1}{(2\pi)^3}\int d^3k\,\frac{\check{\xi}_\Lambda(k)}{2(k^2+\mu^2)}\,e^{ik(x_j-x_i)} = \frac{1}{2}\,\check{\xi}_\Lambda * V_{Yukawa}(x_i-x_j)$$

$$= \Big[N\cdot m \underbrace{- g^2\sum_{i<j}^{N}\check{\xi}_\Lambda * V(x_i-x_j)}_{\text{Potentials due to interaction between the charges}} \underbrace{- g^2\,\frac{N}{2}\,\check{\xi}_\Lambda * V(0)}_{\text{self-interaction}}\Big]\mathrm{Dress}_\Lambda\Omega$$

What happens if we remove the cut-off?

removal of UV cut-off

$$\check{\xi}_\Lambda(x) \longrightarrow \delta^3(x) \quad\Rightarrow\quad \check{\xi}_\Lambda * V(x) \longrightarrow V(x)$$

and therefore the self-interaction blows up $\check{\xi}_\Lambda * V(0) \xrightarrow[\Lambda\to\infty]{} \infty$

Hence, it is not surprising that the Hamiltonian can not be defined without a cut-off.

A simple way to mend the problem is to make the rest energy cut-off dependent and absorb the divergence with it:

$$m_\Lambda = m_{ren} + g^2 \frac{1}{2} \check{\delta}_\Lambda * V(0)$$

rest energy renormalization

This would informally keep the groundstate energy finite, however, even that does not yield a well-defined dynamics. This can be seen from the fact that the dressing operator is only well-defined for finite cut-offs:

representation problem & dressing operator

$$\mathrm{Dress}_\Lambda = \exp\Big(-g\sum_{j=1}^{N}\int d^3k \, \frac{\hat{\gamma}_k^\Lambda}{\omega_k}\big(a_k^+ e^{-ikx_j} - a_k e^{+ikx_j}\big)\Big)$$

$\sim \mathcal{O}_{|k|\to\infty}\left(\frac{1}{|k|^{3/2}}\right)$ when $\Lambda \to \infty$ which is not in L^2!

In conclusion, the ultraviolet problem makes itself felt in a two-fold manner:

two classes of UV problems

- The Hamiltonian generating the dynamics morally evaluates the field at its sources, and there it is ill-defined.
- The representation of eigenstates is not possible in standard Fock space.

The first problem is of conceptual nature and inherited from classical electrodynamics while the second one is man-made as we can easily choose another Fock space to represent our states:

Noting that

Bogulyubov transformation

$$b_k := \mathrm{Dress}_\Lambda \, a_k \, \mathrm{Dress}_\Lambda^* = a_k - g\sum_{j=1}^{N} \frac{\hat{\gamma}_k}{\omega_k} e^{-ikx_j}$$

$$b_k^+ := \mathrm{Dress}_\Lambda \, a_k^+ \, \mathrm{Dress}_\Lambda^* = a_k^+ - g\sum_{j=1}^{N} \frac{\hat{\gamma}_k}{\omega_k} e^{+ikx_j}$$

HW: $[b_k, b_k^+] = \delta^3(k-k)$, $b_k \Omega_k = 0$

we may def. another Fock space $\tilde{\mathcal{F}}_\Lambda$ by

$\tilde{\Omega}_\Lambda = \mathrm{Dress}_\Lambda \Omega$ and b_k, b_k^+ as creation operators

We note further that

transformed & renormalized Hamiltonian

$$\widetilde{H}_\Lambda^{ren} = \text{Dress}_\Lambda^* H_\Lambda^{ren} \text{Dress} = N \cdot m_{ren} + \int d^3k\, \omega_k b_k^+ b_k - g^2 \sum_{i<j}^{N} \check{\xi}_\Lambda * V(x_i - x_j)$$

For $\Lambda < \infty$ both representations are unitary equivalent, for $\Lambda \to \infty$ not anymore. Nevertheless, on $\widetilde{\mathcal{F}}_\infty$ we yield a well-defined dynamics without cut-offs. Since the sources where fixed only the dynamics of the free field on top of the ground state remains:

$$\widetilde{H}^{ren} = N \cdot m_{ren} + \int d^3k\, \omega_k b_k^+ b_k - g^2 \sum_{i<j}^{N} V(x_i - x_j)$$

HW: Could we have guessed Dress_Λ and the transformed Hamiltonian from H_{TOY}?

(complete the "square")

Further reading:

There is a toy model, the so-called "Nelson model", similar to the one considered above in which the fermions are allowed to move according to the non-relativistic Schrödinger dispersion. For this model it has been shown that the ultraviolet cut-off can be removed rigorously:

Interaction of Nonrelativistic Particles with a Quantized Scalar Field,
E. Nelson, JMP, 1964

The employed strategy, however, relies heavily on the non-relativistic dispersion relation and fails in a pseudo-relativistic model such as the Yukawa model:

Ultraviolet Properties of the Spinless, One-Particle Yukawa Model,
D.-A.D. & A. Pizzo, CMP, 2014

A great book on mathematical rigorous treatment of non-relativistic QED (and also on self-interaction problem of classical electrodynamics) is:

Dynamics of charged particles and their radiation fields,
H. Spohn, Cambridge, 2004