Eigenvalue estimates for Dirac and Schrödinger type operators

Jean-Claude Cuenin

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Jean-Claude Cuenin

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Jean-Claude Cuenin

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Contents

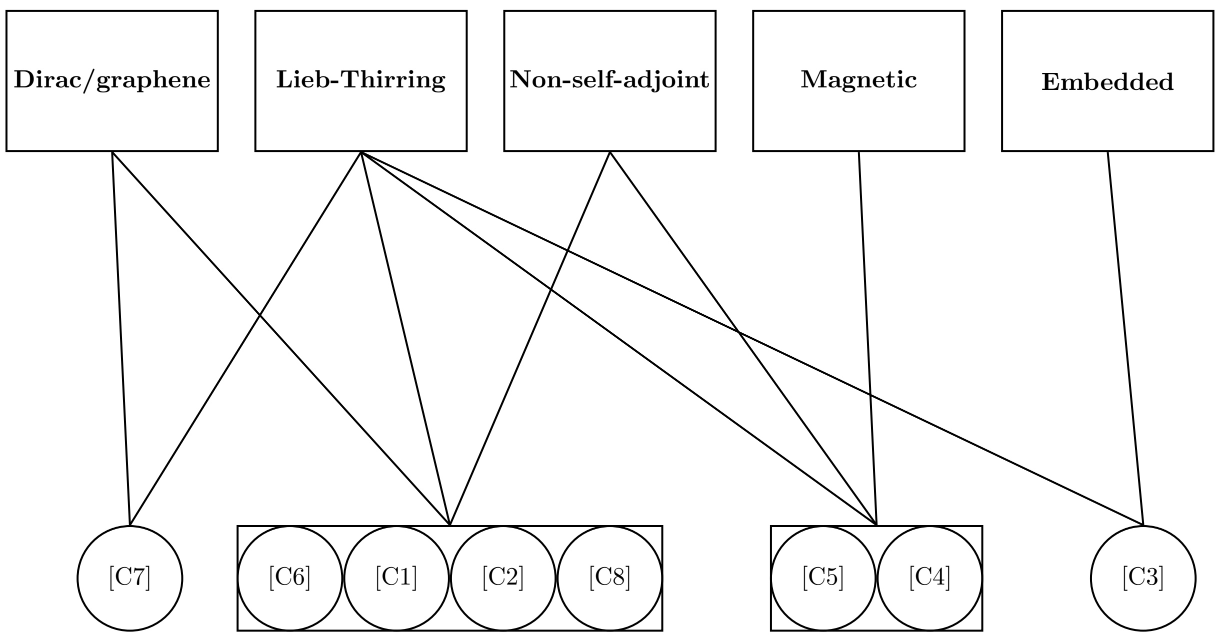
Summary

This is a Habilitation thesis on eigenvalue estimates for Dirac and Schrödinger type operators, i.e. operators of the form $H = H_0 + V$ where H_0 is a self-adjoint differential or pseudodifferential operator and V is potential. We are particularly interested in the case where V is a complex-valued (or non-hermitian matrix-valued when H_0 is the Dirac operator), which renders the operator H non-self-adjoint. In this setting, neither the variational principle nor the spectral theorem are available, and one usually has to resort to much more rudimentary tools such as the Birman-Schwinger principle. The latter reduces the spectral problem of H to the study of the compact operator $|V|^{1/2}(H_0 - z)^{-1}V^{1/2}$. In order to handle unbounded potentials it is thus pertinent to have a good understanding of mapping properties of the free resolvent $(H_0 - z)^{-1}$ between L^p-spaces with $p \neq 2$. In contrast to bound state problems with real-valued potentials the usual Sobolev inequalities are inadequate here due to an unfavorable dependence of the estimate on z. A major part of the effort will thus be spent on proving resolvent estimates that are *uniform* in z.

The thesis consists of the articles [C1, C2, C3, C4, C5, C6, C7, C8] (reprinted in Chapter 2), together with a short overview in Chapter 1. With the exception of [C7] and [C3], all publications are concerned with non-self-adjoint operators in one way or another. A more complete list of topics would roughly include the following.

- Dirac operators and graphene,
- Lieb-Thirring inequalities,
- Non-self-adjoint operators,
- Magnetic Schrödinger operators,
- Embedded eigenvalues.

A schematic picture showing the relation between these topics and the articles of the thesis is depicted on the next page, followed by a bibliography.


Dirac/graphene
Lieb-Thirring
Non-self-adjoint
Magnetic
Embedded
[C7]
[C6]
[C1]
[C2]
[C8]
[C5]
[C4]
[C3]

Articles of the thesis

[C1] J.-C. Cuenin. Estimates on complex eigenvalues for Dirac operators on the half-line. *Integral Equations Operator Theory*, 79(3):377–388, 2014.

[C2] J.-C. Cuenin. Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials. *J. Funct. Anal.*, 272(7):2987–3018, 2017.

[C3] J.-C. Cuenin. Embedded eigenvalues for generalized Schrödinger operators. *ArXiv:1709.06989*, 2017.

[C4] J.-C. Cuenin. Sharp spectral estimates for the perturbed Landau Hamiltonian with L^p potentials. *Integral Equations Operator Theory*, 88(1):127–141, 2017.

[C5] J.-C. Cuenin and C. E. Kenig. L^p resolvent estimates for magnetic Schrödinger operators with unbounded background fields. *Comm. Partial Differential Equations*, 42(2):235–260, 2017.

[C6] J.-C. Cuenin, A. Laptev, and C. Tretter. Eigenvalue estimates for non-selfadjoint Dirac operators on the real line. *Ann. Henri Poincaré*, 15(4):707–736, 2014.

[C7] J.-C. Cuenin and H. Siedentop. Dipoles in graphene have infinitely many bound states. *J. Math. Phys.*, 55(12):122304, 10, 2014.

[C8] J.-C. Cuenin and P. Siegl. Eigenvalues of one-dimensional non-self-adjoint Dirac operators and applications. *Lett. Math. Phys*, Jan 2018.

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Chapter 1

Overview

1.1 Motivation

An important paradigm in the realm of spectral theory of differential operators is that *functional inequalities* give rise to *eigenvalue inequalities*. One of the most successful incarnations of this paradigm is the stability of matter in quantum mechanics. The relevant functional inequalities in this case are Sobolev and Hardy inequalities. These may be seen as manifestations of the *uncertainty principle* and provide an elegant solution to the problem that, classically, electrons would collapse on top of the atomic nucleus.

Let us consider the hydrogen atom. The energy of an electron with wave function ψ in the presence of an attractive nucleus of charge $Z = 1$ is given by

$$\mathcal{E}[\psi] = \int_{\mathbb{R}^3} |\nabla\psi(x)|^2 \mathrm{d}x - Z \int_{R^3} \frac{|\psi(x)|^2}{|x|} \mathrm{d}x. \tag{1.1}$$

Using the sharp version of the Sobolev inequality (see Lieb-Loss [52, Theorem 8.3])

$$\int_{\mathbb{R}^3} |\nabla\psi(x)|^2 \mathrm{d}x \geq (\pi/2)^{4/3} \Big(\int_{R^3} |\psi(x)|^6 \Big)^{1/3}, \tag{1.2}$$

we arrive at the minimization problem

$$E_0 := \inf \mathcal{E}[\psi] \geq \inf \left((\pi/2)^{4/3} \Big(\int_{R^3} |\psi(x)|^6 \Big)^{1/3} - Z \int_{R^3} \frac{|\psi(x)|^2}{|x|} \mathrm{d}x \right), \tag{1.3}$$

where the infimum is taken over all L^2-normalized wave functions. Splitting the Coulomb potential $1/|x|$ into a short-range and a long-range part, one obtains, after a standard application of Hölder's inequality, that $E_0 \geq -Z^2/3$. This is remarkably close to the true ground state energy $E_0 = -Z^2/4$. The fact that the ground state energy is bounded below is called "stability of the first kind".[1] By the variational principle (see [61, Theorem

[1] Roughly speaking, "stability of the second kind" says that the total energy of a quantum mechanical system of N electrons and M nuclei is bounded from below by $-C(N+M)$, where $C > 0$ is some universal constant. We refer to the textbook of Lieb-Seiringer [53] for background on the stability of matter.

XIII.1]) we know that E_0 is the in fact the lowest eigenvalue of the Schrödinger operator $-\Delta - Z|x|^{-1}$.

We now consider a general Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^d)$, $d \geq 1$, on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, where V is a (possibly complex-valued) potential in $L^q(\mathbb{R}^d)$ for some $q \in [1, \infty]$. Our goal is to obtain eigenvalue bounds that depend only on the L^q-norm of V. Here and in the following we use the abbreviation

$$\|V\|_q := \|V\|_{L^q(\mathbb{R}^d)} := \Big(\int_{\mathbb{R}^d} |V(x)|^q \mathrm{d}x \Big)^{1/q},$$

with the obvious modification for $q = \infty$. Since $-\Delta + V$ is no longer self-adjoint, we cannot use the variational principle. Instead, we appeal to the *Birman-Schwinger principle*: $z \in \mathbb{C} \setminus [0, \infty)$ is an eigenvalue of $-\Delta + V$ if and only if -1 is an eigenvalue of the Birman-Schwinger operator $|V|^{1/2}(-\Delta - z)^{-1} V^{1/2}$. Here we define $V^{1/2} := V/|V|^{1/2}$. If -1 is an eigenvalue of an operator, then that operator must have norm at least 1. Put differently, we have the inequality

$$1 \leq \||V|^{1/2}(-\Delta - z)^{-1} V^{1/2}\|_{L^2 \to L^2} \leq \|V\|_q \|(-\Delta - z)^{-1}\|_{L^p \to L^{p'}} \tag{1.4}$$

whenever $1/p - 1/p' = 1/q$. Here p' is the Hölder conjugate of p, that is $1/p + 1p' = 1$. In the second inequality of (1.4) we used that, for multiplication operators A, B, $\|A\|_{L^{p'} \to L^2} = \|A\|_{2q}$ and $\|B\|_{L^2 \to L^p} = \|B\|_{2q}$ by Hölder's inequality. In our case $A := |V|^{1/2}$ and $B := V^{1/2}$. Here and henceforth, $\|A\|_{L^p \to L^r}$ denotes the operator norm of $A : L^p(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$.

For simplicity, we momentarily return to $d = 3$. In this case the integral kernel of the free resolvent $(H_0 - z)^{-1}$ is given by $(4\pi|x - y|)^{-1} \mathrm{e}^{\mathrm{i}\sqrt{z}|x-y|}$, where we use the branch of the square root on $\mathbb{C} \setminus [0, \infty)$ with positive imaginary part. Hence, this kernel is bounded in absolute value by $(4\pi|x - y|)^{-1}$, for arbitrary $z \in \mathbb{C} \setminus [0, \infty)$. Hence, a routine application of the Hardy-Littlewood-Sobolev inequality (see Frank [16]) yields

$$\|(-\Delta - z)^{-1}\|_{L^{6/5}(\mathbb{R}^3) \to L^6(\mathbb{R}^3)} \leq \frac{2^{4/3}}{3\pi^{4/3}}., \quad \text{if} \quad \frac{1}{p} - \frac{1}{p'} = \frac{2}{3}. \tag{1.5}$$

Combining this bound with (1.4), we conclude that $-\Delta + V$ cannot have any eigenvalues in $\mathbb{C} \setminus [0, \infty)$ unless $\|V\|_{3/2} \geq \frac{3\pi^{4/3}}{2^{4/3}}$. For $q \neq 3/2$ the estimate (1.5) is considerably more delicate if one insists on bounds that are uniform in $\arg(z)$. In view of (1.4), an upper bound of the form

$$\|(-\Delta - z)^{-1}\|_{L^p \to L^{p'}} \leq C(z), \quad \text{with} \quad \frac{1}{p} - \frac{1}{p'} = \frac{1}{q}, \tag{1.6}$$

would immediately imply that $C(z)\|V\|_q \geq 1$ if $z \in \mathbb{C} \setminus [0, \infty)$ is an eigenvalue of $-\Delta + V$. A naive estimate yields

$$C(z) \leq S_{d,q} \left\| \frac{-\Delta + 1}{-\Delta - z} \right\|_{L^2 \to L^2}, \tag{1.7}$$

where $q \geq d/2$ and $S_{d,q}$ is the best constant in the Sobolev embedding inequality

$$\|(-\Delta + 1)^{-1}\|_{L^p \to L^{p'}} \leq S_{d,q}. \tag{1.8}$$

The estimate (1.7) is uniform if z lies outside some arbitrary but fixed sector since

$$\sup\left\{\left\|\frac{-\Delta+1}{-\Delta-z}\right\| : |\arg(z)|\geq\varphi,\, |z|\geq\delta\right\}\leq C_{\varphi,\delta},\quad \text{if}\quad \delta>0,\quad 0<\varphi<2\pi.$$

However, the constant $C_{\varphi,\delta}$ blows up as $\varphi\to 0$ or $\varphi\to 2\pi$ (or $\delta\to 0$).

A fundamental insight of Frank [16] was to replace the standard Sobolev inequality (1.8) by a much more refined *uniform Sobolev inequality* due to Kenig-Ruiz-Sogge [45]. A special case of the latter can be stated as

$$\|(-\Delta-z)^{-1}\|_{L^p\to L^{p'}}\leq C_{d,q}|z|^{\frac{d}{2q}-1},\quad \text{if}\quad \frac{2}{d+1}\leq\frac{1}{p}-\frac{1}{p'}:=\frac{1}{q}\leq\frac{2}{d}. \tag{1.9}$$

In contrast to (1.8) the estimate (1.9) is uniform in the argument of z. The same estimate (up to the endpoint $q=(d+1)/2$) was proved independently by Kato-Yajima [42]. In their terminology, any multiplication operator $A\in L^d(\mathbb{R}^d)$ is "super-smooth" with respect to $H_0=-\Delta$. In general, if H_0 is a self-adjoint operator on a Hilbert space $\mathcal{H}$, an operator A is called H_0- *smooth* if

$$|(\operatorname{Im} R_0(z)A^*u, A^*u)|\leq\pi a\|u\|_{\mathcal{H}}^2,\quad u\in D(A^*),\quad \operatorname{Im} z\neq 0 \tag{1.10}$$

and H_0- *super-smooth* if

$$|(R_0(z)A^*u, A^*u)|\leq\pi a\|u\|_{\mathcal{H}}^2,\quad u\in D(A^*),\quad \operatorname{Im} z\neq 0. \tag{1.11}$$

for some constant $a>0$. Here and in the following we set $R_0(z):=(H_0-z)^{-1}$. It is known (see e.g. Reed-Simon [61, Theorem XIII.25]) that A is H_0-smooth if and only if

$$\int_{-\infty}^{+\infty}\|A\mathrm{e}^{\mathrm{i}tH_0}u\|^2\mathrm{d}t\leq 2\pi a\|u\|^2,\quad u\in\mathcal{H}. \tag{1.12}$$

The property (1.12) is called the *smoothing effect* of the propagator $\mathrm{e}^{\mathrm{i}tH_0}$ since it implies that $\mathrm{e}^{\mathrm{i}tH_0}u\in D(A)$ for almost every $t\in\mathbb{R}$. In the case of $A\in L^d(\mathbb{R}^d)$ the smoothing corresponds to a gain of integrability and is closely related to the so-called *Strichartz-estimates.* The equivalence of (1.10) and (1.12) is the reason that uniform resolvent estimates also play a crucial role in scattering theory. In this context, a resolvent estimate that is locally uniform in z up to the positive real axis, say for z in some compact subinterval of $(0,\infty)$, is called a *limiting absorption principle.* Related ideas, based on harmonic analysis techniques, were used by Schlag et al [28, 36], see also the recent overview of Schlag [65].

To get a sense of the strength of condition (1.11), consider the following implication. If $V=BA$ (not necessarily real-valued) and A,B are H_0-smooth, then By Kato's theory [39, Theorem 1.5] (see also Simon [67, Section 14]) the operator $-\Delta+\varepsilon V$ is similar[2] to $-\Delta$ if ε is sufficiently small. Moreover, if V is real-valued, the wave operators $\Omega^{\pm}(-\Delta+V,-\Delta)$ exist and are unitary, so that $-\Delta+V$ has purely absolutely continuous spectrum.

[2]This means that there exists a bounded and boundedly inverticle operator W on $L^2(\mathbb{R}^d)$ such that $W(-\Delta+\varepsilon V)W^{-1}=-\Delta$. In particular, $-\Delta+\varepsilon V$ has the same spectrum as $-\Delta$.

1.2 Caricature of the main problem

Motivated by the previous example we now discuss some of the main questions addressed in the thesis. In an attempt to describe the overarching theme the exposition starts out quite general, but becomes more specific in subsequent sections.

Consider a self-adjoint differential or pseudodifferential operator H_0 on $L^2(\mathbb{R}^d)$ and a decaying potential V. Later H_0 will be either the Laplacian, the Dirac operator, the fractional Laplacian, the harmonic oscillator or a magnetic Schrödinger operator. A significant difference to the hydrogen example is that we allow the potential V to be *complex-valued*. This means that $H = H_0 + V$ will generally not be a self-adjoint operator.

From the point of view of mathematical physics an important motivation to consider non-self-adjoint operators comes from the study of resonances. In fact, the complex scaling method introduced by Aguilar-Combes [2] identifies resonances with eigenvalues of a complex dilation of the Hamiltonian. Another method to investigate resonances, also relying on non-self-adjoint operators, is that of complex absorbing potentials. We refer to Riss-Meyer [62] for an introduction and to Zworski [76] for a mathematically rigorous treatment. A large range of other applications of non-self-adjoint operators may be found in the book of Embree-Trefethen [72].

Assume that we can make sense of the sum $H = H_0 + V$ as a closed operator and that the essential spectra $\sigma_{\mathrm{e}}(H_0)$ and $\sigma_{\mathrm{e}}(H)$ coincide. Then the discrete spectrum $\sigma_{\mathrm{d}}(H)$ consists of at most countably many isolated eigenvalues of finite algebraic multiplicities that can accumulate only at $\sigma_{\mathrm{e}}(H_0)$. We are interested in quantitative bounds on the location and the distribution of these eigenvalues that depend on V only through its L^q-norms. In anticipation of the results ahead let us introduce a subset $\tau(H_0)$ of the spectrum $\sigma(H_0)$. In our applications $\tau(H_0)$ will either be the set of critical values of the symbol of H_0 (when H_0 is translation-invariant) or the set of eigenvalues of H_0 (when H_0 has pure point spectrum). Let $\Phi, \Psi : \mathbb{C} \to [0, \infty)$ be continuous functions (not identically zero) that vanish on $\tau(H_0)$. Given $q \in [1, \infty]$ we want to find the best possible (i.e. as large as possible) choice for Φ, Ψ such that the following two questions have an affirmative answer:

(Q1) Does there exist a universal constant $C_{d,q} > 0$ such that the inequality

$$\Phi(z) \leq C_{d,q} \int_{\mathbb{R}^d} |V(x)|^q \mathrm{d}x \tag{1.13}$$

holds for every $z \in \sigma_{\mathrm{d}}(H)$ and for every $V \in L^q(\mathbb{R}^d)$?

(Q2) Does there exist a universal constant $C_{d,q} > 0$ such that the inequality

$$\Big(\sum_{z \in \sigma_{\mathrm{d}}(H)} \Psi(z)^\alpha \Big)^{1/\alpha} \leq C_{d,q} \int_{\mathbb{R}^d} |V(x)|^q \mathrm{d}x \tag{1.14}$$

holds for every $V \in L^q(\mathbb{R}^d)$, for some $\alpha > 0$?[3]

[3]If Ψ is fixed, then (1.14) would be stronger the smaller α is since the left hand side of (1.14) is the ℓ^α-norm of Ψ over the countable set $\sigma_{\mathrm{d}}(H)$, and $\ell^\alpha \subset \ell^\beta$ for $\alpha < \beta$.

By "universal" we mean that the constant $C_{d,q}$ is independent of z and V. These two questions lie at the heart of[4] [C6, C1, C2, C3, C4] and are somewhat peripheral to [C5, C8]. Clearly, if $\Psi = \Phi$, then (1.14) is stronger than (1.13). However, in a non-self-adjoint setting, Φ and Ψ will usually not coincide. The third question asks whether or not we can have $\Phi(z_0) = 1$ in (Q1) for some fixed $z_0 \in \mathbb{C} \setminus \tau(H_0)$. In other words:

(Q3) Given $z_0 \in \mathbb{C} \setminus \tau(H_0)$, does there exist a constant $C_{d,q,z_0} > 0$ (possibly depending on z_0, but not on V) such that

$$C_{d,q,z_0} \int_{\mathbb{R}^d} |V(x)|^q \mathrm{d}x < 1 \implies z_0 \notin \sigma(H_0 + V).$$

This question is central to [C3]. Closely related to the last question is the following:

(Q3') Fix some sufficiently "nice" potential V. Does $H_0 + \varepsilon V$ have an eigenvalue in $\mathbb{C} \setminus \sigma(H_0)$ for arbitrary small $\varepsilon > 0$? If so, what is its leading asymptotic behavior as $\varepsilon \to 0$? Is it possible that $H_0 + \varepsilon V$ has infinitely many eigenvalues for any $\varepsilon > 0$?

This question is important in [C7, C8].

1.3 Non-self-adjoint Schrödinger operators

To make the discussion of the previous paragraph more concrete let us consider the case $H_0 = -\Delta$. This will serve the dual purpose of illustrating the basic problems as well as describing the state of the art of the subject. The sharpest results in this direction have been obtained by Frank [16, 17], Frank-Sabin [22] and Frank-Simon [24]. In connection with the first question (Q1) it was shown in [16] that, if $\gamma \leq 1/2$, all eigenvalues of the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^d)$ lie in a disk whose radius is bounded by the $L^{d/2+\gamma}$-norm[5] of V. More precisely, any eigenvalue $z \in \mathbb{C}$ satisfies the bound

$$|z|^\gamma \leq D_{d,\gamma} \int_{\mathbb{R}} |V(x)|^{\gamma + d/2} \mathrm{d}x \tag{1.15}$$

for $\gamma = 1/2$ if $d = 1$ and any $0 < \gamma \leq 1/2$ if $d \geq 2$, and with a constant $D_{d,\gamma}$ independent of V and z. The bound for $d = 1$ with the sharp constant $D_{1,1/2} = 1/2$ is due to Abramov-Aslanyan-Davies [1]. Similar estimates for Schrödinger operators on the half-line, where the constant $1/2$ has to be replaced by 1, were established by Frank-Lieb-Seiringer [21]. In the case of the half-line operator with Dirichlet boundary conditions, the sharp estimate

$$|z|^{1/2} \leq \frac{1}{2} g(\cot(\theta/2)) \int_0^\infty |V(x)| \, \mathrm{d}x \tag{1.16}$$

[4] The articles [C1]–[C8] that are the subject of the thesis, are listed on page ix.
[5] It is customary to write $q = d/2 + \gamma$ where $\gamma \geq 0$.

holds, where $z = |z|e^{i\theta}$ and $g(b) := \sup_{y\geq 0} |e^{iby} - e^{-y}| \in [1,2]$. Originally, the bound (1.15) was stated for $z \in \mathbb{C} \setminus [0,\infty)$, but it was later realized [24] that embedded eigenvalues can also be accommodated. In [17] this bound was extended to "long-range" potentials, i.e. to the case $\gamma > 1/2$. The result is that

$$\delta(z)^{\gamma-1/2}|z|^{1/2} \leq C_{d,\gamma} \int_{\mathbb{R}} |V(x)|^{\gamma+d/2} \mathrm{d}x \tag{1.17}$$

for $d \geq 1$ and $\gamma > 1/2$, with $\delta(z) := \operatorname{dist}(z, [0,\infty))$. Changing variables $q = d/2 + \gamma$ and restricting our attention to $d \geq 2$, we thus see that (1.13) holds with

$$\Phi(z) := \begin{cases} |z|^{q-d/2} & \text{if } d/2 < q \leq (d+1)/2, \\ \delta(z)^{q-(d+1)/2}|z|^{1/2} & \text{if } (d+1)/2 < q. \end{cases}$$

We note that the origin is the only critical value for the symbol $|\xi|^2$ of the Laplacian, i.e. $\tau(H_0) = \{0\}$ here. The second question (Q2) was addressed in [22] for $\gamma \leq 1/2$ and in [17] for $\gamma > 1/2$. The situation here is less well understood and the results are more complicated to state. The simplest bound from [22] is

$$\left(\sum_{z \in \sigma_{\mathrm{d}}(H)} \frac{\delta(z)}{|z|^{1/2}} \right)^{2\gamma} \leq L_{d,\gamma} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \mathrm{d}x, \quad 0 < \gamma < \frac{d}{2(2d-1)}, \quad d \geq 2. \tag{1.18}$$

This is of the form (1.14) with $\Psi(z)^\alpha := \delta(z)|z|^{-1/2}$ and $1/\alpha := 2\gamma$. Observe that the estimates (1.15)–(1.18) are scale-invariant. We will briefly explain what this means. Given $\lambda > 0$ consider the scaling transformation $(U_\lambda \psi)(x) := \lambda^{-d/2}\psi(\lambda x)$. Then the operator $H_\lambda := \lambda^{-2} U_\lambda^* H U_\lambda$ again takes the form of a Schrödinger operator $H_\lambda = -\Delta + V_\lambda$, where $V_\lambda(x) = \lambda^{-2}V(\lambda^{-1}x)$. Since U_λ is unitary on $L^2(\mathbb{R}^d)$, we have $\sigma(H) = \sigma(\lambda^2 H_\lambda)$, and similarly for $\sigma_{\mathrm{d}}(H)$, etc. Hence, applying (1.15)–(1.18) to H_λ yields unchanged inequalities for z, after cancelling a common factor on both sides. If this were not the case, we could improve the inequalities simply by rescaling. Therefore, scale-invariant inequalities are of particular importance in analysis.

The inequalities (1.15)–(1.18) are motivated by the self-adjoint case (i.e. when V is real-valued) where they are completely understood (up to the important question of optimal constants). In this case the estimate (1.15) for the lowest negative eigenvalue z and $|V(x)|$ replaced by $V_-(x) := \max(-V(x), 0)$ was first proved by Keller [44] for $d = 1$, $\gamma \geq 1/2$ and later generalized by several authors to the so-called *Lieb-Thirring inequalities*

$$\sum_{z \in \sigma_{\mathrm{d}}(H)} |z|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V_-(x)^{\gamma+d/2} \mathrm{d}x, \tag{1.19}$$

where $\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ if $d \geq 3$. For $\gamma > 0$ and $d \geq 2$ or $\gamma > 1/2$ and $d = 1$ the estimates (1.19) can be proved with the methods developed by Lieb-Thirring [54, 55]. The case $\gamma = 1/2$ and $d = 1$ is due to Weidl [73]. The case $\gamma = 0$ and $d \geq 3$ is a bound on the number of negative eigenvalues and was independently proved by Cwikel [7], Lieb [51] and Rosenblum [63]. The right hand side of (1.19) is proportional to the phase space integral $\int\int_{\mathbb{R}^d\times\mathbb{R}^d}(|\xi|^2 - V_-(x))_-^\gamma \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi)^d}$. In fact, if one assumes that $V \in L^{d/2+\gamma}(\mathbb{R}^d)$ and replaces V by αV, then (1.19) becomes an equality (modulo lower order terms) in the semiclassical limit $\alpha \to \infty$, where $L_{d,\gamma}$ is replaced by the semiclassical constant $L_{d,\gamma}^{\mathrm{cl}} = (2\pi)^{-d}\int_{\mathbb{R}^d}(|\xi|^2-1)_-^\gamma \mathrm{d}\xi$. This is known as a Weyl type asymptotic formula.

Comparing (1.19) to (1.15)–(1.18) one notes that the former is in general much stronger. This begs the question whether the latter can still be improved, or if there are counterexamples that rule out such improvements. Laptev-Safronov [50] conjectured that (1.15) holds for $0 < \gamma \leq d/2$. For radial potentials the conjecture was proved by Frank-Simon [24]. In the same paper the authors provide evidence that the conjecture is false for non-radial potentials if $\gamma > 1/2$. In fact, their counterexamples show that the range $\gamma \leq 1/2$ is sharp for embedded eigenvalues. Bögli [5] showed that (1.15) is false for $\gamma > d/2$. Actually, her example shows that the failure is rather dramatic since, even for a single potential with arbitrary small $L^{d/2+\gamma}$-norm for fixed $\gamma > d/2$, eigenvalues may accumulate at every point of the positive real axis. The conjecture is still open in the non-radial case. It is less clear what the conjectured result should be in the case of eigenvalue sums as in (1.14). The reason is simply that, for negative z, one has $\delta(z) = |z|$, so one can come up with different generalizations from the the self-adjoint model (1.19). A possible modification, proposed by Demuth-Hansmann-Katriel [11], is as follows:

$$\sum_{z\in\sigma_{\mathrm{d}}(H)} \frac{\delta(z)^{\gamma+d/2}}{|z|^{d/2}} \leq C_{d,\gamma} \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma} dx. \tag{1.20}$$

The proof or disproof of (1.20) was left as an open problem. Related inequalities were obtained in [19, 9, 10]. We also mention the remarkable paper [20] where a bound on the number of eigenvalues of Schrödinger operators with exponentially decaying complex potentials is proved in all odd dimensions. Recent generalizations of (1.15), (1.17) or (1.18) were obtained by Mizutani [58] for non-self-adjoint Schrödinger operators with inverse square potentials and by Guillarmou-Hassell-Krupchyk [29] for non-self-adjoint Schrödinger operators on conical manifolds with non-trapping metrics.

1.4 Non-self-adjoint Dirac operators

In this section we summarize the main results of the papers [C6], [C1], [C2], [C8] in the thesis.

The free Dirac operator on $\mathbb{R}^d$ is a first-order matrix-valued differential operator given by (in units where $\hbar = c = 1$)

$$H_0 := -\mathrm{i}\sum_{j=1}^d \alpha_j\partial_j + m\beta. \tag{1.21}$$

Here, $m \geq 0$ is the mass and $\alpha_1, \ldots, \alpha_d$, $\alpha_{d+1} = \beta$ are $n \times n$ matrices (n can be chosen as $2^{d/2}$ if d is even and $2^{(d+1)/2}$ if d is odd) satisfying the Clifford algebra relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = -2\delta_{jk}\mathbf{1}_{n\times n}, \quad j,k = 1,\ldots,d+1. \tag{1.22}$$

In dimensions $d=1$ or $d=2$ one may take $\alpha_j=\sigma_j$, where

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{1.23}$$

are the standard Pauli matrices. We consider (1.21) as an unbounded operator on $L^2(\mathbb{R}^d;\mathbb{C}^n)$ with domain the first order Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$. From (1.22) it readily follows that

$$\sigma(H_0)=(-\infty,-m]\cup[m,\infty).$$

For this section we define

$$\|V\|_q^q:=\int_{\mathbb{R}^d}\|V(x)\|^q\mathrm{d}x,\quad 1\leq q<\infty,$$

with the usual modification for $q=\infty$. Here, $\|V(x)\|$ is the operator norm of $V(x)$ on $\mathbb{C}^n$. We also abbreviate $L^q(\mathbb{R}^d;\mathrm{Mat}(n\times n;\mathbb{C}))\equiv L^q$, i.e. $V\in L^q \iff \|V\|_q<\infty$.

1.4.1 Dimension $d=1$

The article [C6] treats the Dirac operator on the whole real line and focuses on question (Q1). The first result of [C6] is the analogue of the bound (1.15) in one dimension for the Dirac operator. Consider the Dirac operator H_0 with mass $m\geq 0$ in $L^2(\mathbb{R};\mathbb{C})$, given by

$$H_0=-\mathrm{i}\partial_{x_1}\sigma_1+m\sigma_3. \tag{1.24}$$

Theorem 1.4.1 (Theorem 2.1 in [C6]). *Let $d=1$, $m\geq 0$, and let H_0 be the Dirac operator (1.24) on $L^2(\mathbb{R};\mathbb{C}^2)$. If $V\in L^1$ with $\|V\|_1<1$, then all eigenvalues of H_0+V lie in the union of two closed disks in the complex plane,*

$$\sigma_{\mathrm{d}}(H_0+V)\subset\overline{B}_{mr_0}(mx_0)\,\dot\cup\,\overline{B}_{mr_0}(-mx_0), \tag{1.25}$$

where

$$x_0:=\sqrt{\frac{\|V\|_1^4-2\|V\|_1^2+2}{4(1-\|V\|_1^2)}+\frac{1}{2}},\quad r_0:=\sqrt{\frac{\|V\|_1^4-2\|V\|_1^2+2}{4(1-\|V\|_1^2)}-\frac{1}{2}},$$

and $\overline{B}_{mr_0}(\pm mx_0):=\{z\in\mathbb{C}:|z\mp mx_0|\leq mr_0\}$. Moreover, if $m=0$ and $\|V\|_1<1$, then the discrete spectrum of H_0+V is empty.

The smallness assumption $\|V\|_1<1$ is natural since L^1 is *scaling-critical* for the Dirac operator. This means that $\|V_\lambda\|_1=\|V\|_1$ if $V_\lambda(x)=\lambda^{-1}V(\lambda^{-1}x)$. It is somewhat surprising that the free Dirac operator can accomodate L^1 potentials since H_0 is of the same order as the Riesz potential $(-\Delta)^{1/2}$. One would perhaps expect that the minimal condition for defining a closed operator H_0+V should be $V\in L^p$ for some $p>1$. The reason that the endpoint $p=1$ is allowed here is that the resolvent kernel of H_0 is bounded, due to cancellations from positive and negative energies. In contrast, the kernel of $(-\Delta)^{-1/2}$ has a logarithmic singularity at the origin in one dimension.

The original formulation of Theorem 1.4.1 excluded embedded eigenvalues. However, these can be included using the same argument [24, Proposition 3.1] as for the Schrödinger operator. An explicit example with a delta potential shows that (1.25) is sharp and that no spectral estimate can be obtained solely in terms of the L^1 norm if $\|V\|_1 \geq 1$. However, for purely imaginary potentials $V = \mathrm{i}W$ with $W \geq 0$ the estimate (1.25) can be improved [C6, Theorem 3.2]. This remains true for all subsequent estimates where complex potentials are considered (see e.g. Theorem [C1, Theorem 6.3]), and we will not mention this special case separately.

In the massless case, the resolvent estimate in [C6] shows that $|V|^{1/2}$ is H_0-smooth in the sense of Kato [40] if $V \in L^1$. It follows from Kato's theory that $H_0 + V$ is similar to H_0. The absence of non-real eigenvalues is a consequence of this similarity. If V is an electric potential, i.e. $V = qI$ with a function $q : \mathbb{R} \to \mathbb{C}$, then the operators are similar even without the smallness condition $\|V\|_1 < 1$ [C6, Remark 2.3]. The second result of [C6] concerns slowly decaying potentials $V \in L^1 + L_0^\infty$. This means that $V = V_1 + V_2$ with $V \in L^1$ and $V_2 \in L_0^\infty$. Here, L_0^∞ is the space of bounded (matrix-valued) functions that vanish at infinity.

Theorem 1.4.2 (Theorem 4.3 in [C6])**.** *Let H_0 be the Dirac operator* (1.24), $V \in L^1 + L_0^\infty$, *and define the positive, decreasing convex function*

$$F_V(s) := \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \|V(x)\| \, \mathrm{e}^{-s|x-y|} \mathrm{d}x, \qquad s > 0.$$

Let $z \in \mathbb{C} \setminus \sigma(H_0)$. If

$$\frac{1}{\sqrt{2}} \sqrt{1 + \frac{|z|^2 + m^2}{|z^2 - m^2|}} \, F_V\left(\operatorname{Im} \sqrt{z^2 - m^2}\right) < 1,$$

then $z \notin \sigma(H_0 + V)$. Moreover, if the equation $F_V(\mu) = \mu/m$ has a solution $\mu_0 \in (-m, m)$, it is unique and $\sigma(H_0 + V) \cap (-\sqrt{m^2 - \mu_0^2}, \sqrt{m^2 - \mu_0^2}) = \emptyset$.

An application of the previous results to resonances is given in [C6, Theorem 5.3].

We now turn our attention to the half-line case that was studied in [C1]. Here the action of H_0 is again given by (1.24), but now considered as an operator on $L^2(\mathbb{R}_+, \mathbb{C}^2)$ subject to separated boundary conditions at zero:[6]

$$\psi_1(0)\cos(\alpha) - \psi_2(0)\sin(\alpha) = 0, \quad \alpha \in [0, \pi/2].$$

Theorem 1.4.3 (Theorem 2.1 in [C1])**.** *Let H_0 be as above, and let $\|V\|_1 < 1/\sqrt{2}$. Then any eigenvalue $z \in \mathbb{C}$ of $H_0 + V$ is contained in the disjoint union of two closed disks in the complex plane,*

$$\sigma_{\mathrm{d}}(H_0 + V) \subset \overline{B}_{mr_0}(mx_0) \,\dot\cup\, \overline{B}_{mr_0}(-mx_0),$$

where

$$x_0 := 1 + \frac{2\|V\|_1^4}{1 - 2\|V\|_1^2}, \quad r_0 := 2\|V\|_1 \frac{1 - \|V\|_1^2}{1 - 2\|V\|_1^2}.$$

Moreover, if $m = 0$ and $\|V\|_1 < 1/\sqrt{2}$, then the discrete spectrum is empty.

[6]Here, $\psi_1 = \frac{\mathbf{1}+\sigma_3}{2}\psi$ and $\psi_2 = \frac{\mathbf{1}-\sigma_3}{2}\psi$.

In [C1] we were also interested in the nonrelativistic limit [C1, Proposition 3.1]. In fact, if one restores the speed of light in (1.21) subtracts mc^2 from H_0 and sends $c \to +\infty$, then one recovers the result (1.16) for the Schrödinger operator. Similarly, the bound (1.15) for the whole-line case ($d = 1$) is recovered from Theorem 1.4.1.

For Dirichlet boundary conditions the result of Theorem 1.4.3 may be obtained from the whole-line case by a suitable reflection of the potential about the origing. This is a well-known trick for Schrödinger operators; for Dirac operators the parity operator is involved. Theorem 1.4.3 is an improvement over this simple argument. The upshot is that disks are no longer optimal in the half-line case. Instead, the spectrum is located in some tear-drop shaped region as in (1.16).

To complete the discussion of the one-dimensional Dirac operator we state two theorems from [C8] that complement Theorem 1.4.1. Here one needs to assume some additional L^p integrability of the potential, e.g. $V \in L^1 \cap L^p$ for some $p > 1$. For simplicity we only consider $p = 2$. The first theorem of [C8], related to question (Q3') on page 5, concerns the weak coupling limit,[7] i.e. the case when the potential V is replaced by εV, where ε tends to zero.

Theorem 1.4.4 (Theorem 2.2 in [C8]). *Let H_0 be the Dirac operator* (1.24) *with $m > 0$ on $L^2(\mathbb{R};\mathbb{C}^2)$, $V \in L^1 \cap L^2$, and consider the 2×2 matrix $U := \int_{\mathbb{R}} V(x)\,\mathrm{d}x$. If $\operatorname{Re} U_{11} < 0$, then, for all sufficiently small $\varepsilon > 0$, there exists an eigenvalue $z_+(\varepsilon)$ of $H + \varepsilon V$ satisfying*

$$z_+(\varepsilon) = m - \frac{m}{2} U_{11}^2 \varepsilon^2 + o(\varepsilon^2), \quad \varepsilon \to 0+ . \tag{1.26}$$

Similarly, if $\operatorname{Re} U_{22} > 0$ then, for all sufficiently small $\varepsilon > 0$, there exists an eigenvalue $z_-(\varepsilon)$ of $H + \varepsilon V$ satisfying

$$z_-(\varepsilon) = -m + \frac{m}{2} U_{22}^2 \varepsilon^2 + o(\varepsilon^2), \quad \varepsilon \to 0+ . \tag{1.27}$$

The asymptotics (1.26)–(1.27) show that the estimate (1.25) is sharp in the weak coupling regime since the latter can be stated as

$$|z \mp m| \leq \frac{m}{2} \varepsilon^2 \|V\|_1^2 + o(\varepsilon^2), \quad \varepsilon \to 0+ .$$

Hence, if we fix V with $U_{11} < 0$ and $U_{22} > 0$ and replace εV by $\varepsilon e^{i\theta} V$, with $\theta \in [0, \pi/2)$, then $z_{\pm}(\varepsilon, \theta)$ parametrize two half-circles in the upper half-plane. Reversing the sign of V gives the corresponding half-circles in the lower half-plane. The second result of [C8] that we state is a special case of [C8, Theorem 2.4]. It addresses question (Q2) on page 5 in the case of the massless Dirac operator on the whole line.

Theorem 1.4.5 (Theorem 2.4 in [C8]). *Let H_0 be the Dirac operator* (1.24) *with $m = 0$. Then the eigenvalues $z \in \mathbb{C} \setminus \sigma(H_0)$ repeated according to their algebraic multiplicity satisfy*

$$\sum_{z \in \sigma_{\mathrm{d}}(H_0 + V)} \frac{\delta(z)}{(|z| + 1)^2} \leq C(1 + \|V\|_2^4) \|V\|_1^2, \tag{1.28}$$

where $\delta(z) := \operatorname{dist}(z, \sigma(H_0))$. The constant $C > 0$ is independent of V.

[7] In fact, the weak-coupling limit and the nonrelativistic limit are in some sense equivalent, see [C6].

Note that the estimate (1.28) is not scale-invariant. A scale-invariant estimate could be obtained by similar arguments as in [C8], but at the expense of increasing the power of $\delta(z)$ on the left hand side.

As a corollary of (1.28) one can obtain upper bounds for the number of eigenvalues N_K of $H_0 + V$ in compact subsets K of $\mathbb{C} \setminus \sigma(H_0)$ such as

$$N_{K_{\delta,R}} \leq C\delta^{-1}(1 + R^2)(1 + \|V\|_2^4)\|V\|_1^2,$$

where $K_{\delta,R} = \{z \in \mathbb{C} : \delta(z) \geq \delta, |z| \leq R\}$. Similar but more complicated bounds were proved in [C8] for $m > 0$. These bounds improve previous results by Dubuisson [13].

1.4.2 Graphene waveguides

The above results on the one-dimensional Dirac operator were applied to the damped wave equation [C8, Theorem 3.1] and to armchair waveguides in graphene [C8, Theorem 3.6]. Here we only discuss the latter as it relates to question (Q3') on page 5. The operator H_0 defined below can be viewed as an intermediate case between one-dimensional and higher-dimensional Dirac operators. We consider an infinite two-dimensional straight graphene waveguide $\Omega = (-a, a) \times \mathbb{R}$ and the corresponding Dirac operator

$$H_0 = \begin{pmatrix} 0 & \tau^* & 0 & 0 \\ \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau \\ 0 & 0 & -\tau^* & 0 \end{pmatrix} \quad \text{in } L^2(\Omega; \mathbb{C}^4), \tag{1.29}$$

where $\tau := -\mathrm{i}\partial_1 + \partial_2$ and $\tau^* := -\mathrm{i}\partial_1 - \partial_2$ is the formal adjoint. The domain of H_0 consists of spinors $\psi \in H^1(\Omega; \mathbb{C}^4)$ satisfying so-called "armchair boundary conditions":

$$\psi_i(-a, x_2) = \psi_{i+2}(-a, x_2), \quad \psi_i(a, x_2) = e^{\mathrm{i}\Theta}\psi_{i+2}(a, x_2), \quad i = 1, 2,$$

where $0 \leq \Theta < 2\pi$ depends on the geometry of the waveguide. Freitas-Siegl [25] proved that H_0 is self-adjoint and that its spectrum is given by

$$\sigma(H_0) = \sigma_{\mathrm{e}}(H_0) = (-\infty, -E_0] \cup [E_0, \infty), \quad E_0 := \min_{n \in \mathbb{Z}} \left| \frac{\pi n}{2a} - \frac{\Theta}{4a} \right|.$$

Set $\xi_0 := \Theta/(4a)$. Theorem 3.6 in [C8] establishes the weak coupling asymptotics

$$z_\pm(\varepsilon) = \mp\xi_0 \pm \frac{\xi_0}{2}(u^\pm)^{\mp 2}\varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0+,$$

where, for sufficiently small $\varepsilon > 0$, $z_\pm(\varepsilon)$ is the unique eigenvalue of $H_0 + V$, provided that $\pm\mathrm{Re}\, u_\pm > 0$. Here $u_\pm$ are solutions of a quadratic equation involving the matrix elements of V. In the simplest case, where $V = \mathrm{diag}(v_1, v_2, v_3, v_4)$ with $v_j = v_j(x_2)$, $j = 1, \ldots, 4$, and $a = \Theta^2/(8\sin^2(\Theta/4))$, one has

$$u^+ = u^- = -\Big(\int_{\mathbb{R}} \mathrm{Tr}(V(x_2))\, \mathrm{d}x_2\Big)^{-1}.$$

1.4.3 Dimension $d \geq 2$

We summarize those results of [C2] that pertain to Dirac operators. A crucial difference to the one-dimensional case is that the free resolvent does not remain uniformly bounded as an operator from L^p to $L^{p'}$ for any p as $|z|$ tends to infinity.[8] This is obvious from scaling since the Dirac operator is of order one. It is also the reason why the results for Dirac are weaker than for Schrödinger.

The first two theorems below concern estimates for individual eigenvalues, i.e. are connected to question (Q1) on page 5. In contrast to the one-dimensional case, eigenvalues need no longer be confined to a compact set. It is thus of particular interest to know how large the confining set can effectively be and what its boundary curve looks like asymptotically as $|\mathrm{Re}\, z| \to \infty$. Naive estimates would give $|\mathrm{Im}\, z| \leq C|\mathrm{Re}\, z|$, i.e. a double sector in $\mathbb{C}$. Theorem 1.4.6 below improves this. Since the estimates for the massless ($m = 0$) and massive ($m > 0$) case only differ in the vicinity of $\pm m$, the distinction between these cases is of no consequence in the asymptotic regime. Therefore we restrict ourselves to the massless case here. The following theorem is an improved version of [C2, Theorem 6.1 b)], for the special case $s = 1$ there.[9]

Theorem 1.4.6 (Version of Theorems 6.1 b) in [C2])**.** *Let $d \geq 2$, and let H_0 be the massless Dirac operator (1.21) on $L^2(\mathbb{R}^d;\mathbb{C}^N)$. Assume that $d \leq q \leq \infty$, and write $q = d+\gamma$, $\gamma \geq 0$. Then every eigenvalue $z \in \mathbb{C}$ of $H_0 + V$ satisfies*

$$|\mathrm{Im}\, z|^{\gamma+\frac{d-1}{2}}|z|^{-\frac{d-1}{2}} \leq C_{d,\gamma}\|V\|_{d+\gamma}^{d+\gamma}, \tag{1.30}$$

where the constant $C_{d,\gamma} > 0$ is independent of V and z.

In particular, as $|z| \to \infty$ inside a fixed sector, say $|\arg(z)| \leq \pi/8$, then $|z|$ may be replaced by $|\mathrm{Re}\, z|$ in (1.30). For $\gamma > 0$ we have that $\beta := (d-1)/(2\gamma + d - 1) < 1$, and (1.30) has the asymptotic form $|\mathrm{Im}\, z| \leq M|\mathrm{Re}\, z|^\beta$ (where we absorbed the norm of the potential into the constant M).

We now describe the results of [C2] for the Dirac operator in relation to question (Q2).

Theorem 1.4.7 (Theorems 2.3, 6.8, 6.9 in [C2])**.** *Let $d \geq 2$, $m \geq 0$, and let H_0 be the Dirac operator (1.21) on $L^2(\mathbb{R}^d;\mathbb{C}^N)$. Then, for any $V \in L^d \cap L^{\frac{d+1}{2}}$ and $\varepsilon > 0$, we have the estimates*

$$\sum_{z\in\sigma_{\mathrm{d}}(H_0+V)} \delta(z)(1+|z|)^{-d-\varepsilon} < \infty \quad (m = 0) \tag{1.31}$$

and

$$\sum_{z\in\sigma_{\mathrm{d}}(H_0+V)} \delta(z)|z^2 - m^2|^{\frac{d-1}{2}+\varepsilon}(1+|z|)^{-2d+1-\varepsilon} < \infty \quad (m > 0). \tag{1.32}$$

In particular, if $(z_n)_n$ is a sequence of discrete eigenvalues of $H_0 + V$ that converges to a point z^* in the essential spectrum of H_0 (if $m > 0$ we assume that $z^* \neq \pm m$), then $(\delta(z_n))_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$. Moreover, if $\|V\|_d + \|V\|_{(d+1)/2}$ is sufficiently small, then $H_0 + V$ has no eigenvalues.

[8] We repeat that p' is the Hölder conjugate of p, that is $1/p + 1p' = 1$.

[9] However, it follows from the resolvent estimates proved in [C2].

It would be desirable to have an effective bound for the left hand sides of (1.31)–(1.32), i.e. a bound that only depends on the L^d- and $L^{\frac{d+1}{2}}$-norms of V, but not on V itself. Such bounds can be obtained if one replaces the scaling-critical L^d-norm by an L^q-norm with $q > d$. The arguments are analogous to those of [C2, Theorem 6.6], where we considered the fractional Laplacian.

1.5 Other non-self-adjoint operators

In this section we will summarize the results of [C5, C4] as well as those of [C2] that we not already discussed. We will focus on question (Q1) here and only discuss some special cases related to (Q2). We will consider the following operators H_0:

- Fractional Laplacian $(-\Delta)^{s/2}$, $s > 0$ [C2],
- Schrödinger operator with constant magnetic field [C5, C4],
- Harmonic oscillator [C5].

The fractional Bessel operator $(\mathbf{1} - \Delta)^{s/2} - \mathbf{1}$, $s > 0$, is also considered in [C2], but we will not discuss it here. We begin with an improved version of Theorem 6.1 in [C2] for the fractional Laplacian.[10]

Theorem 1.5.1 (Version of Theorem 6.1 in [C2])**.** *Let $d \geq 1$, $H_0 = (-\Delta)^{s/2}$, $s > 0$ and $q \geq q_s$, where*[11]

$$q_s := \begin{cases} d/s & \text{if } s < d, \\ 1+ & \text{if } s = d, \\ 1 & \text{if } s > d. \end{cases} \tag{1.33}$$

We also write $q = d/s + \gamma$, where $\gamma \geq 0$.

(i) *Let $q \leq (d+1)/2$. Then any eigenvalue $z \in \mathbb{C}$ of $H_0 + V$ satisfies*

$$|z|^\gamma \leq C_{d,s,\gamma} \|V\|_{d/s+\gamma}^{d/s+\gamma}. \tag{1.34}$$

(ii) *Let $\frac{2d}{d+1} \leq s < d$ and $V \in L^{d/s}(\mathbb{R}^d)$. If $\|V\|_{L^{d/s}}$ is sufficiently small, then $H_0 + V$ is similar to H_0 and thus has no eigenvalues.*

(iii) *Let $q > (d+1)/2$. Then any eigenvalue $z \in \mathbb{C}$ of $H_0 + V$ satisfies*

$$\delta(z)^{\gamma + \frac{d}{s} - \frac{d+1}{2}} |z|^{\frac{s(d+1)-2d}{2s}} \leq C_{d,s,\gamma} \|V\|_{d/s+\gamma}^{d/s+\gamma}, \tag{1.35}$$

where $\delta(z) := \operatorname{dist}(z, [0,\infty))$. The constants $C_{d,s,\gamma} > 0$ are independent of V and z.

[10] The improvement primarily lies in the second part of the theorem (the "long-range" case $q > (d+1)/2$). Observe that if $s < 2d/(d+1)$, then one is always in the long-range case since $(d+1)/2 < q_s$.

[11] We recall that $1+$ means $1 + \varepsilon$ for arbitrary but fixed $\varepsilon > 0$.

The proof of (1.34), (1.35) hinges on the resolvent estimates (1.70) and (1.71), respectively.

The following theorem gives an answer to (Q2) when H_0 is the fractional Laplacian. It is a generalization of the bound (1.18) of Frank-Sabin [22]. For simplicity we only state the special case corresponding to the sets A1 and A5 in [C2, Theorem 6.6].

Theorem 1.5.2 (Special case of Theorem 6.6 in [C2]). *Let $d \geq 2$, $H_0 = (-\Delta)^{s/2}$, $\frac{2d}{d+1} < s < \frac{4d}{2d+1}$ and $\frac{d}{s} < q < \frac{2d^2-2d+ds}{2ds-s}$. Then, with $\gamma = q - d/2$, we have the estimates*

$$\left(\sum_{z\in\sigma_{\mathrm{d}}(H_0+V)} \frac{\delta(z)}{|z|^{1/2}}\right)^{s\gamma} \leq L_{\gamma,d,s} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/s} \mathrm{d}x, \tag{1.36}$$

where $L_{\gamma,d,s} > 0$ is independent of V.

Next, we consider the *harmonic oscillator*

$$H_0 = -\Delta + |x|^2 \tag{1.37}$$

and, when the dimension $d = 2n$ is even, the Schrödinger operator with constant magnetic field

$$H_0 = \sum_{j=1}^{n} \left[\left(-\mathrm{i}\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\right)^2 + \left(-\mathrm{i}\frac{\partial}{\partial y_j} + \frac{1}{2}x_j\right)^2\right]. \tag{1.38}$$

Here we denoted the independent variable by $(x, y) \in \mathbb{R}^{2n}$. This operator, often called the *Landau Hamiltonian*,[12] is one of the in mathematical physics. In contrast to the cases considered so far, the spectra of (1.37)–(1.38) are pure point,

$$\sigma(H_0) = 2\mathbb{N} + m(d),$$

where $m(d) = d$ in the case of (1.37) and $m(d) = d/2$ in the case of (1.38). We note that in the case of the Landau Hamiltonian every eigenvalue has infinite multiplicity, but this will not play a role here. Concerning our question (Q1) the best one can hope for in the present situation is that the complex eigenvalues of H_0+V lie in a neighborhood of $\sigma(H_0)$. In other words, the set $\tau(H_0)$ featuring in (Q1) is now the entire spectrum $\tau(H_0) = \sigma(H_0)$, whereas before we had $\tau(H_0) = \{0\}$ or $\tau(H_0) = \{\pm m\}$ (for the massive Dirac operator). The following is a simplified version of [C5, Theorem 5.1]. It says that all eigenvalues of $H_0 + V$ must lie in a neighborhood of $\sigma(H_0)$ whose size depends only on the L^q-norm of V. For $q = \infty$ this follows from standard perturbation theory [41, V.3.5]. It is thus the case $q < \infty$ that is mainly of interest. The hard part is to prove that the imaginary part of a sequence of eigenvalues $(z_n)_{n\in\mathbb{N}}$ remains bounded as $\operatorname{Re} z_n \to +\infty$. One of the main concerns of [C5] was to also allow gradient perturbations $A \cdot \nabla$. This will be discussed in Subsection 1.8.2.

[12] As pointed out by Pushnitski-Rozenblum [59] the "Landau levels" were computed by Fock [15] two years before Landau [49].

Theorem 1.5.3 (Version of Theorem 5.1 in [C5])**.** *Let $d \geq 2$ and let H_0 be either the harmonic oscillator* (1.37) *or the Landau Hamiltonian* (1.38) *(when d is even). Let $q \geq \max(d/2, 1+)$ and let $a > 0$ be fixed. Then every eigenvalue $z \in \mathbb{C}$ of H_0+V with $|\mathrm{Im}\, z| \geq a$ satisfies*

$$|\mathrm{Im}\, z|^{1-\frac{d}{2q}} \leq C_{d,q,a}\|V\|_q. \tag{1.39}$$

The constant $C_{d,q,a} > 0$ is independent of V and z.

A similar estimate holds for the Pauli operator [C5, Corollary 5.3]. In $d = 2$ dimensions, the latter is given by

$$H_0 = \begin{pmatrix} (-\mathrm{i}\nabla + A(x))^2 + B_0 & 0 \\ 0 & (-\mathrm{i}\nabla + A(x))^2 - B_0 \end{pmatrix} \tag{1.40}$$

on $L^2(\mathbb{R}^2, \mathbb{C}^2)$. Here, $A(x, y) = \frac{B_0}{2}(-y, x)$.

For the Landau Hamiltonian, estimate (1.39) was refined in [C4]. Roughly speaking, the refinement concerns the size of neighborhoods of the k-th Landau level $\lambda_k = 2k + n \in \sigma(H_0)$ where eigenvalues of the perturbed operator may be located. More precisely, for $n = d/2 \in \mathbb{N}$, $k \in \mathbb{N}_0$, let

$$\Lambda_k := \{z \in \mathbb{C} : |\lambda_k - \mathrm{Re}\, z| \leq 1\}, \quad \lambda_k := 2k + n. \tag{1.41}$$

We define

$$\nu(q, d) := \begin{cases} \frac{d}{2q} - 1 & \text{if } \frac{d}{2} \leq q \leq \frac{d+1}{2}, \\ -\frac{1}{2q} & \text{if } \frac{d+1}{2} \leq q \leq \infty. \end{cases} \tag{1.42}$$

Theorem 1.5.4 (Theorem 2.1 in [C4])**.** *Let $d = 2n$, $n \geq 1$, and let $q \geq \max(d/2, 1+)$. Then there exist $K, C > 0$ such that for $k \geq K$ we have*

$$\sigma(H_0 + V) \cap \Lambda_k \subset \{z \in \mathbb{C} : \delta(z) \leq C\|V\|_q \lambda_k^{\nu(q,d)}\}, \tag{1.43}$$

where $\delta(z) := \mathrm{dist}(z, \sigma(H_0))$. Moreover, the estimate is sharp in the following sense: For every k as above there exists $V \in L^q(\mathbb{R}^d)$, real-valued and $V \leq 0$, such that

$$\sigma(H_0 + V) \cap \{z \in \mathbb{R} : C^{-1}\|V\|_q \lambda_k^{\nu(q,d)} \leq |z - \lambda_k| \leq C\|V\|_q \lambda_k^{\nu(q,d)}\} \neq \emptyset. \tag{1.44}$$

1.6 Dipoles in graphene

In this section we return to the Dirac operator (1.21), but this time with a hermitian potential, i.e. we will deal with self-adjoint operators. In [C7], we considered the two-dimensional Dirac operator H_γ on $L^2(\mathbb{R}^2; \mathbb{C}^2)$, initially given on the dense domain $C_0^\infty(\mathbb{R}^2 \setminus \{-x_0, x_0\}; \mathbb{C}^2)$ as

$$\begin{aligned} H_\gamma &:= H_0 + \gamma V, \\ H_0 &:= -\mathrm{i}\sigma \cdot \nabla + m\sigma_3 \\ V(x) &:= |x - x_0|^{-1} - |x + x_0|^{-1}. \end{aligned} \tag{1.45}$$

Here, $x_0 \in \mathbb{R}^2 \setminus \{0\}$ is an arbitrary point, $\gamma > 0$ is a coupling constant, $\sigma = (\sigma_1, \sigma_2)$ and $\sigma_1, \sigma_2, \sigma_3$ are the standard Pauli matrices (1.23). The operator H_γ serves as an effective model for an electron in a sheet of strained graphene in the presence a dipole potential V (see [8]). The main results of [C7], motivated by an article of De Martino et al [8], are summarized in the following theorem.

Theorem 1.6.1 (Theorems 1–3 in [C7])**.** *Assume that $\gamma < 1/2$. Then there exists a unique self-adjoint extension of H_γ with domain contained in $H^{1/2}(\mathbb{R}^2, \mathbb{C}^2)$.*[13] *For any $\gamma > 0$, H_γ has infinitely many eigenvalues $(E_n)_{\in \mathbb{N}}$ in $(-m, m)$. Moreover, these eigenvalues cluster to the edges $\pm m$ of the spectrum faster than any power, i.e.*

$$\sum_{n=1}^{\infty} (m - |E_n|)^\delta < \infty \quad \textit{for any } \delta > 0. \tag{1.46}$$

Let us mention some generalizations of Theorem 1.6.1. The proof of existence of a distinguished self-adjoint extension in [C7, Theorem 1] can accommodate finitely many sub-critical ($\gamma < 1/2$) Coulomb singularities, not just two. A quantitative bound for the right hand side of (1.46) was proved in [C7, Theorem 3], involving the best constant in Herbst's inequality [31]. The number of bound states depends on the long-range attraction of the potential rather than on its singularities. In fact, for a potential V generated by a nice charge distribution ρ we proved [C7, Theorem 4] that the non-vanishing of either the total charge or the dipole moment of ρ is necessary and sufficient for the existence of infinitely many bound states. Regarding (1.46), De Martino et al [8] in fact predicted exponential clustering of eigenvalues at the band edges $\pm m$. Rademacher-Siedentop [60] proved this for a dipole potential without Coulomb singularities. Later, Dorsch [12] proved exponential clustering for the potential V in (1.45).

1.7 Birman-Schwinger principle reloaded

The proofs of virtually all known results surrounding question (Q1) in the non-self-adjoint case conspicuously rely on the Birman-Schwinger principle (1.4). In this section we comment on a version that is suitable for question (Q2). Before we do so, we recall that in the self-adjoint case, the Birman-Schwinger principle is usually stated in the following form (see e.g. [61, Theorem X.III.10]): Suppose that $H_0 = -\Delta$ and that V is a nice enough potential, say $V \in C_c^\infty(\mathbb{R}^d)$. If $N_E(V)$ denotes the number of bound states of $H_0 + V$ below $E < 0$ and $n(Q(E); 1)$ denotes the number of singular values $(s_j)_{j \in \mathbb{N}}$ of the compact operator $Q(E) := V_-^{1/2}(H_0 - E)^{-1} V_-^{1/2}$ above 1, then

$$N_E(V) \leq n(Q(E); 1).$$

The important observation is that, for any $\alpha > 0$,

$$n(Q(E); 1) = \sum_{s_j \geq 1} 1 \leq \sum_j s_j^\alpha =: \|Q(E)\|_{\mathfrak{S}^\alpha}^\alpha = \mathrm{Tr}[Q(E)^* Q(E)]^{\alpha/2}. \tag{1.47}$$

[13] By abuse of notation we will continue to denote this self-adjoint extension by H_γ.

The space $\mathfrak{S}^\alpha \equiv \mathfrak{S}^\alpha(\mathcal{H})$, with $\mathcal{H} = L^2(\mathbb{R}^d)$ here, is called the *Schatten space* of order α. It is an ideal (called a "trace ideal") in the space $\mathfrak{S}^\infty$ of all compact operators on $\mathcal{H}$ with $\|\cdot\|_{\mathfrak{S}^\infty}$ being the operator norm. Note, in particular, that (1.47) implies that $N_E(V) = 0$ whenever $\|Q(E)\|_{\mathfrak{S}^\infty} < 1$. Hence, in the self-adjoint case, (1.47) is a generalization of the simple version of the Birman-Schwinger principle (1.4). It lies at the core of several estimates for the number of bound states of self-adjoint Schrödinger operators (see e.g. [61, Section XIII.3]). The simplest proofs of the Lieb-Thirring inequalities (1.19) also rely on the Birman-Schwinger principle (1.47) (see e.g. [52]).

The proof of (1.47) relies on the monotonicity of $Q(\cdot)$. In the non-self-adjoint setting, $E < 0$ is replaced by the complex number $z \in \mathbb{C}\backslash\sigma(H_0)$ and hence this method breaks down. An alternative is furnished by complex analysis arguments in the following fashion (we refer to [10] for details): Set $Q(z) := |V|^{1/2}(H_0 - z)^{-1}V^{1/2}$, and suppose that $Q(z) \in \mathfrak{S}^\alpha$. The main idea is that eigenvalues $z \in \mathbb{C} \setminus \sigma(H_0)$ of $H_0 + V$ correspond to zeros of the analytic function

$$\mathbb{C} \setminus \sigma(H_0) \ni z \mapsto h(z) := \mathrm{Det}_{\lceil\alpha\rceil}(1 + Q(z)) \in \mathbb{C}, \tag{1.48}$$

see [66, 22]. Here, $\lceil\alpha\rceil$ is the smallest integer which is $\geq \alpha$ and Det_n, $n \in \mathbb{N}$, is a *regularized Fredholm determinant*, defined for $A \in \mathfrak{S}^n(\mathcal{H})$ by

$$\mathrm{Det}_n(1 + A) := \prod_k \left[(1 + \lambda_k(A)) \exp\Big(\sum_{j=1}^{n-1}(-1)^j j^{-1}\lambda_n(A)^j\Big)\right]$$

see e.g. [66, Chapter 9]. Here $(\lambda_k(A))_{k\in\mathbb{N}}$ are the eigenvalues of A. This observation opens the possibility to use classical theorems in complex analysis on the distribution of zeros of analytic functions in the quest of solving (Q2). The most basic estimate is *Jensen's inequality* which states that if h is a bounded analytic function on the unit disk, then

$$\sum_{h(z)=0} (1 - |z|) < \infty.$$

There are two obstacles when trying to apply Jensen's inequality to the funtion h in (1.48). First, this function is not defined on the unit disk but on $\mathbb{C} \setminus \sigma(H_0)$. Second, h will usually not be bounded. The first obstacle can be easily dealt with by a conformal mapping, at least if $\mathbb{C} \setminus \sigma(H_0)$ is simply connected. Note that $\sigma(H_0)$ is mapped to the boundary of the unit disk. The second obstacle is more serious. A pivotal result in this area is a deep theorem due to Borichev-Golinskii-Kupin [6] on zeros of analytic functions that may blow up at the boundary.[14] This theorem is sensitive to the rate of blowup at "non-generic" points of the boundary. By virtue of the bound (see e.g. [14, Lemma XI.9.22])

$$\log|h(z)| \leq \Gamma_\alpha \|Q(z)\|_{\mathfrak{S}_\alpha}^\alpha, \tag{1.49}$$

it is clear that Schatten norm estimates for $Q(z)$ that are *uniform* in z dramatically improve the output of the Borichev-Golinskii-Kupin theorem.

[14]Their result is stated in the paragraph preceeding (5.9) in [C1].

1.8 Uniform resolvent estimates

As already emphasized, a crucial input for the Birman-Schwinger principle to work its magic are resolvent estimates for H_0 that have a good behavior in the spectral parameter z. In the case of (Q1) one needs $L^p \to L^{p'}$ resolvent estimates since these immediately imply $L^2 \to L^2$ estimates for the Birman-Schwinger operator. In the case of (Q2) one needs stronger trace ideal bounds, as discussed in the previous section. The latter were first established for $H_0 = -\Delta$ in the pioneering work of Frank-Sabin [22] for "short-range" potentials and later by Frank [17] for "long-range potentials".[15] Their results may be viewed as special cases of Proposition 1.8.3 below for $s = 2$.

1.8.1 Translation-invariant operators

We now summarize the main results of [C2] concerning uniform resolvent estimates for a translation-invariant operator $T(D)$ (a Fourier multiplier), acting as

$$\widehat{T(D)f}(\xi) = T(\xi)\widehat{f}(\xi). \tag{1.50}$$

Here $T : \mathbb{R}^d \to \mathbb{R}$ is a polynomially bounded function and

$$\widehat{f}(\xi) := \int \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x$$

is the Fourier transform of $f \in L^1$.

In applications, $H_0 := T(D)$ plays the role of the kinetic energy for the generalized Schrödinger operator $H = H_0 + V$ on $L^2(\mathbb{R}^d)$. In this case H_0 is defined on its maximal domain

$$D(H_0) := \{\psi \in L^2(\mathbb{R}^d) : T(D)\psi \in L^2(\mathbb{R}^d)\} \tag{1.51}$$

and $V \in L^q(\mathbb{R}^d)$. For simplicity we also assume that V is bounded, so that H may be defined as an operator sum. Other examples of kinetic energies, besides the ones already considered, that play a role in mathematical physics are

- $T(\xi) = (m^2 + |\xi|^2)^{1/2}$,
- $T(\xi) = \sum_{j=1}^d (1 - \cos(\xi_j))$,
- $T(\xi) = (|\xi|^2 - \mu)\frac{\mathrm{e}^{\beta(|\xi|^2-\mu)}+1}{e^{\beta(|\xi|^2-\mu)}-1}$.

The first is the symbol of the relativistic kinetic energy of a particle of mass m, the second is the symbol of the discrete Laplacian on the cubic lattice $\mathbb{Z}^d$, and the third is related to the BCS (Bardeen-Cooper-Schrieffer) theory of superconductivity (here $\beta > 0$ is the inverse temperature and $\mu > 0$ is the chemical potential), see e.g. [18].

[15] We recall that in the present case "short-range" means $V \in L^q(\mathbb{R}^d)$ with $q \leq (d+1)/2$, while "long-range" amounts to $q > (d+1)/2$.

Fix $\lambda \in \mathbb{R}$ and consider the *Fermi surface* at energy λ:

$$M_\lambda := \{\xi \in \mathbb{R}^d : T(\xi) = \lambda\}. \tag{1.52}$$

We will assume that T is a smooth function in a neighborhood of M_λ. The key technical result in [C2] is Lemma 4.3 there. It yields uniform estimates in Schatten spaces for a frequency-localized resolvent. The frequency-localization cuts out a precompact neighborhood of the Fermi surface M_λ where λ is, roughly speaking, the real part of the spectral parameter. Since we are only interested in frequency-localized estimates here, by modifying T away from M_λ, we may restrict our attention to globally smooth functions. A point $\xi \in \mathbb{R}^d$ is called a *critical point* if $\nabla T(\xi) = 0$. The set of *critical values* of T is defined as

$$\kappa(T) := \{\lambda \in \mathbb{R} : \exists\, \xi \in \mathbb{R}^d \text{such that } T(\xi) = \lambda,\ \nabla T(\xi) = 0\}. \tag{1.53}$$

The set of *regular values* is the complement in $\mathbb{R}$ of $\kappa(T)$. If λ is a regular value, then M_λ is a smooth $d-1$ dimensioanl submanifold of $\mathbb{R}^d$ (a hypersurface). Fix $\xi_0 \in M_\lambda$. By normalizing T, we may assume without loss of generality that $|\nabla T(\xi_0)| = 1$ on M_λ. The *principal curvatures* of M_λ at ξ_0 are the eigenvalues $\kappa_1, \ldots, \kappa_{d-1}$ of the curvature form

$$\mathbb{R}^d \ni v \mapsto \sum_{i,j=1}^{d} \frac{\partial^2 T(\xi_0)}{\partial \xi_i \partial \xi_j} v_i v_j \in \mathbb{R},$$

restricted to the tangent space of M_λ at ξ_0. The *Gaussian curvature* of M_λ at ξ_0 is the product $\kappa_1 \cdot \ldots \cdot \kappa_{d-1}$.

For completeness we recall some basic facts about the spectral theory of $T(D)$ (see e.g. [3, Section 7.6.2]):

1. $\sigma(H_0) = \overline{T(\mathbb{R}^d)}$, and H_0 has purely a.c. spectrum in $\sigma(H_0) \setminus \kappa(T)$. It is purely a.c. if $T^{-1}(\kappa(T)) \subset \mathbb{R}^d$ has measure zero.

2. By Sard's theorem (see e.g. Guillemin-Pollack [30]), $\kappa(T) \subset \mathbb{R}$ has measure zero. If T is analytic, then $\kappa(T)$ is discrete. If T is a polynomial, then $\kappa(T)$ is a finite set.

3. Assume that

$$|T(\xi)| + |\nabla T(\xi)| \to \infty \quad \text{if} \quad |\xi| \to \infty. \tag{1.54}$$

   Then $\kappa(T) \subset \mathbb{R}$ is closed.

We state here a slight generalization of [C2, Lemma 4.3]. It will be convenient to have this version available in view of the discussion in Section 1.9 below. The original version corresponds to the special case $k = d-1$, that is the case where the Fermi surface M_λ in (1.52) has everywhere non-vanishing Gaussian curvature.

Proposition 1.8.1 (Generalization of Lemma 4.3 in [C2])**.** *Let*[16] $T \in C^\infty(\mathbb{R}^d;\mathbb{R})$ *and let* $\chi \in C_c^\infty(\mathbb{R}^d)$ *be such that* T *has no critical points in* $\operatorname{supp}(\chi)$*. Assume that* M_λ *has at*

[16] The assumption in [C2, Lemma 4.3] is that T is C^2. This is enough to define curvature of M_λ, but one needs a bit more regularity to make the stationary phase argument work. This does not affect the results of the paper since in all applications T is smooth near the Fermi surface M_λ.

least $k \leq d-1$ nonvanishing principal curvatures at every point in some neighborhood of $M_\lambda \cap \mathrm{supp}(\chi)$. Let $1 \leq q \leq (k+2)/2$. Let $H_0 := T(D)$ with domain given by (1.51), and set $R_0(z) := (H_0 - z)^{-1}$. Then there exists a constant $C > 0$ such that for all $A, B \in L^{2q}(\mathbb{R}^d)$ and all $z \in \mathbb{C} \setminus \sigma(H_0)$ we have the estimate

$$\|A\chi(D)R_0(z)B\|_{\mathfrak{S}^{\alpha_q}} \leq C\|A\|_{2q}\|B\|_{2q}, \tag{1.55}$$

where[17]

$$\alpha_q := \begin{cases} \frac{2(d-1-k/2)q}{d-q}, & \text{if } \frac{d}{d-k/2} \leq q \leq \frac{k+2}{2}, \\ \frac{qk+}{qk-d(q-1)}, & \text{if } 1 \leq q < \frac{d}{d-k/2}. \end{cases}$$

Since Proposition 1.8.1 is local in Fourier space it does not depend on ellipticity of T. The dependence on T is only via the local geometry (curvature) of the Fermi surfaces M_λ. On the other hand, if T satisfies the weak ellipticity assumption (1.54), then M_λ is compact, and one can sum up the local estimates. In combination with the standard Kato-Seiler-Simon inequality [66, Theorem 4.1] this then yields bounds for the full resolvent $R_0(z)$. Note that, in the complement of a compact neighborhood of M_λ, the symbol $T(\xi) - z$ is nonzero, uniformly for z near λ, so that $(T(\cdot) - z)^{-1}$ will have uniformly bounded L^p-norm for suitable p.

The estimate (1.55) is the main ingredient in the proofs of Theorems 1.4.6, 1.4.7, 1.5.1 above. The proof of (1.55) relies on a factorization of the symbol $T(\xi)$, stationary phase estimates (this is where the curvature assumption comes in) and a suitable version of Stein's interpolation theorem for analytic families of operators [68], see also [70, Section IX.1.2.5].

In their proof, Frank-Sabin [22] used a pointwise estimate from Kenig-Ruiz-Sogge [45] on complex powers of the resolvent kernel of $(-\Delta - z)^{-1}$. This estimate is derived from an explicit formula for the Fourier transform of a quadratic form in terms of modified Bessel functions.[18] Such an explicit formula is not available in the case considered here. As a replacement, the following stationary phase lemma, which is a formalization of the arguments in the proof of [C2, Lemma 4.3], is used:

Lemma 1.8.2 (Pointwise bounds on complex powers)**.** *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth real-valued function and let $\psi \in C^\infty(\mathbb{R}^n)$. Assume that*

$$\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}xh(\eta)}\psi(\eta)\mathrm{d}\eta = \mathcal{O}(|x|^{-r}), \quad \text{as} x \to \pm\infty, \tag{1.56}$$

for some $r > 0$. Given $a \in [1, 1+r]$, $t \in \mathbb{R}$, define the tempered distributions

$$u_{a,t}^{\pm}(\xi,\eta) := \mathrm{e}^{\pi^2(a+\mathrm{i}t)^2}\psi(\eta)(\xi - h(\eta) \pm \mathrm{i}0)^{-a-\mathrm{i}t}, \quad \xi \in \mathbb{R}, \quad \eta \in \mathbb{R}^n. \tag{1.57}$$

Then the $(n+1)$-dimensional inverse Fourier transform $v_{a,t}^{\pm} := \mathcal{F}^{-1}u_{a,t}^{\pm}$ satisfies the pointwise estimate

$$\sup_{t\in\mathbb{R}} \sup_{(x,y)\in\mathbb{R}\times\mathbb{R}^n} (1 + |x| + |y|)^{1+r-a}|v_{a,t}^{\pm}(x,y)| < \infty.$$

[17] We recall that the Schatten norms $\|\cdot\|_{\mathfrak{S}^{\alpha_q}}$ were defined in (1.47) and that $qk+ = qk + \varepsilon$ for arbitrary but fixed $\varepsilon > 0$.

[18] See formula (2.21) in [45] or pages 288-289 in Gelfand-Shilov [26].

We sketch the main steps in the proof of Proposition 1.8.1 above: First, by the Phragmén-Lindelöf maximum principle (see e.g. [64, Section 5.3]) it is sufficient to prove the claim for the incoming or outgoing resolvents

$$(T(D)-(\lambda\pm\mathrm{i}0))^{-1}:=\lim_{\varepsilon\to+0}(T(D)-(\lambda\pm\mathrm{i}\varepsilon))^{-1}. \tag{1.58}$$

These are defined, when λ is a regular value, as maps from $C_c^\infty(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ (see e.g. [34, Lemma 6.2.2]). The estimate (1.55) is trivial for λ outside a compact neighborhood of $T(\mathrm{supp}(\chi)) := \{T(\xi)\in\mathbb{R}:\xi\in\mathbb{R}^d\}\subset\mathbb{R}$, so we may concentrate on a fixed $\lambda\in T(\mathrm{supp}(\chi))$. Set $p(\xi):=T(\xi)-\lambda$. By a partition of unity argument we may assume that either $|p|\geq 1/C$ or else that $|\nabla p|\geq 1/C$. In the first case, the estimate (1.55) easily follows from Young's inequality. We turn to the second case. For each $\xi_0\in\mathrm{supp}(\chi)$ there is a $j\in\{1,\dots,d\}$ such that $|\partial_j p(\xi)|\geq 1/(\sqrt{d}C)$. By a linear change of coordinates we may always assume that $j=1$. By the implicit function theorem, we may then assume that the Fermi surface M_λ (see (1.52)) is given as the graph of a smooth function h,

$$M_\lambda=\{(h(\xi'),\xi'):\xi'\in U\}, \tag{1.59}$$

where $U\subset\mathbb{R}^{d-1}$. Then we have the factorization

$$p(\xi)=e(\xi)(\xi_1-h(\xi')),\quad (\xi_1,\xi')\in\mathbb{R}\times U\subset\mathbb{R}^d, \tag{1.60}$$

where $e(\xi)\neq 0$. By [33, Lemma 6.2.2] we have that

$$(p(\xi)\pm\mathrm{i}0)^{-1}=e(\xi)^{-1}(\xi_1-h(\xi')\pm\mathrm{i}0)^{-1}\quad\text{in }\mathcal{S}'(\mathbb{R}^d). \tag{1.61}$$

We pick $\psi\in C_c^\infty(\mathbb{R}^{d-1})$ such that $\psi(\xi')=1$ whenever $\xi\in\mathrm{supp}(\chi)$, and thus $\chi(\xi)=\chi(\xi)\psi(\xi')$. Discarding of χ/e by means of Young's inequality, it remains to prove that

$$\|A\psi(D')(D_1-h(D')\pm\mathrm{i}0)^{-1}B\|_{\mathfrak{S}^{\alpha_q}}\leq C\|A\|_{2q}\|B\|_{2q}.$$

Let $\mathrm{d}\sigma$ be the induced Lebesgue measure on M_λ and let $\beta\in C_c^\infty(\mathbb{R}^d)$. Then $\mathrm{d}\mu:=\beta\mathrm{d}\sigma$ is a compactly supported measure on M_λ with Fourier transform

$$\widehat{\mathrm{d}\mu}(x)=\int_{M_\lambda}\mathrm{e}^{-\mathrm{i}x\cdot\xi}\mathrm{d}\mu(x).$$

In particular, if $\mathrm{supp}(\beta)\subset\mathbb{R}\times U$, then

$$\widehat{\mathrm{d}\mu}(x)=\int_U\mathrm{e}^{-\mathrm{i}x\cdot(h(\xi'),\xi')}\beta((h(\xi'),\xi'))\sqrt{1+|\nabla h(\xi')|^2}\mathrm{d}\xi'.$$

Littman [56] proved that

$$|\widehat{\mathrm{d}\mu}(x)|\leq C(1+|x|)^{-k/2}.$$

This implies (1.56) with $r=k/2$, $n=d-1$. Lemma 1.8.2 thus yields the kernel bound

$$\sup_{t\in\mathbb{R}}|\mathrm{e}^{\pi^2(a+\mathrm{i}t)^2}\psi(D')(D_1-h(D')\pm\mathrm{i}0)^{-(a+\mathrm{i}t)}(x,y)|\leq C(1+|x-y|)^{a-1-k/2}. \tag{1.62}$$

Consider the family of operators

$$T_z := A^z e^{\pi^2 z^2} \psi(D')(D_1 - h(D') \pm i0)^{-z} B^z, \quad 0 \le \mathrm{Re}\, z \le 1 + k/2.$$

For simple functions f, g the map $z \mapsto \langle f, T_z g \rangle$ is bounded and continuous in the strip $0 \le \mathrm{Re}\, z \le 1 + k/2$ and analytic in its interior. Moreover, for any $t \in \mathbb{R}$, we have

$$\|T_{a+it}\|_{\mathfrak{S}^2} \le C \|A\|^a_{\frac{2ad}{d-1-k/2+a}} \|B\|^a_{\frac{2ad}{d-1-k/2+a}}, \quad 1 \le a \le 1 + k/2, \tag{1.63}$$

$$\|T_{it}\|_{\mathfrak{S}^\infty} \le C. \tag{1.64}$$

The first bound (1.63) follows from (1.62) together with the Hardy-Littlewood-Sobolev inequality (see e.g Lieb-Loss [52, Theorem 4.3]). The second bound (1.64) follows from Plancherel's theorem. Let $\theta = \frac{1-0}{a-0}$ and $\frac{1}{\alpha_q} = \frac{1}{\infty} + \theta \left(\frac{1}{2} - \frac{1}{\infty} \right)$, i.e. $\theta = \frac{1}{a}$, $\alpha_q = 2a$. Complex interpolation between (1.63) and (1.64) (see e.g. Gohberg-Krein [27, Theorem 13.1]) yields

$$\|T_1\|_{\mathfrak{S}^{\alpha_q}} \le C \|A\|^{\theta a}_{\frac{2ad}{d-1-k/2+a}} \|B\|^{\theta a}_{\frac{2ad}{d-1-k/2+a}}.$$

Changing variables

$$q = \frac{2ad}{d - 1 - k/2 + a} \iff a = \frac{q(d - 1 - k/2)}{d - q} \tag{1.65}$$

and observing that

$$1 \le a \le 1 + k/2 \iff \frac{d}{d - k/2} \le q \le 1 + k/2$$

yields the claim (1.55) for q as in (1.65). The proof for the case $1 \le q < \frac{d}{d-k/2}$ relies on the bound

$$\|A\phi(D) dE(\lambda) \phi(D) B\|_{\mathfrak{S}^1} \le C \|A\|_2 \|B\|_2, \tag{1.66}$$

where $dE(\lambda)$ denotes the spectral measure associated to the operator $T(D)$. To prove (1.66), one writes $dE(\lambda)/d\lambda = \mathcal{R}(\lambda)^* \mathcal{R}(\lambda)$, where $\mathcal{R}(\lambda)$ is the Fourier restriction operator to the Fermi surface M_λ,

$$\mathcal{R}(\lambda) f := \widehat{f}|_{M_\lambda}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The operators $A\phi(D)\mathcal{R}(\lambda)^*$ and $\mathcal{R}(\lambda)\phi(D)B$ are Hilbert-Schmidt since their kernels, given by $(2\pi)^{-d} A(x) \phi(\xi) e^{ix \cdot \xi}$ and $e^{-ix \cdot \xi} \phi(\xi) B(x)$, respectively, are in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Since $\|\phi\|_2 \le C$, the Hilbert-Schmidt norms are bounded by $C\|A\|_2$ and $C\|B\|_2$, respectively. Hölder's inequality in trace ideals (see e.g. [66, Theorem 2.8]) thus implies (1.66). Next, write

$$R_0(z)^{b+it} = \int_{\mathbb{R}} (\lambda - z)^{-b-it} dE(\lambda), \quad z \in \mathbb{C} \setminus \mathbb{R} \quad 0 < b < 1.$$

Using the local integrability of $(\cdot - z)^{-b-it}$ (uniformly in z) and (1.66), we then obtain, for $0 < b < 1$,

$$\|A\phi(D) R_0(\lambda \pm i0)^{b+it} \phi(D) B\|_{\mathfrak{S}^1} \le C e^{\pi t} (1 - b)^{-1} \|A\|_2 \|B\|_2. \tag{1.67}$$

By (1.62), we have the Hilbert-Schmidt bound

$$\|A\phi(D)R_0(\lambda\pm \mathrm{i}0)^{1+k/2+\mathrm{i}t}\phi(D)B\|_{\mathfrak{S}^2}\leq C\mathrm{e}^{\pi^2t^2}\|A\|_2\|B\|_2. \tag{1.68}$$

For fixed $0<b<1$ consider the analytic family

$$z\mapsto A\phi(D)R_0(\lambda\pm \mathrm{i}0)^z\phi(D)B,\quad b\leq \mathrm{Re}\, z\leq 1+k/2.$$

Complex interpolation between (1.67) and (1.68) yields

$$\|A\phi(D)R_0(\lambda\pm \mathrm{i}0)\phi(D)B\|_{\mathfrak{S}^{1+}}\leq C\|A\|_2\|B\|_2. \tag{1.69}$$

We repeat that 1+ means $1+\varepsilon$ for any $\varepsilon>0$; of course, the constant C depends on ε. Estimate (1.55) for $1\leq q<\frac{d}{d-k/2}$ now follows from interpolation between (1.69) and the the part of (1.55) for $q\geq \frac{d}{d-k/2}$ already proven.

We now state the relevant part of [C2, Theorem 3.1] for the full resolvent of the fractional Laplacian.

Proposition 1.8.3. *Let $d\geq 1$, $H_0=(-\Delta)^{s/2}$, $s>0$ and let $q\geq q_s$, where q_s is defined in* (1.33)*. Then the following estimates hold for $R_0(z):=((-\Delta)^{s/2}-z)^{-1}$.*

(i) If $q\leq (d+1)/2$, then we have the estimate

$$\|R_0(z)\|_{L^p\to L^{p'}}\leq C_{d,s,q}|z|^{\frac{d}{sq}-1},\quad \frac{1}{p}-\frac{1}{p'}:=\frac{1}{q}. \tag{1.70}$$

(ii) If $q>(d+1)/2$, then we have the estimate

$$\|R_0(z)\|_{L^p\to L^{p'}}\leq C_{d,s,q}\delta(z)^{-1+\frac{d+1}{2(\gamma+d/s)}}|z|^{\frac{2d-s(d+1)}{2s(\gamma+d/s)}},\quad \frac{1}{p}-\frac{1}{p'}:=\frac{1}{q}. \tag{1.71}$$

The constant $C_{d,s,q}>0$ is independent of z.

Proof. By scaling we may assume that $|z|=1$. Let χ be a bump function adapted to the unit sphere S^{d-1}.[19] Since the latter has everywhere non-vanishing Gaussian curvature we may apply Proposition 1.8.1 with $k=d-1$. This yields (1.70) with $\chi(D)R_0(z)$ instead of $R_0(z)$. The estimate for $(1-\chi(D))R_0(z)$ follows from Sobolev embedding, together with standard estimates for the Bessel potential $(\mathbf{1}-\Delta)^{s/2}$, see e.g. [69, Section V.3]. The same argument yields (1.71) for that part of the resolvent, and it remains to prove (1.71) for $\chi(D)R_0(z)$. This follows by interpolation of (1.70) with $d=(d+1)/2$ and the trivial estimate $\|\chi(D)R_0(z)\|_{L^2\to L^2}\leq \delta(z)^{-1}$. □

[19] For example $\chi\in C^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\chi)\subset\{\xi\in\mathbb{R}^d: 1/2\leq|\xi|\leq 3/2\}$ and $\chi(\xi)=1$ for $3/4\leq|\xi|\leq 5/4$.

1.8.2 Magnetic Schrödinger operators

We now turn to the resolvent estimates of [C5, C4] where H_0 is no longer translation-invariant. In fact, H_0 will be an electromagnetic Schrödinger operator

$$H_0 = (-\mathrm{i}\nabla + A_0(x))^2 + V_0(x) \tag{1.72}$$

on $L^2(\mathbb{R}^d)$ with $d \geq 2$. Here, $A_0 : \mathbb{R}^d \to \mathbb{R}^d$ is the vector potential and $V_0 : \mathbb{R}^d \to \mathbb{R}$ is the electric potential. The perturbed operator H will also be of the form (1.72), but with A_0, V_0 replaced by $A := A_0 + A_1, V := V_0 + V_1$. In fact, the techniques in [C5] are general enough to allow for more general "gradient perturbations", i.e.

$$H = H_0 + L, \quad L = a \cdot \nabla + U.$$

Of course, in the case above, we would have

$$a = -2\mathrm{i}A_1, \quad U = -\mathrm{i}(\nabla \cdot A_1) + A_1^2 + V_1.$$

The "background potentials" A_0, V_0 are assumed to be smooth but may be unbounded at infinity. The "perturbation potentials" A_1, V_1 are assumed to decay at infinity in a suitable sense but may be rough.

The aim is to prove $L^p \to L^{p'}$ ($p < 2$) resolvent estimates[20] for H that are uniform away from the spectrum. We will state these estimates in the form

$$\|u\|_{p'} \leq C\|(H - z)u\|_p, \quad \frac{1}{p} - \frac{1}{p'} := \frac{1}{q}, \tag{1.73}$$

where $q \geq \max(d/2, 1+)$ and the constant is supposed to be uniform for $\mathrm{dist}(z, \sigma(H)) \geq a$, with $a > 0$ fixed. We will only be dealing with self-adjoint operators here, so $\sigma(H) \subset \mathbb{R}$. For simplicity, we will assume $|\mathrm{Im}\, z| \geq a$. Since H may have eigenvalues it will generally not be possible to obtain a limiting absorption principle as in the previous subsection, i.e. the estimates cannot be uniform up to the spectrum. Indeed, if z is an eigenvalue with corresponding eigenfunction u, then the right hand side of (1.73) is zero. If $\mathrm{Re}\, z$ is bounded from above or if z lies outside some fixed sector, a routine application of the diamagnetic inequality (see e.g. Lieb-Seiringer [53, Theorem 4.4]) and Sobolev embedding would prove (1.73) for $1/p - 1/p' \leq 2/d$. The case where $\mathrm{Re}\, z$ is large and positive and $|\mathrm{Im}\, z| = \mathcal{O}(1)$ is much harder. It may be viewed as a semiclassical problem with semiclassical parameter $1/\mathrm{Re}\, z$.

The precise assumptions on A, V are as follows. In the following, $\varepsilon > 0$ is a yet undetermined constant that will later be chosen sufficiently small.

(A1) $A_0 \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and for every $\alpha \in \mathbb{N}^n$, $|\alpha| \geq 1$, there exist constants $C_\alpha, \varepsilon_\alpha > 0$ such that

$$|\partial_x^\alpha A_0(x)| \leq C_\alpha, \quad |\partial^\alpha B_0(x)| \leq C_\alpha \langle x \rangle^{-1-\varepsilon_\alpha}, \quad x \in \mathbb{R}^d. \tag{1.74}$$

Here, $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $B_0 = (B_{0,j,k})_{j,k=1}^n$ is the magnetic field, i.e.

$$B_{0,j,k}(x) = \partial_j A_{0,k}(x) - \partial_k A_{0,j}(x).$$

[20] To be consistent with the previous subsection we change the notation from [C5, C4] from q' to p and from r to q.

(A2) $V_0 \in C^\infty(\mathbb{R}^d;\mathbb{R})$ and for every $\alpha \in \mathbb{N}^n$, $|\alpha| \geq 2$, there exist constants $C_\alpha > 0$ such that

$$|\partial_x^\alpha V_0(x)| \leq C_\alpha, \quad x \in \mathbb{R}^d. \tag{1.75}$$

(A3) $A_1 \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and there exists $\delta > 0$ such that

$$|A_1(x)| \leq \varepsilon \langle x \rangle^{-1-\delta} \quad \text{for almost every } x \in \mathbb{R}^d.$$

Moreover, assume that one of the following additional assumptions holds (with $\delta > 0$ as above):

(A3a) A_1 is Lipschitz and

$$|\nabla A_1(x)| \leq \varepsilon \langle x \rangle^{-1-\delta} \quad \text{for almost every } x \in \mathbb{R}^d.$$

(A3b) There exists $\delta' \in (0,\delta)$ such that[21] $\langle x \rangle^{1+\delta'} A_1 \in \dot{W}^{\frac{1}{2},2d}(\mathbb{R}^d;\mathbb{R}^d)$, with

$$\|\langle x \rangle^{1+\delta'} A_1\|_{\dot{W}^{\frac{1}{2},2d}} \leq \varepsilon.$$

(A4) $V_1 \in L^q(\mathbb{R}^d;\mathbb{R})$, with $\|V_1\|_q \leq \varepsilon$, for some $q \geq \max(d/2, 1+)$.

One of the main challenges in proving $L^p \to L^{p'}$ resolvent estimates is the presence of the gradient perturbation $a \cdot \nabla$. It is well known, even for $H_0 = -\Delta$, that uniform $L^p \to L^{p'}$ resolvent estimates with a gain of a full derivative cannot hold (see e.g. [4] and [37]). This is a major obstacle for proving (1.73) perturbatively. One way to circumvent the problem is to combine the $L^p \to L^{p'}$ estimates for the free operator with weighted L^2 estimates. The local smoothing effect allows one to gain a full derivative.

In order to state the resolvent estimates it is convenient to introduce the following spaces: Let $E_{1/2}$ be the Weyl quantization of the symbol $|(x,\xi)|^{1/2}$. The space X is defined to be the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_X := \|u\|_2 + \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_2 + \|u\|_{p'},$$

where $0 < \mu \leq \delta$ ($\delta > 0$ from (A3)) is fixed and $1/p - 1/p' := 1/q$, with $q \geq \max(d/2, 1+)$ as above. The topological dual of X is the space of distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{X'} := \inf_{f = f_1 + f_2 + f_3} \left(\|f_1\|_2 + \|\langle x \rangle^{\frac{1+\mu}{2}} E_{-1/2} f_2\|_2 + \|f_3\|_p \right)$$

is finite.

Theorem 1.8.4 (Theorem 2.2 in [C5])**.** *Assume that A, V satisfy Assumptions (A1)–(A4) with A_0, V_0 fixed. Moreover, let $a > 0$ be fixed. Then there exist constants $C, \varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, then we have the estimate*

$$\|(H - z)^{-1}\|_{X' \to X} \leq C \tag{1.76}$$

for all $z \in \mathbb{C}$ with $|\operatorname{Im} z| \geq a$. The constants C, ε_0 depend on d, q, μ, δ, a and on finitely many seminorms C_α in (1.74) *and* (1.75).

[21] $\dot{W}^{\frac{1}{2},2d}(\mathbb{R}^d;\mathbb{R}^d)$ is the homogeneous Sobolev space $(-\Delta)^{-1/4}(L^{2d}(\mathbb{R}^d;\mathbb{R}^d))$.

Note in particular that (1.76) contains the $L^p \to L^{p'}$ estimate. The corresponding estimates for H_0 (see (1.72)) are a consequence of Strichartz estimates for the inhomogeneous Schrödinger equation

$$\mathrm{i}\partial_t u - H_0 u = f, \quad u|_{t=0} = u_0, \quad (t,x) \in [0,T] \times \mathbb{R}^d,$$

where $T > 0$ is sufficiently small. These, in turn, follow by the abstract Keel-Tao argument [43] and the following short-time dispersive estimate

$$\|\mathrm{e}^{\mathrm{i}tH_0}\|_{L^1 \to L^\infty} \leq Ct^{-d/2}, \quad t \leq T,$$

due to Yajima [75].[22] The smoothing estimate

$$\|\langle x\rangle^{-\frac{1+\mu}{2}} E_{1/2}(H_0 - z)^{-1} E_{1/2} \langle x\rangle^{-\frac{1+\mu}{2}}\|_{L^2 \to L^2} \leq C \tag{1.77}$$

is proved by means of a positive commutator argument (see [C5, Lemma 3.2]). The latter is proved by pseudodifferential operator techniques (a suitable version of the sharp Gårding inequality). Having established (1.76) for H_0, the corresponding estimate for H is proved perturbatively. The main tool here are commutator estimates for pseudodifferential operators with limited smoothness, in the spirit of the Coifman-Meyer theorem, see e.g. Taylor [71, Section 4.1].

For the Landau Hamiltonian (1.38) we have an improvement of the $L^p \to L^{p'}$ bound of Theorem 1.8.4.

Proposition 1.8.5 (Proposition 2.2 in [C4])**.** *Let $d = 2n$, $n \geq 1$, and let H_0 be the Landau Hamiltonian* (1.38)*. Let $q \geq \max(d/2, 1+)$ and $1/p - 1/p' := 1/q$. Then we have the estimate*

$$\|(H_0 - z)^{-1}\|_{L^p \to L^{p'}} \leq C_{d,q}(1 + |\mathrm{Re}\, z|)^{\rho(p')}(1 + \delta(z)^{-1}), \tag{1.78}$$

where

$$\rho(p') := \begin{cases} \frac{1}{p'} - \frac{1}{2} & \text{if } 2 \leq p' \leq \frac{2(d+1)}{d-1}, \\ \frac{d-2}{2} - \frac{d}{p'} & \text{if } \frac{2(d+1)}{d-1} \leq p', \end{cases} \tag{1.79}$$

and $\delta(z) := \mathrm{dist}(z, \sigma(H_0))$. The constant $C_{d,q} > 0$ is independent of z.

The main advantage of (1.78) over (1.76) is that, in the range $2 < p' < 2d/(d-2)$[23], the exponent $\rho(p')$ is negative, and hence the right hand side of (1.78) tends to 0 as $\mathrm{Re}\, z \to +\infty$.

The proof of Proposition 1.8.5 is based on spectral projection estimates of Koch-Ricci [46] and dispersive estimates of Koch-Tataru [47]. The latter have a local character, but since the Landau Hamiltonian is translation-invariant up to a phase shift, the local estimates may be glued together to yield a global estimate (see [C4, (2.14)]).

The resolvent estimate (1.78) is optimal in the sense that the exponent $\rho(p')$ cannot be improved. This follows from the optimality of the spectral projection estimates of [46]. It is noteworthy that the resolvent estimates (1.76) and (1.78) may be used to prove spectral projection estimates as the following exemplary result shows. For simplicity we assume that $d = 2n \geq 4$, but the $d = 2$ case may be proved along the same lines.

[22] This is where Assumptions (A1)-(A2) come from.

[23] This corresponds to $d/2 < q < \infty$ in the assumption of Proposition 1.8.5.

Corollary 1.8.6. *Let $d = 2n \geq 4$. Under the assumptions of Proposition 1.8.5 consider $H = H_0 + V$ where $V \in L^q(\mathbb{R}^d)$ is real-valued. Then there exists $C > 0$ such that for $\lambda > 1$ we have the spectral projection estimates*

$$\|\mathbf{1}_{[\lambda^2-1,\lambda^2+1]}(H)\|_{L^p \to L^2} \leq C_{d,q}\lambda^{\rho(p')}. \tag{1.80}$$

Proof. By a result of Frank-Schimmer [23, Lemma 10] the spectral projection estimate is implied by the resolvent estimate

$$\|(H - (\lambda + \mathrm{i})^2)^{-1}\|_{L^p \to L^{p'}} \leq C\lambda^{2\rho(p')}. \tag{1.81}$$

By interpolation it is sufficient to prove (1.81) for the endpoints $p' = 2$, $p' = 2(d+1)/(d-1)$ and $p' = 2d/(d-2)$.[24] The estimate for $p' = 2$ is trivial. The estimate for $p' = 2d/(d-2)$ follows from (1.76). It remains to prove (1.81) for $p' = 2(d+1)/(d-1)$. By (1.78) we may assume that λ is so large that $\|V\|_q \|(H_0 - (\lambda + \mathrm{i})^2)^{-1}\| \leq 1/2$. The resolvent identity and (1.78) then yield

$$\begin{aligned}\|(H - (\lambda + \mathrm{i})^2)^{-1}\|_{L^p \to L^{p'}} &\leq \|(H_0 - (\lambda + \mathrm{i})^2)^{-1}\|_{L^p \to L^{p'}} \sum_{k \geq 0} \|V(H_0 - (\lambda + \mathrm{i})^2)^{-1}\|^k_{L^p \to L^p} \\ &\leq 2\|(H_0 - (\lambda + \mathrm{i})^2)^{-1}\|_{L^p \to L^{p'}} \leq C\lambda^{2\rho(p')}. \qquad \square\end{aligned}$$

1.9 Embedded eigenvalues for generalized Schrödinger operators

We return to the study of generalized Schrödinger operators $H = T(D) + V$ with $T(D)$ given by (1.50). To motivate the results of [C3], we start with a classical result of Kato [38] on the absence of embedded eigenvalues of Schrödinger operators with short range potentials. More precisely, assume that $|V(x)| \leq C(1 + |x|)^{-1-\varepsilon}$ for some $\varepsilon > 0$. Then $-\Delta + V$ has no embedded eigenvalue. This is sharp[25] in view of the Wigner-von Neumann example [74].

We would now like to trade pointwise decay of V at infinity for some average decay. In the L^q-scale, larger q means less decay. It is therefore of interest to find the largest q such that $V \in L^q(\mathbb{R}^d)$ implies the absence of embedded eigenvalues. Koch-Tataru [48] proved that $-\Delta + V$ has no embedded eigenvalue if $V \in L^{\frac{d+1}{2}}(\mathbb{R}^d)$. This is also sharp[26] in view of an example of Ionescu-Jerison [35].

There is no hope that a generalization of the Kato or Koch-Tataru result might hold for general $H_0 = T(D)$. For example, it is well-known (see e.g. Herbst-Skibsted [32]) that there are compactly supported potentials V such that $\Delta^2 + V$ has embedded eigenvalues. Therefore, we ask the following question, related to (Q3) on page 5. Assuming that $\lambda \in \mathbb{R} \setminus \kappa(T)$ is a fixed regular value[27] of T, what is the largest $q_c \in [1, \infty)$ such that the following holds:

[24] The assumption $d \geq 3$ is only used in this interpolation argument.
[25] In the sense that the conclusion fails for $\varepsilon = 0$.
[26] In the sense that the conclusion fails for $q > \frac{d+1}{2}$.
[27] In the terminology of Section 1.2: $\tau(H_0) = \kappa(T)$. See (1.53) for the definition of $\kappa(T)$.

1. $\exists\varepsilon_0$ such that $\|V\|_{q_c} + \|V\|_\infty \leq \varepsilon_0$ implies $\lambda \notin \sigma_{\rm p}(H_0 + V)$,

2. $\exists (V_n)_{n\in\mathbb{N}}$ such that for all $q > q_c$: $\lim_{n\to\infty} \|V_n\|_q = 0$ and $\lambda \in \sigma_{\rm p}(H_0 + V_n)$.

Here, $\sigma_{\rm p}(H)$ denotes the point spectrum of H, i.e. the set of eigenvalues. Note that we did not assume that V is real-valued. The case of real-valued V will only be considered for a specific $T(D)$ (the Chandrasekhar-Herbst operator). For simplicity we only consider bounded potentials here, i.e. $V \in L^\infty(\mathbb{R}^d)$. Under additional ellipticity assumptions on T one can easily allow local singularities of V.

Frank-Simon [24] proved that for $H_0 = -\Delta$ one has $q_c = \frac{d+1}{2}$.

In the following we will assume that $d \geq 2$. The statements are still true for $d = 1$ but yield nothing new.

Theorem 1.9.1 (Theorems 1.1 and 1.2 in [C3]). [28] *Let $d \geq 2$, and let $T : \mathbb{R}^d \to \mathbb{R}$ be smooth and polynomially bounded, and let $\lambda \in \mathbb{R}$ be a regular value of T. Denote by M_λ the Fermi surface* (1.52).

(i) If M_λ has at most $k \leq d-1$ non-vanishing principal curvatures at some point, then one has the upper bound $q_c \leq \frac{k+2}{2} + \frac{d-1-k}{3}$.

(ii) If M_λ has at least $k \leq d-1$ non-vanishing principal curvatures at every point, then one has the lower bound $q_c \geq \frac{k+2}{2}$.

Assume that M_λ has *exactly* k non-vanishing principal curvatures *at every point*. In the case $k = d-1$ the upper and lower bounds of Theorem 1.9.1 match up and we get $q_c = \frac{d+1}{2}$. If $k < d-1$ there is a gap between the upper and the lower bound. This means that the vanishing or nonvanishing of principal curvatures is not enough to determine q_c completely in this case. Instead, one would also have to take into account the order of vanishing. For instance, if $T(\xi) = \xi_1^2 + \ldots + \xi_{k+1}^2$ where $k < d-1$ (i.e. T depends on less than d variables), then for any $\lambda > 0$ the Fermi surface M_λ is a cylinder. Using the methods of [C3] one can construct potentials V_n such that

$$|V_n(x)| \leq C_m(n + |x_1|^2 + \ldots + |x_k|^2 + |x_{k+1}|^m + \ldots + |x_{d-1}|^m + |x_d|)^{-1}$$

for arbitrary positive m. Then $V_n \in L^q(\mathbb{R}^d)$ for all $q > (k+2)/2 + (d-1-k)/m$. This shows that the upper bound $q_c \leq (k+2)/2$ is in general optimal. The problem for the lower bound is probably much more difficult.

The assumptions on the kinetic energy in Theorem 1.9.1 are too general to allow for counterexamples with real-valued potentials. The minimal assumption seems to be that T is time-reversal symmetric, i.e. $T(\xi) = T(-\xi)$. This implies that generalized eigenfunctions of $T(D)$ can be chosen real-valued, see [C3, Lemma 4.1]. We conjecture that this assumption is sufficient. At present, we only have examples for specific kinetic energies. The following theorem generalizes a recent result of Lorinczi-Sasaki [57] on the Chandrasekhar-Herbst operator to nonradial potentials.

[28] Part (ii) follows from Proposition 1.8.1.

Theorem 1.9.2 (Theorem 1.4 in [C3]). *Let $d \geq 2$, and let $H_0 = \sqrt{-\Delta + 1} - 1$ be the Chandrasekhar-Herbst operator with mass 1, and let $\lambda > 0$. Then there exists a sequence of smooth potentials $V_n : \mathbb{R}^d \to \mathbb{R}$, $n \in \mathbb{N}$, satisfying*

$$|V_n(x)| \leq C(n + |x_1|^2 + \ldots + |x_{d-1}|^2 + |x_d|)^{-1}, \tag{1.82}$$

such that λ is an eigenvalue of $H_0 + V_n$ in $L^2(\mathbb{R}^d)$, for every $n \in \mathbb{N}$. The constant $C > 0$ is independent of n, but may depend on λ.

In comparison with the radial potential of [57], which decays like $1/|x|$, the (non-radial) potential in Theorem 1.9.2 exhibits this decay only in a parabolic tube about a single coordinate direction.

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