A constructive proof of the minimax theorem

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The von Neumann minimax theorem

Theorem 1 (classical)

Let A be an $n \times m$ matrix. Then

$$\max_{\mathbf{y}\in S^m} \min_{\mathbf{x}\in S^n} \mathbf{x}^T A\mathbf{y} = \min_{\mathbf{x}\in S^n} \max_{\mathbf{y}\in S^m} \mathbf{x}^T A\mathbf{y},$$

where S^n is the n-dimensional simplex.

- S^n and S^m are inhabited compact convex subsets of normed spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively;
- $(\mathbf{x},\mathbf{y}) \mapsto \mathbf{x}^T A\mathbf{y}$ is a uniformly continuous function from $S^n \times S^m$ into $\mathbf{R}$;
- $(\cdot)^T A\mathbf{y} : S^n \to \mathbf{R}$ is convex for each $\mathbf{y} \in S^m$;
- $\mathbf{x}^T A(\cdot) : S^m \to \mathbf{R}$ is concave for each $\mathbf{x} \in S^n$.

Inhabited sets and constructive suprema

Definition 2

A set S is inhabited if there exists x such that $x \in S$.

Definition 3

Let S be a subset of $\mathbf{R}$. Then $s = \sup S \in \mathbf{R}$ is a supremum of S if $\forall x \in S(x \leq s)$ and $\forall \epsilon > 0 \exists x \in S(s < x + \epsilon)$.

Proposition 4

An inhabited subset S of $\mathbf{R}$ with an upper bound has a supremum if and only if for each $a, b \in \mathbf{R}$ with $a < b$, either $\exists x \in S(a < x)$ or $\forall x \in S(x < b)$.

Metric spaces

Definition 5

A metric space is a set X equipped with a metric $d : X \times X \to \mathbf{R}$ such that

- $d(x, y) = 0 \leftrightarrow x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, y) \leq d(x, z) + d(z, y)$,

for each $x, y, z \in X$.

Remark 6

Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Then $X \times Y$ is a metric space with a metric

$$d_{X \times Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$$

for each $(x, y), (x', y') \in X \times Y$.

Closures and closed subsets

Definition 7

The closure $\overline{S}$ of a subset S of a metric space X is defined by

$$\overline{S} = \{x \in X \mid \forall \epsilon > 0 \exists y \in S[d(x,y) < \epsilon]\}.$$

A subset S of a metric space X is closed if $\overline{S} = S$.

Definition 8

A sequence (x_n) of X converges to $x \in X$ if

$$\forall \epsilon > 0 \exists N \forall n \geq N[d(x_n, x) < \epsilon].$$

Remark 9

A subset S of a metric space X is closed if and only if $x \in S$ whenever there exists a sequence (x_n) of S converging to x.

Uniform continuity and total boundedness

Definition 10

A mapping f between metric spaces X and Y is uniformly continuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall xy \in X[d(x, y) < \delta \to d(f(x), f(y)) < \epsilon].$$

Definition 11

A metric space X is totally bounded if for each $\epsilon > 0$ there exist $x_1, \ldots, x_n \in X$ such that $\forall y \in X \exists i \in \{1, \ldots, n\}[d(x_i, y) < \epsilon]$.

Proposition 12

If f is a uniformly continuous function from an inhabited totally bounded metric space X into $\mathbf{R}$, then $\sup_{x \in X} f(x)$ exists.

Complete and compact metric spaces

Definition 13

A sequence (x_n) of a metric space is a Cauchy sequence if

$$\forall \epsilon > 0 \exists N \forall mn \geq N[d(x_m, x_n) < \epsilon].$$

A metric space is complete if every Cauchy sequence converges.

Definition 14

A metric space is compact if it is totally bounded and complete.

Convex sets and convex functions

Definition 15

A subset C of a linear space is convex if $\lambda x + (1-\lambda)y \in C$ for each $x, y \in C$ and $\lambda \in [0,1]$.

Definition 16

A function f from a convex subset C of a linear space into $\mathbf{R}$ is

- convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

  for each $x, y \in C$ and $\lambda \in [0,1]$;

- concave if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$

  for each $x, y \in C$ and $\lambda \in [0,1]$.

Normed spaces

Definition 17

A normed space is a linear space E equipped with a norm $\|\cdot\| : E \to \mathbf{R}$ such that

- $\|x\| = 0 \leftrightarrow x = 0$,
- $\|ax\| = |a|\|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$,

for each $x, y \in E$ and $a \in \mathbf{R}$.

Note that a normed space E is a metric space with the metric

$$d(x, y) = \|x - y\|.$$

Definition 18

A Banach space is a normed space which is complete with respect to the metric.

The minimax theorem

Theorem 19

Let K and C be inhabited totally bounded convex subsets of normed spaces E and F, respectively, and let $f : K \times C \to \mathbf{R}$ be a uniformly continuous function such that

- $f(\cdot, y) : K \to \mathbf{R}$ *is convex for each* $y \in C$;
- $f(x, \cdot) : C \to \mathbf{R}$ *is concave for each* $x \in K$.

Then

$$\sup_{y \in C} \inf_{x \in K} f(x, y) = \inf_{x \in K} \sup_{y \in C} f(x, y).$$

General lemmata

Lemma 20

Let X and Y be inhabited totally bounded metric spaces, and let $f : X \times Y \to \mathbf{R}$ be a uniformly continuous function. Then $\sup_{y \in Y} \inf_{x \in X} f(x, y)$ and $\inf_{x \in X} \sup_{y \in Y} f(x, y)$ exist, and

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) \leq \inf_{x \in X} \sup_{y \in Y} f(x, y).$$

Lemma 21

Let X and Y be inhabited totally bounded metric spaces, let $f : X \times Y \to \mathbf{R}$ be a uniformly continuous function, and let $c \in \mathbf{R}$. If $c < \inf_{x \in X} \sup_{y \in Y} f(x, y)$, then there exist $y_1, \ldots, y_n \in Y$ such that

$$c < \inf_{x \in X} \max\{f(x, y_i) \mid 1 \leq i \leq n\}.$$

Fan's theorem for inequalities

Theorem 22

Let K be an inhabited totally bounded convex subset of a normed space E, let $f_1, \ldots, f_n$ be uniformly continuous convex functions from K into $\mathbf{R}$, and let $c \in \mathbf{R}$. Then

$$c < \inf_{x \in K} \max\{f_i(x) \mid 1 \leq i \leq n\}$$

if and only if there exist nonnegative numbers $\lambda_1, \ldots, \lambda_n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$c < \inf_{x \in K} \sum_{i=1}^{n} \lambda_i f_i(x).$$

Proof.

We will give a proof after a proof of the minimax theorem. □

A proof of the minimax theorem

Proof.

By Lemma 20, it suffices to show that

$$\sup_{y \in C} \inf_{x \in K} f(x,y) \geq \inf_{x \in K} \sup_{y \in C} f(x,y).$$

Let $c = \sup_{y \in C} \inf_{x \in K} f(x,y)$, and suppose that

$$c < \inf_{x \in K} \sup_{y \in C} f(x,y).$$

Then, by Lemma 21, there exist $y_1, \ldots, y_n \in C$ such that

$$c < \inf_{x \in K} \max\{f(x,y_i) \mid 1 \leq i \leq n\}.$$

□

A proof of the minimax theorem

Proof.

Therefore, by Theorem 22, there exist nonnegative numbers $\lambda_1, \ldots, \lambda_n$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$c < \inf_{x \in K} \sum_{i=1}^n \lambda_i f(x, y_i).$$

Since $f(x, \cdot) : C \to \mathbf{R}$ is concave for each $x \in K$, we have

$$c < \inf_{x \in K} \sum_{i=1}^n \lambda_i f(x, y_i) \le \inf_{x \in K} f(x, \sum_{i=1}^n \lambda_i y_i) \le \sup_{y \in C} \inf_{x \in K} f(x, y),$$

a contradiction. □

Hilbert spaces

Definition 23

An inner product space is a linear space E equipped with an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathbf{R}$ such that

- $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \leftrightarrow x = 0$,
- $\langle x, y \rangle = \langle y, x \rangle$,
- $\langle ax, y \rangle = a\langle x, y \rangle$,
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

for each $x, y, z \in E$ and $a \in \mathbf{R}$.

Note that an inner product space E is a normed space with the norm

$$\|x\| = \langle x, x \rangle^{1/2}.$$

Definition 24

A Hilbert space is an inner product space which is a Banach space.

Hilbert spaces

Remark 25

Let E be an inner product space. Then

- $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$,
- $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

for each $x, y \in E$.

Example 26

Define an inner product on $\mathbf{R}^n$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=0}^{n} x_i y_i$$

for each $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$.
Then $\mathbf{R}^n$ is a Hilbert space.

Closest points

Lemma 27

Let C be a convex subset of a Hilbert space H, and let $x \in H$ be such that $d = d(x, C) = \inf\{\|x - y\| \mid y \in C\}$ exists. Then there exists $z \in \overline{C}$ such that $\|x - z\| = d$.

Proof.

We may assume without loss of generality that $x = 0$. Let (y_n) be a sequence in C such that $\|y_n\| \to d$ as $n \to \infty$. Then

$$\begin{aligned} \|y_m - y_n\|^2 &= 2\|y_m\|^2 + 2\|y_n\|^2 - 4\|(y_m + y_n)/2\|^2 \\ &\leq 2\|y_m\|^2 + 2\|y_n\|^2 - 4d^2 \to 0 \end{aligned}$$

as $m, n \to \infty$, and hence (y_n) is a Cauchy sequence in H. Therefore (y_n) converges to a limit $z \in \overline{C}$, and so $\|z\| = d$. $\square$

A separation theorem

Proposition 28

Let C be a convex subset of a Hilbert space H, and let $x \in H$ be such that $d = d(x, C)$ exists and $0 < d$. Then there exists $z_0 \in H$ such that $\|z_0\| = 1$ and $d + \langle z_0, x\rangle \leq \langle z_0, y\rangle$ for each $y \in C$.

A separation theorem

Proof.

We may assume without loss of generality that $x = 0$. By Lemma 27, there exists $z \in \overline{C}$ such that $\|z\| = d$. Note that $\|z\| \leq \|y\|$ for each $y \in \overline{C}$. Let $z_0 = z/d$, and let $y \in C$. Then $\|z_0\| = 1$. Since

$$\|z\|^2 \leq \|(1 - 1/n)z + (1/n)y\|^2 = \|z + (1/n)(y - z)\|^2$$
$$= \|z\|^2 + (2/n)\langle z, y - z\rangle + (1/n^2)\|y - z\|^2,$$

we have $0 \leq (2/n)\langle z, y - z\rangle + (1/n^2)\|y - z\|$ for each n, and hence

$$0 \leq \langle z, y - z\rangle + (1/2n)\|y - z\|^2$$

for each n. Therefore, letting $n \to \infty$, we have $0 \leq \langle z, y - z\rangle$, and so $d^2 \leq \langle z, y\rangle$. Thus $d \leq \langle z_0, y\rangle$. □

Fan's theorem for inequalities

Theorem 29

Let K be an inhabited totally bounded convex subset of a normed space E, let $f_1, \ldots, f_n$ be uniformly continuous convex functions from K into $\mathbf{R}$, and let $c \in \mathbf{R}$. Then

$$c < \inf_{x \in K} \max\{f_i(x) \mid 1 \leq i \leq n\}$$

if and only if there exist nonnegative numbers $\lambda_1, \ldots, \lambda_n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$c < \inf_{x \in K} \sum_{i=1}^{n} \lambda_i f_i(x).$$

A proof of Fan's theorem

Since the "if" part is trivial, we show the "only if" part.

Define a subset C of $\mathbf{R}^{n+1}$ by

$$C = \{(u_1, \ldots, u_{n+1}) \in \mathbf{R}^{n+1} \mid \exists x \in K \forall i \in \{1, \ldots, n\}(f_i(x) \leq u_i + u_{n+1} + c)\}.$$

Lemma 30

C is inhabited.

Proof.

Let $x_0 \in K$, and let $t_0 \in \mathbf{R}$ be such that $0 < t_0$ and $\max\{f_i(x_0) \mid 1 \leq i \leq n\} < t_0 + c$. Then $(0, \ldots, 0, t_0) \in C$. □

A proof of Fan's theorem

Lemma 31

C is a convex subset of $\mathbf{R}^{n+1}$.

Proof.

Let $\mathbf{u} = (u_1, \ldots, u_{n+1}), \mathbf{v} = (v_1, \ldots, v_{n+1}) \in C$, and let $\lambda \in [0, 1]$. Then there exist x and y in K such that $f_i(x) \leq u_i + u_{n+1} + c$ and $f_i(y) \leq v_i + v_{n+1} + c$ for each $i \in \{1, \ldots, n\}$, and, since f_i is convex for each $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned} f_i(\lambda x + (1-\lambda)y) &\leq \lambda f_i(x) + (1-\lambda) f_i(y) \\ &\leq (\lambda u_i + (1-\lambda) v_i) + (\lambda u_{n+1} + (1-\lambda) v_{n+1}) + c \end{aligned}$$

for each $i \in \{1, \ldots, n\}$. Therefore, since K is convex, we have $\lambda \mathbf{u} + (1-\lambda)\mathbf{v} \in C$. □

A proof of Fan's theorem

Suppose that $d = d(0, C)$ exists and $0 < d$.
Then, by Proposition 28, there exists $(\alpha_1, \ldots, \alpha_{n+1}) \in \mathbf{R}^{n+1}$ such that $\sum_{i=1}^{n+1} |\alpha_i|^2 = 1$ and

$$d \leq \sum_{i=1}^{n+1} \alpha_i u_i$$

for each $\mathbf{u} \in C$.

Lemma 32

$0 < \alpha_{n+1}$ and $0 \leq \alpha_i$ for each $i \in \{1, \ldots, n\}$.

Proof.

Since $(0, \ldots, 0, t_0) \in C$, we have $d \leq \alpha_{n+1} t_0$, and hence $0 < d/t_0 \leq \alpha_{n+1}$. Since $(0, \ldots, 0, m, 0, \ldots, 0, t_0) \in C$ for each m, we have $d \leq m\alpha_i + \alpha_{n+1} t_0$ for each $i \in \{1, \ldots, n\}$ and m, and hence $0 \leq \alpha_i$ for each $i \in \{1, \ldots, n\}$. □

A proof of Fan's theorem

Let $\lambda_i = \alpha_i / \alpha_{n+1}$ for each $i \in \{1, \ldots, n\}$, and recall that

$$C = \{(u_1, \ldots, u_{n+1}) \in \mathbf{R}^{n+1} \mid \exists x \in K \forall i \in \{1, \ldots, n\}(f_i(x) \leq u_i + u_{n+1} + c)\}.$$

Then, for each $r \in \mathbf{R}$ and $x \in K$, since

$$(f_1(x) - c + r, \ldots, f_n(x) - c + r, -r) \in C,$$

we have

$$d \leq \sum_{i=1}^{n} \alpha_i (f_i(x) - c + r) - \alpha_{n+1} r,$$

and hence

$$\frac{d}{\alpha_{n+1}} \leq \sum_{i=1}^{n} \lambda_i (f_i(x) - c) + r \left(\sum_{i=1}^{n} \lambda_i - 1 \right).$$

A proof of Fan's theorem

Lemma 33

$\sum_{i=1}^{n} \lambda_i = 1.$

Proof.

For $r > 0$ and $x = x_0$, we have

$$\frac{d}{\alpha_{n+1} r} \leq \frac{1}{r} \sum_{i=1}^{n} \lambda_i (f_i(x_0) - c) + \left(\sum_{i=1}^{n} \lambda_i - 1 \right),$$

and hence, letting $r \to \infty$, we have $1 \leq \sum_{i=1}^{n} \lambda_i$.

For $r < 0$ and $x = x_0$, we have

$$\frac{d}{\alpha_{n+1} r} \geq \frac{1}{r} \sum_{i=1}^{n} \lambda_i (f_i(x_0) - c) + \left(\sum_{i=1}^{n} \lambda_i - 1 \right),$$

and hence, letting $r \to -\infty$, we have $1 \geq \sum_{i=1}^{n} \lambda_i$. □

A proof of Fan's theorem

Lemma 34

$c < \inf_{x \in K} \sum_{i=1}^{n} \lambda_i f_i(x)$.

Proof.

By Lemma 33, we have

$$\frac{d}{\alpha_{n+1}} \leq \sum_{i=1}^{n} \lambda_i (f_i(x) - c) = \sum_{i=1}^{n} \lambda_i f_i(x) - c$$

for each $x \in K$, and hence $c < \inf_{x \in K} \sum_{i=1}^{n} \lambda_i f_i(x)$. □

It remains to show that $d = d(0, C)$ exists and $0 < d$.

A proof of Fan's theorem

Lemma 35

There exists $d' > 0$ such that $d' < \|\mathbf{u}\|$ for each $\mathbf{u} \in C$.

Proof.

Let $d' \in \mathbf{R}$ be such that

$$c < 4d' + c < \inf_{x \in K} \max\{f_i(x) \mid 1 \leq i \leq n\},$$

and let $\mathbf{u} = (u_1, \ldots, u_{n+1}) \in C$. If $|u_i| < 2d'$ for each $i = 1, \ldots, n+1$, then there exists $x' \in K$ such that

$$\begin{aligned} f_i(x') \leq u_i + u_{n+1} + c &< 2d' + 2d' + c = 4d' + c \\ &< \inf_{x \in K} \max\{f_i(x) \mid 1 \leq i \leq n\} \end{aligned}$$

for each $i \in \{1, \ldots, n\}$, a contradiction. Therefore $d' < |u_i|$ for some $i = 1, \ldots, n+1$, and so $d' < (\sum_{i=1}^{n+1} |u_i|^2)^{1/2} = \|\mathbf{u}\|$. □

A proof of Fan's theorem

Lemma 36

$d = d(0, C)$ exists.

Proof.

Since $\{\|\mathbf{u}\| \mid \mathbf{u} \in C\}$ is inhabited and has a lower bound 0, it suffices, by Proposition 4, to show that for each $a, b \in \mathbf{R}$ with $a < b$, either

- $\|\mathbf{u}\| < b$ for some $\mathbf{u} \in C$, or
- $a < \|\mathbf{u}\|$ for each $\mathbf{u} \in C$.

Let $a, b \in \mathbf{R}$ with $a < b$, and let $\epsilon = (b - a)/5$. □

A proof of Fan's theorem

Proof.

Then, since $f_1, \ldots, f_n$ are uniformly continuous, there exists $\delta > 0$ such that

$$\forall xy \in K \forall i \in \{1, \ldots, n\}(\|x - y\| < \delta \rightarrow |f_i(x) - f_i(y)| < \epsilon).$$

Since K is totally bounded, there exist $y_1, \ldots, y_m \in K$ such that

$$\forall x \in K \exists j \in \{1, \ldots, m\}(\|x - y_j\| < \delta).$$

Also, since

$$B = \{\mathbf{w} \in \mathbf{R}^{n+1} \mid \|\mathbf{w}\| < b\}$$

is a totally bounded subset of $\mathbf{R}^{n+1}$, there exist $\mathbf{w}^1 = (w_1^1, \ldots, w_{n+1}^1), \ldots, \mathbf{w}^l = (w_1^l, \ldots, w_{n+1}^l) \in B$ such that

$$\forall \mathbf{u} \in B \exists k \in \{1, \ldots, l\}(\|\mathbf{u} - \mathbf{w}^k\| < \epsilon).$$

A proof of Fan's theorem

Proof.

Either

- $\forall i \in \{1, \ldots, n\}(f_i(y_j) < w_i^k + w_{n+1}^k + c)$ for some $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, l\}$, or
- $\exists i \in \{1, \ldots, n\}(f_i(y_j) + \epsilon > w_i^k + w_{n+1}^k + c)$ for each $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, l\}$.

In the former case, $\|\mathbf{u}\| < b$ for some $\mathbf{u} \in C$.

In the latter case, assume that $\|\mathbf{u}\| < a + \epsilon$ for some $\mathbf{u} = (u_1, \ldots, u_{n+1}) \in C$. □

A proof of Fan's theorem

Proof.

Then there exists $x \in K$ such that

$$\forall i \in \{1, \ldots, n\}(f_i(x) \leq u_i + u_{n+1} + c).$$

Therefore there exists $j \in \{1, \ldots, m\}$ such that $\|x - y_j\| < \delta$, and so

$$\forall i \in \{1, \ldots, n\}(f_i(y_j) < f_i(x) + \epsilon).$$

Let $\mathbf{u}' = (u_1, \ldots, u_n, u_{n+1} + 4\epsilon)$. Then

$$\|\mathbf{u}'\| = (\sum_{i=1}^{n} |u_i|^2 + |u_{n+1} + 4\epsilon|^2)^{1/2} \leq (\|\mathbf{u}\|^2 + 8\epsilon\|\mathbf{u}\| + (4\epsilon)^2)^{1/2}$$
$$= \|\mathbf{u}\| + 4\epsilon < a + 5\epsilon = b,$$

and hence there exists $k \in \{1, \ldots, l\}$ such that $\|\mathbf{u}' - \mathbf{w}^k\| < \epsilon$. □

A proof of Fan's theorem

Proof.
Therefore, since

$$f_i(y_j) + 3\epsilon < f_i(x) + 4\epsilon \leq u_i + (u_{n+1} + 4\epsilon) + c < w_i^k + w_{n+1}^k + c + 2\epsilon$$

for each $i \in \{1, \ldots, n\}$, we have

$$\forall i \in \{1, \ldots, n\}(f_i(y_j) + \epsilon < w_i^k + w_{n+1}^k + c),$$

a contradiction.
Thus $a < a + \epsilon \leq \|\mathbf{u}\|$ for each $\mathbf{u} \in C$. □

A generalization

Definition 37

Let X and Y be metric spaces. Then a function $f : X \times Y \to \mathbf{R}$. is convex-concave like if

- for each $x, x' \in X$ and $\lambda \in [0, 1]$, there exists $z \in X$ such that

  $$f(z, y) \leq \lambda f(x, y) + (1 - \lambda) f(x', y)$$

  for each $y \in Y$, and
- for each $y, y' \in Y$ and $\lambda \in [0, 1]$, there exists $z \in Y$ such that

  $$f(x, z) \geq \lambda f(x, y) + (1 - \lambda) f(x, y')$$

  for each $x \in X$.

A generalization

Definition 38
A set $\{f_i \mid i \in I\}$ of functions between metric spaces X and Y is uniformly equicontinuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall i \in I \forall xy \in X[d(x,y) < \delta \to d(f_i(x), f_i(y)) < \epsilon].$$

A generalization

Theorem 39

Let X and Y be metric spaces, and let $f : X \times Y \to \mathbf{R}$ be a convex-concave like function such that the set $\{f(\cdot, y) \mid y \in Y\}$ of functions from X into $\mathbf{R}$ is uniformly equicontinuous.
If X is totally bounded, and $\sup_{y \in Y} \inf_{x \in X} f(x, y)$ and $\inf_{x \in X} \sup_{y \in Y} f(x, y)$ exist, then

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y).$$

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