# Constructive Operations Research

## *Bridging Logic, Mathematics, Computer Science, and Philosophy*

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# What is Operations Research?


Chair for Operations Research,Technical University of Munich

**Since the 2nd World War, *operations research* has been an *interdisciplinary research field* of *mathematics*, *computer science*, and *management* which should *optimize algorithms* with *economic* and *societal relevance*.**

# Why Constructive Operations Research (CORE)?

***Operations research* aims at *efficient* and *reliable* *algorithms* to *solve problems in economy* and *society*.**

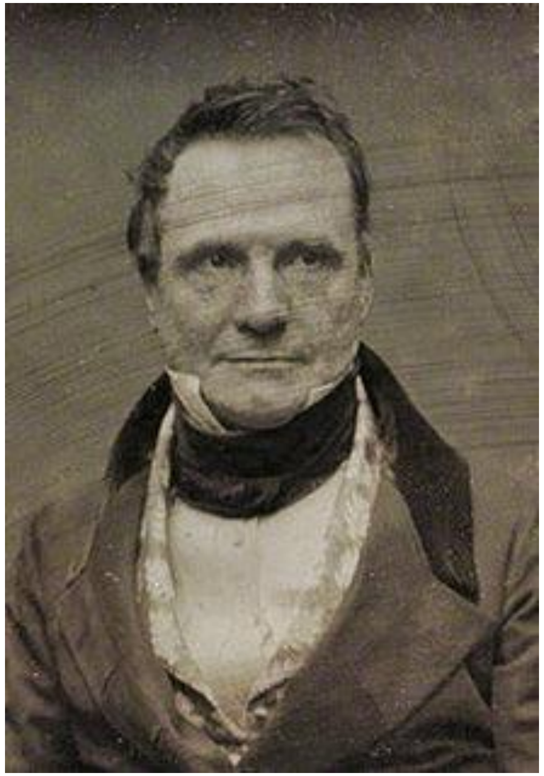
***Trust & security* can only be guaranteed by *constructive proofs*.**

**Therefore, besides *mathematics*, *computer science* and *management*, also *logic* and *philosophy* come in.**

1. **Basics of Computability**
2. **Computability in Higher Types**
3. **Proof  Mining and Program Extraction**
4. **From Brouwer‘s Creative Subject to Fan Theorem**
5. **Reverse Mathematics and Constructivity**
6. **Intuitionistic Type Theory and Proof Assistants**
7. **Univalent Foundation of Mathematics**
8. **Proof-of-Work and Financial Trust**
9. **Bridging Logic, Mathematics, Computer Science and Philosophy**

# 1. Basics of Computability

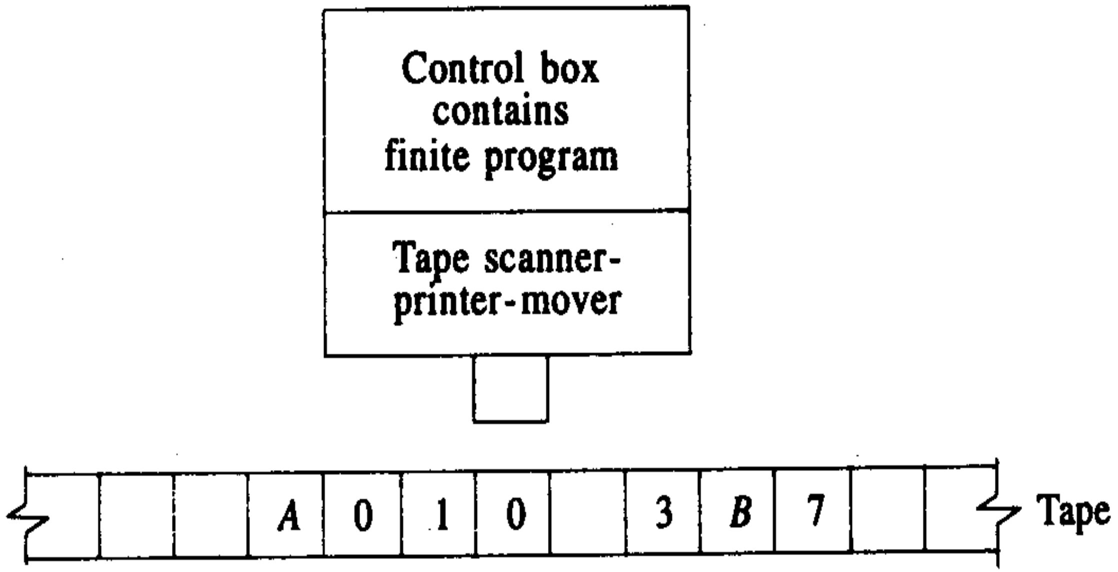
# C. Babbage: Computer and Operations Research



**The British polymath Charles Babbage (1791-1871), mathematician, philosopher, inventor, and mechanical engineer, is well-known as „*father of the computer*" who originated the concept of a *digital programmable computer* with his first mechanical models (e.g., difference machine, analytical engine).**

**Babbage also published *On the Economy of Machinery and Manufactures* (1832), on the organization of industrial production. It was an influential early work of *operational research* with great impact on *political economy*.**

# Turing's Theory of Computability




Every *computable procedure* (algorithm) can be simulated by a *Turing machine* (*Church's thesis*). Every *Turing program* can be simulated by a *universal Turing machine* (*general purpose computer*).

A *Turing machine* is a *formal procedure*, consisting of

a) a *control box* in which a *finite program* is placed,

b) a potential *infinite tape*, divided lengthwise into squares,

c) a device for *scanning*, or *printing* on one square of the tape at a time, and for *moving* along the tape or *stopping*, all under the command of the *control box*.

# 2. Computability in Higher Types

# Approximations of Computable Functionals in Information Systems

In order to describe *approximations* of *abstract objects* like *functionals* by *finite* ones, we use an *information system* with a *countable set A* of *bits* of *data* ("*tokens*") (Scott 1982, Schwichtenberg/Wainer 2012). *Approximations* need *finite sets U* of *data* which are *consistent* with each other. An "*entailment relation*" expresses the fact that the *information* of a *consistent set U* of data is *sufficient* to compute a *bit of information* (*"token"*) :

An *information system* is a *structure* $(A, \mathrm{Con}, \vdash)$ with a *countable set A* ("*tokens*"), *non-empty set* Con of *finite* ("*consistent*") *subsets* of *A* and *subset* $\vdash$ of $\mathrm{Con} \times A$ ("*entailment relation*") with

*i.* $U \subseteq V \in \mathrm{Con} \Longrightarrow U \in \mathrm{Con}$

*ii.* $\{a\} \in \mathrm{Con}$

*iii.* $U \vdash a \in \mathrm{Con} \Longrightarrow U \vdash a$

*iv.* $a \in U \in \mathrm{Con} \Longrightarrow U \vdash a$

*v.* $U, V \in \mathrm{Con} \Longrightarrow \forall a \in V\ (U \vdash a) \Longrightarrow (V \vdash b \Longrightarrow U \vdash b)$

The *ideals* („*objects*") of an *information system* $(A, \mathrm{Con}, \vdash)$ are defined as *subjects x of A* with

i. $U \subseteq x \Longrightarrow U \in \mathrm{Con}$ ($x$ is *consistent*)

*ii.* $x \supseteq U \vdash a \Longrightarrow a \in x$ ($x$ is *deductively closed*)

__Example__: The *deductive closure* $\overline{U} \coloneqq \{a \in A | U \vdash a\}$ of $U \in \mathrm{Con}$ is an *ideal.*

# Computable Partial Continuous Functionals of Finite Type

*Types* are built from *base types* by the formation of *function types* $\rho \to \sigma$. For every *type* $\rho$, the *information system* $\mathcal{C}_\rho = (C_\rho, \mathrm{Con}_\rho, \vdash_\rho)$ can be *defined*.
The *ideals* $x \in |\mathcal{C}_\rho|$ are the *partial continuous functionals of type* $\rho$ .
Since $\mathcal{C}_{\rho\to\sigma} = \mathcal{C}_\rho \to \mathcal{C}_\sigma$, the *partial continuous functionals* of *type* $\rho \to \sigma$ will correspond to the *continuous functions* from $|\mathcal{C}_\rho|$ to $|\mathcal{C}_\sigma|$ with respect to the *Scott topology*.

A *partial continuous functional* $x \in |\mathcal{C}_\rho|$ is *computable* iff it is *recursive enumerable* as *set of tokens*.

*Partial continuous functionals* of *type* $\rho$ can be used as *semantics* of a *formal functional programming language* :
Every *closed term* of *type* $\rho$ in the *programming language* denotes a *computable partial continuous functional* of type $\rho$, i.e. a *recursive enumerable consistent* and *deductively closed set* of *tokens*.

Another approach uses *recursive equations* to *define computable functionals* (Berger, Eberl, Schwichtenberg 2003).

# 3. Proof  Mining and Program Extraction

# Proofs as Verification of Truth

**Theorem**: *There are infinitely many prime numbers.*

The predicate $P(x) \equiv' x\ is\ a\ prime\ number'$ can be expressed in a *quantifier-free* way as *primitive recursive predicate.*

**Euclid's Proof (reductio ad absurdum)**: Elements IX Prop. 20; M. Aigner/G.M. Ziegler 2001, pp. 3-6; U. Kohlenbach 2008, p. 15)

*Assume* there are *finitely many prime numbers* $p \leq x$

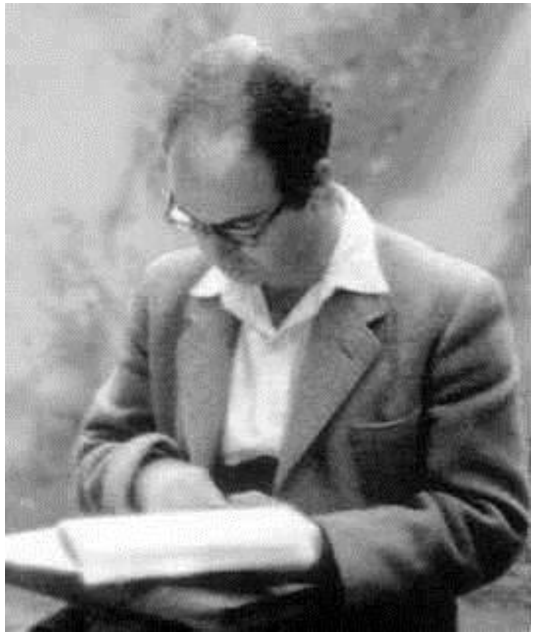
construct $a := 1 + \prod_{\substack{p \leq x \\ p\ prime}} p$

⇨ $a$ *cannot* be *prime* (because $a > p$ for all $p \leq x$)

⇨ $a$ contains a *prime factor* (by the decomposition of every number into prime factors) $q \leq a$ with $q > x$ ( otherwise $q$ is a prime factor $q \leq x$ )

⇨ *contradiction to assumption* !

# Proofs are more than Verification!



*„What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?"*

G. Kreisel: *Unwinding of Proofs*

U. Kohlenbach: *Proof Mining* (cf. „Applied Proof Theory" 2008, Chapter 2)

Consider an *existential theorem* $A \equiv \exists x\, B(x)$ (closed):

A weaker requirement is to *construct* a *list of terms* $t_1, \ldots, t_n$ which are candidates for *A*, such that $B(t_1) \vee \cdots \vee B(t_n)$ holds. More general:

If $A \equiv \forall x \exists y\, B(x, y)$, then one can ask for an *algorithm p* such that

$\forall x\, B\big(x, p(x)\big)$ holds

or – weaker - for a *bounding function b* that $\forall x\, \exists y \leq b(x)\, B(x, y)$.

# Automatic Program Extraction with MINLOG

MINLOG is an *interactive proof system* which is equipped with *tools* to extract *functional programs* directly from *proof terms*. The system is supported by *automatic proof search* and *normalization* by *evaluation* as an *efficient term rewriting device*.

**Example: Existence proof for "list reversal"**

Write $vw$ for the result $v * w$ of appending the list $w$ to the list $v$,
Write $vx$ for the result $v * x$ of appending one element list $x$: to the list $v$,
Write $xv$ for the result $x :: v$ of writing one element $x$ in front of a list $v$
Assume Init Rev: $R$(nil, nil)
GenRev: $\forall v, w, x\ (Rvw \rightarrow R(vx, xw))$

**Proposition**: $\forall v \in T\ \exists w \in T\ Rvw$ („Existence of list *w* with reverse order of given list *v*")
**Proof**: Induction on the length of $v$

# Goal of Program Extraction in Computer Science: *Metatheorems of Software*

*Customers* want *software* which *solves a problem*. Thus, they require a *proof* that it *works*. *Suppliers* answer with a *proof of the existence of a solution* to the specification of the problem. The proof has been *automatically extracted* from the *formal specification* of the problem by a *proof mining software* (e.g., MINLOG).

But, the question arises whether the *extraction mechanism of the proof* is *itself*, *in general*, *correct*. The *metatheorem of soundness* guarantees that *every formal proof* can be realized by a *normalized extracted term*.

# 4. From Brouwer‘s Creative Subject to FanTheorem

# Intuitionistic Philosophy of Creative Subject

According to Brouwer, *mathematical truth* is founded by *construction of a creative subject* . Following Kant, *mathematical construction* can only be realized in a *finite process, step by step in time* like counting in arithmetic. Thus, for Brouwer, *mathematical truth* depends on *finite stages of realization in time by a creative subject* (in a definition of Kripke and Kreisel 1967) :

The creative subject has a proof of proposition *A* at stage *m* $(\sum \vdash_m A)$ iff

(CS1) For any proposition *A* , $\sum \vdash_m A$ is a *decidable* function of *A* , i.e. $\forall x \in \mathbb{N} \ (\sum \vdash_x A \vee \neg \sum \vdash_x A)$

(CS2) $\forall x, y \in \mathbb{N} \ (\sum \vdash_x A \rightarrow (\sum \vdash_{x+y} A)$

(CS3) $\exists x \in \mathbb{N} \ (\sum \vdash_x A) \leftrightarrow A$

A weaker version of CS3 is G. Kreisel's "*Axiom of Christian Charity*" (1967)

(CC) $\neg \exists x \in \mathbb{N} \ (\sum \vdash_x A) \rightarrow \neg A.$

# Fan Principle and Fan Theorem

The *fan principle* states that for every fan *T* in which every *branch* at some point satisfies a property *A*, there is a *uniform bound* on the depth at which this property is met. Such a property is called a *bar* of *T*.

**FAN Principle:**

$\forall\alpha \in T\, \exists n\, A\big(\alpha(\overline{n})\big) \rightarrow \exists m\, \forall\alpha \in T\, \exists n \leq m\, A\big(\alpha(\overline{n})\big)$

with $\alpha$ *choice sequences* and $\alpha(\overline{n})$ the *initial segment* of $\alpha$ with the first $m$ elements.

**FAN Theorem:**

Every *continuous real function* on a *closed interval* is *uniformly continuous*.

**Proof:** *Fan Principle*

# Brouwer‘s Bar Principle for the Universal Spread

The *bar principle* provides *intuitionistic mathematics* with an *induction principle* for trees. It expresses a *well-foundedness principle* for *spreads* with respect to *decidable properties* :

$$\forall\alpha\,\forall n\, A\big(\alpha(\overline{n})\big) \vee \neg A(\alpha(\overline{n}))) \wedge \forall\alpha\,\exists n\, A\big(\alpha(\overline{n})\big) \wedge \forall\alpha\,\forall n\, A\big(\alpha(\overline{n})\big) \to B\big(\alpha(\overline{n})\big)) \wedge$$
$$\forall\alpha\,\forall n(\forall m\, B(\alpha(\overline{n}) \cdot m) \to B\big(\alpha(\overline{n})\big))) \to B(\varepsilon)$$

with $\varepsilon$ empty sequence.

# Intuitionism and Functional Interpretation

In *functional interpretation* of *proofs* , the *intuitionistically unexplained* notion of *construction* is defined by *computable functionals of finite type* with sets of *finite approximations* which are (*primitive* ) *recursively enumerable*.

Thus, the *intuitionistically unexplained* notion of a "*constructive proof at a finite stage* " is explained by *finite approximation of some computable functional* at a *finite stage*.

As *computable functionals of finite type* are (mathematical) *ideals of information systems, constructions* do not depend on "*mental actions of human creative subjects*" (i.e. psychology and epistemology), but on *finite processes* of (*mathematically definable* ) *information systems*. *Humans* (Brouwer's "creative subject") and *computers* ("machines") are only *epistemical* resp. *technical examples* of *information systems* :

***Constructivity and Computability are founded in Information Systems!***

# 5. Reverse Mathematics and Constructivity

# Reverse Mathematics in Antiquity

Since Euclid (Mid-4th century – Mid 3rd century BC), *axiomatic mathematics* has started with *axioms* to *deduce* a *theorem*. But the "*forward*" *procedure* from *axioms* to *theorems* is not always obvious. How can we *find appropriate axioms* for a proof starting with a *given theorem* in a „*backward*" (*reverse*) *procedure*?

Pappos of Alexandria (290-350 AC) called the "*forward*" *procedure* as "*synthesis*" with respect to Euclid's *logical deductions* from *axioms* of geometry and *geometric constructions* (Greek: "*synthesis*") of corresponding figures. The *reverse search procedure of axioms* for a given theorem was called "*analysis*" with respect to *decomposing* a *theorem* in its *necessary* and *sufficient conditions* and the *decomposition* of the *corresponding figure* in its *building blocks*.

# Classical Reverse Mathematics

*Reverse mathematics* is a modern *research program* to determine the *minimal axiomatic system* required to *prove theorems*. In general, it is *not possible* to start from a *theorem* $\tau$ to prove a *whole axiomatic subsystem* $T_1$. A *weak base theory* $T_2$ is required to *supplement* $\tau$:

If $T_2 + \tau$ can prove $T_1$, this *proof* is called a *reversal*.
If $T_1$ *proves* $\tau$ and $T_2 + \tau$ is a *reversal*, then $T_1$ and $\tau$ are said to be *equivalent over* $T_2$.

*Reverse mathematics* allows to determine the *proof-theoretic strength* resp. *complexity* of *theorems* by *classifying* them with respect to *equivalent theorems* and *proofs*. Many *theorems* of *classical mathematics* can be *classified* by *subsystems* of *second-order arithmetic* $\mathbb{Z}_2$ with *variables* of *natural numbers* and *variables* of *sets of natural numbers*.

# $\mathcal{Z}_2$ - Subsystems and Philosophical Research Programs

The *five* most commonly used $\mathcal{Z}_2$ - *subsystems* in *reverse mathematics* correspond to *philosophical programs* in *foundations of mathematics* with *increasing proof-theoretic power* starting with the *weakest* $\mathrm{RCA}_0$*-subsystem*.

$\mathrm{RCA}_0$: *Turing's computability*
$\mathrm{WKL}_0$: *Hilbert's finitistic reductionism*
$\mathrm{ACA}_0$: *Weyl's & Lorenzen's predicativity* (*with classical logic*)
$\mathrm{ATR}_0$: *Friedman's & Simpson's predicative reductionism*
$\prod_1^1 - \mathrm{CA}_0$: *impredicativity*

$\Delta_1^1 - \mathrm{CA}_0$ yields *systems* of *hyperarithmetic analysis* (Feferman et al.) with $\Delta_1^1$-*predicativism*:

*T* is a *theory* of *hyperarithmetic analysis* iff

i. its $\omega$-*models* are *closed* under *joins* and *hyperarithmetic reducibility*

ii. it *holds* in $\mathrm{HYP}(x)$ for all *x*

# Constructive Reverse Mathematics

*Classical reverse mathematics* (Friedmann/Simpson) uses *classical logic* and *classification of proof-theoretic strength* with $\mathrm{RCA}_0$ ($\Delta_1^0$-*recursive comprehension*) as *weakest subsystem*.

*Constructive reverse mathematics* (Ishihara et al.) uses *intuitionistic logic* and *Bishop's constructive mathematics* (BISH) as *weakest subsystem* of a *constructive classification* (Bishop/Bridges/Vita/Richman).

BISH = $\mathcal{Z}_2$ + *Intuitionistic Logic* + *Axioms of Countable*, *Dependent* and *Unique Choice*

Intuitionistic Mathematics (Brouwer, Heyting et al.):

INT = BISH + *Axiom of Continuous Choice* + *Fan Theorem*

Constructive Recursive Mathematics (Markov et al.):

RUSS = BISH + *Markov's Principle* + *Church's Thesis*

Classical Mathematics (Hilbert et al.):

CLASS = BISH + *Principle of Excluded Middle* + *Full Axiom of Choice*

# 6. Intuitionistic Type Theory and Proof Assistants

# Curry-Howard Correspondance

**In 1969, the logician W.A. Howard observed that Gentzen's *proof system of natural deduction* can be directly interpreted in its *intuitionistic version* as a *typed variant* of the mode of *computation* known as *lambda calculus*.**

**According to Church, $\lambda a.\, b$ means a *function* mapping an element $a$ onto the function value $b$ with $\lambda a.\, b[a] = b$. In the following, *proofs* are represented by terms $a, b, c, \ldots$ ; *propositions* are represented by $A, B, C, \ldots$ .**

**Examples:**

$$\dfrac{\begin{matrix} [A] \\ \lambda a(\lambda b.\, a) \quad \vdots \\ \underline{B \to A} \end{matrix}}{} \qquad (\to \mathrm{I}) \quad A \to (B \to A)$$

$$\dfrac{\begin{matrix} [A] \\ \lambda a.\, b \vdots \\ \underline{B} \end{matrix}}{} \qquad (\to \mathrm{I}) \quad A \to B$$

***A proof is a program, and the formula it proves is the type for the program.***

# Martin-Löf's Intuitionistic Type Theory

**In addition to the *type formers* of the *Curry-Howard interpretation*, the logician and philosopher P. Martin-Löf extended the *basic intuitionistic type theory* (containing *Heyting's arithmetic of higher types* HA$^{\omega}$ and *Gödel's system* T *of primitive recursive functions of higher type*) with *primitive identity types*, *well founded tree types*, *universe hierarchies* and general notions of *inductice* and *inductive–recursive definitions*.**

***His extension increases the <u>proof-theoretic strength</u> of the theory and its application to <u>programming</u> as well as to <u>formalization of mathematics</u>.***

# Calculus of Constructions (CoC)

**CoC is a *type theory* of Thierry Coquand which can serve as *typed programming language* as well as *constructive foundation of mathematics*. With *inductive types*, the *calculus of inductive constructions* (CiC) removes *impredicativity* (cf. Weyl, Lorenzen). It extends the *Curry-Howard isomorphism* to *proofs* in the *full intuitionistic predicate calculus.* Coc has very few basic operators (e.g., λ,∀):**

**logical operators:**

$$
\begin{aligned}
A \Rightarrow B &\equiv \forall x{:}A.B \quad (x \notin B)\\
A \wedge B &\equiv \forall C{:}P.(A \Rightarrow B \Rightarrow C) \Rightarrow C\\
A \vee B &\equiv \forall C{:}P.(A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C\\
\neg A &\equiv \forall C{:}P.(A \Rightarrow C)\\
\exists x{:}A.B &\equiv \forall C{:}P.(\forall x{:}A(B \Rightarrow C)) \Rightarrow C
\end{aligned}
$$

**data types:**

| | |
|---|---|
| **booleans:** | $\forall A{:}P.A \Rightarrow A \Rightarrow A$ |
| **naturals:** | $\forall A{:}P.(A \Rightarrow A) \Rightarrow (A \Rightarrow A)$ |
| **product** $A \times B$: | $A \wedge B$ |
| **disjoint union** $A + B$: | $A \vee B$ |

# The Coq Proof Assistant

**Coq implements a *program specification* which is based on the *Calculus of Inductive Constructions* (CiC) combining both a *higher-order logic* and a *richly-typed functional language*.**

**The commands of Coq allow**

- **to *define functions* or *predicates* (that can be evaluated efficiently)**
- **to *state mathematical theorems* and *software specifications***
- **to *interactively develop formal proofs* of these *theorems***
- **to *machine-check* these *proofs* by a relatively small certification (kernel)**
- **to *extract certified programs* to languages (e.g., Objective Caml, Haskell, Scheme)**

**Coq provides *interactive proof methods, decision* and *semi-decision algorithms*. Connections with *external theorem provers* is available.**

**Coq is a platform for the formalization of mathematics as well as the development of programs.**

# The Language of Proof Assistant Coq

**The heart of Coq is a *type-checking algorithm* in the language of CiC:**

**Coq objects are sorted into the `Prop` *sort* and the `Type` *sort*:**

- `Prop` **is the sort of propositions (with *type* `Prop`):**

  e.g., $\forall$ `A,B:Prop, A /\ B -> A \/ B`

  ***New predicates* can be defined either *inductively* or by *abstracting* over existing propositions:**

  e.g., `Definition divide (x y:N) :=`$\exists$`z, x * z = y`

- `Type` **is the sort for datatypes and mathematical structures (with *type* `Type`):**

  **Types can be *inductive structures* or *types for tuples* or a *form for subsets* (Σ-types):**

  e.g., the type of even natural numbers {n: N|even n}

**Coq implements a functional programming language supporting these types:**

e.g., the pairing function of type `z->z*z` is written `fun x =>(x, x)`

# 7. Univalent Foundations of Mathematics

# From Leibniz’ Analysis Situs to Algebraic Topology

**In 1679, Leibniz wrote a letter to Huygens which described his „analysis situs“:**

**Examples of Leibniz‘s <u>characteristica geometrica propria</u>:**

$A \infty B$ **(coincidence)**

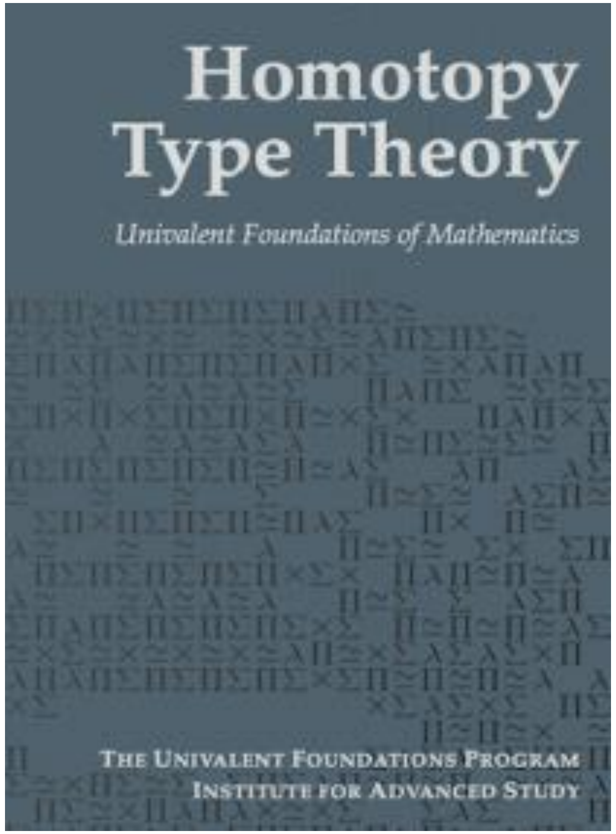
$A \sim B$ **(similarity)**

$A \, 8 \, B$ **(congruence)**

> I am still not satisfied with algebra, because it does not give the shortest methods or the most beautiful constructions in geometry. This is why I believe that, so far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express *situation* [*situs*] directly as algebra expresses *magnitude* directly. And I believe that I have found the way and that we can represent figures and even machines and movements by characters, as algebra represents numbers or magnitudes. I am sending you an essay which seems to me to be important. (1; 382)

**Euler's *Seven Bridges of Königsberg Problem* and *Polyhedron Formula* are the field's first theorems.**

**In modern mathematics, *algebraic topology* uses tools of *abstract algebra* to study *topological spaces*. The basic goal is to find *algebraic invariants* that classify *topological spaces* up to *homeomorphism*, though usually most classify up to *homotopy equivalence*.**




**Since their very beginning, *data types* play a crucial role in *computer languages*:**

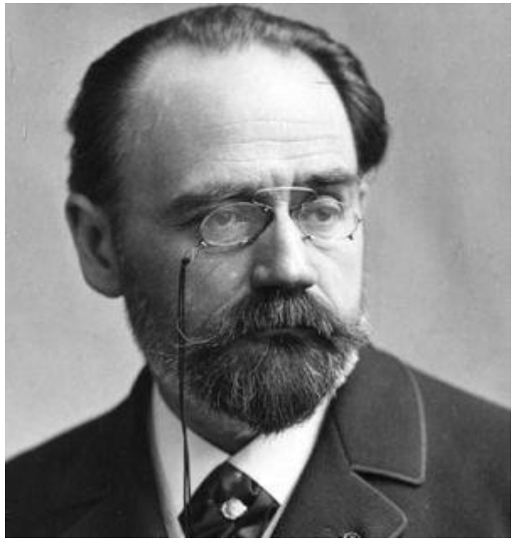
***How far can mathematical objects be represented with types of computer languages*?**

***Homotopy theory* is an outgrowth of *algebraic topology* and *homological algebra* with relationships to higher *category theory* which can be considered as *fundamental concepts of mathematics*.**

***Type theory* is a branch of *mathematical logic* and *theoretical computer science*.**

***Homotopy type theory* (HoTT) interprets *types* as *objects of abstract homotopy theory*. Therefore, HoTT tried to develop a *universal* („*univalent*“) *foundation of mathematics* as well as *computer language* with respect to the *proof assistant Coq*.**

# Trust & Security in Mathematics



**Nowadays, *mathematical arguments* had become so *complicated* that a *single mathematician* rarely can examine them in detail: They trust in the *expertise of their colleagues*. The situation is completely similar to *modern industrial labor world*: According to the French sociologist Emile Durkheim (1858-1917), modern industrial production is so *complex* that it must be organized on the *principle of division of labor and trust in expertise*, but *nobody* has the *total survey*.**

**On the background of *critical flaws* overlooked by the *scientific community*, Vladimir Voevodsky (1966-2017) *no longer trusted* in the principle of "job-sharing". Humans could not keep up with the *ever-increasing complexity of mathematics*. *Are computers the only solution*? Thus, his foundational program of univalent mathematics is inspired by the idea of a *proof-checking software* to *guarantee trust & security in mathematics*.**

# Intuitionistic Type Theory and Homotopy Theory

| Intuitionistic Type Theory | Homotopy Theory |
|---|---|
| types $A$ | spaces $A$ |
| terms $a$ | points $a$ |
| $a: A$ | $a \in A$ |
| dependent type $x: A \vdash B(x)$ | fibration $B \to A$ |
| identity type $\mathrm{Id}_A(a, b)$ | space of paths from $a$ to $b$ |
| $p: \mathrm{Id}_A(a, b)$ | path $p: a \longmapsto b$ |
| $\alpha: \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p, q)$ | homotopy $\alpha: p \Rightarrow q$ |

In *intuitionistic type theory*, a term $a: A$ can be understood as an <u>element of the type</u> $A$ or a <u>proof of the proposition</u> $A$ or, in *homotopy type theory*, as a <u>point of the space</u> $A$. *Proofs* $p$ of *identity* between two *elements* $a, b$ of a *type* $A$ are *geometrically illustrated* as *paths* connecting the corresponding *points*.

In *intuitionistic type theory*, all *proofs of an identity* are *not* forced to be *equal* (Hofmann/Streicher 1998): This was shown by a model where *each type* is interpreted as a *groupoid* (i.e. a *category*, in which each *morphism* is *invertible* resp. an *isomorphism*).

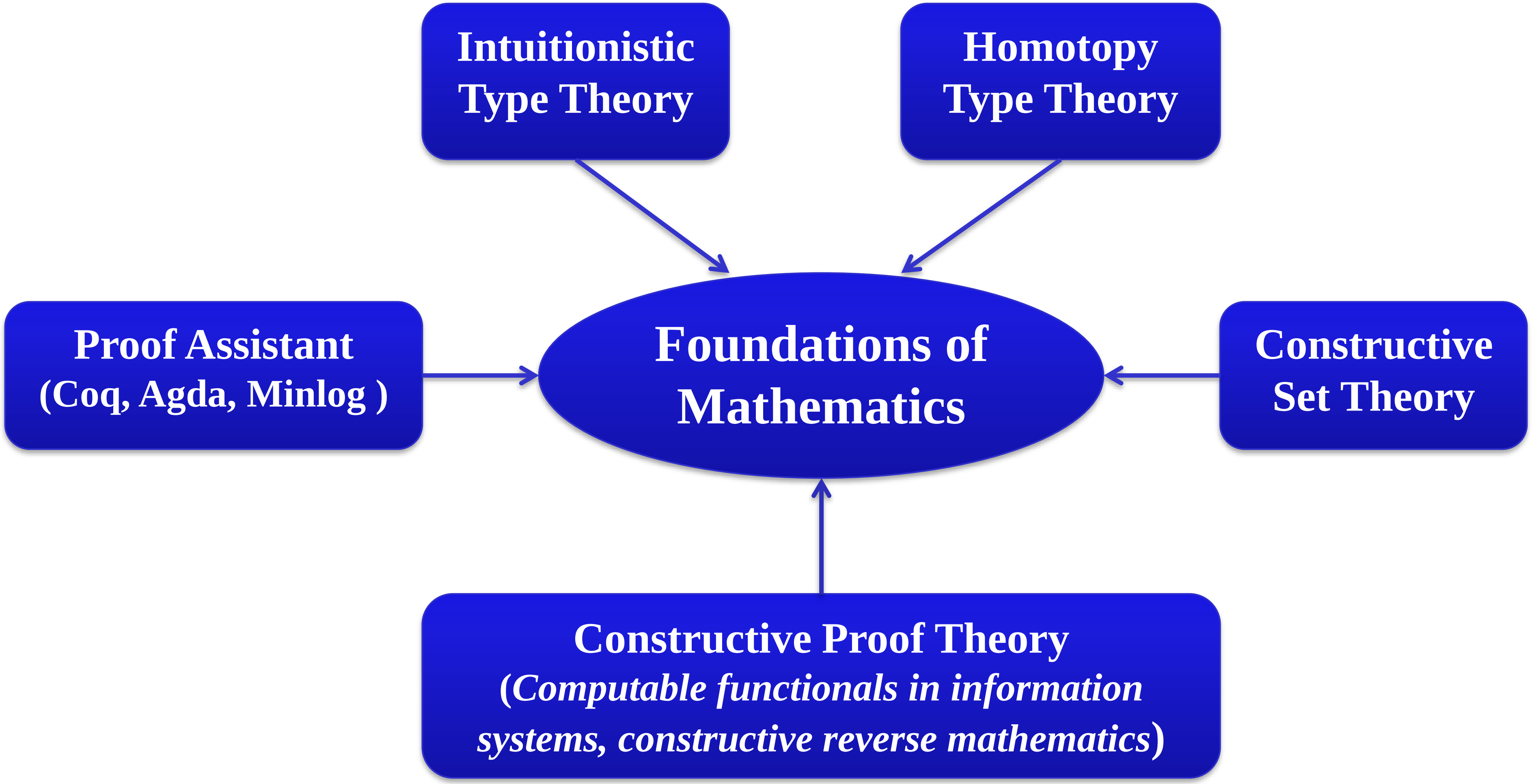
# Provability, Constructivity, and Computability in HoTT

**HoTT allows mathematical proofs to be translated into a computer programming language for *computer proof assistants* (e.g., Coq) even for advanced mathematical categories with "*isomorphism as equality*"(UA). Therefore, an essential goal of HoTT is :**

***type checking ⇒ proof checking in higher categories („difficult proofs")***

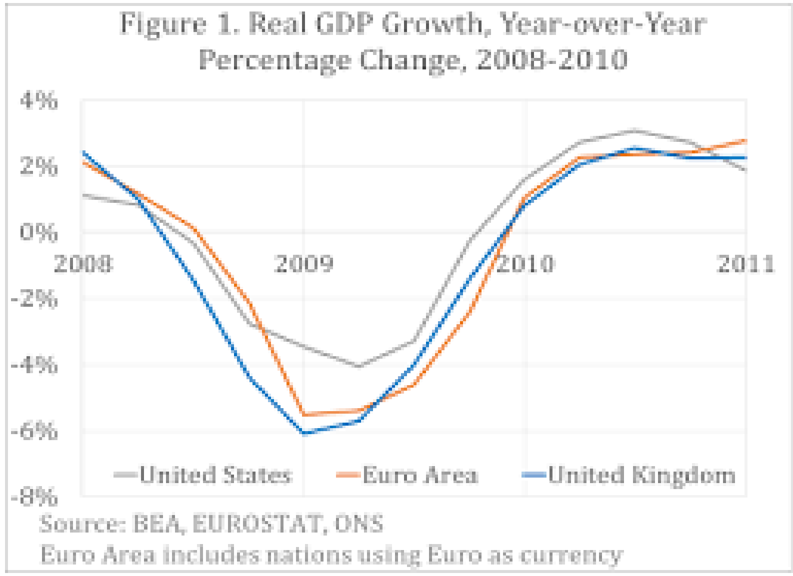
**Besides UA, HoTT is extended by *higher inductively defined structures* (e.g. *inductively defined spaces* with collections of *points*, *paths*, *higher paths* et al.) which can be characterized by appropriate *induction principles*. HoTT is *consistent* with respect to a *model* in the *category of Kan complexes* (V. Voevodsky). Thus, it is consistent relative to ZFC (with as many *inaccessible cardinals* which are necessary for *nested univalent universes*).**

**But it is still an *open question* whether it is possible to provide a constructive justification of the Univalence Axiom (UA).**


Intuitionistic Type Theory
Homotopy Type Theory
Proof Assistant (Coq, Agda, Minlog )
Foundations of Mathematics
Constructive Set Theory
Constructive Proof Theory (*Computable functionals in information systems, constructive reverse mathematics*)

# 8. Proof-of-Work and Financial Trust

# From Financial Crisis to Crisis of Trust in Banks




**How far were *shortcomings* of the *policy frameworks* of the *major central banks* responsible for the *financial crisis* 2008?**

***“The root problem with conventional currency is all the trust that is required to make it work. The central bank must be trusted not to debase the currency, but the history of fiat currencies is full of breaches of that trust. Banks must be trusted to hold our money and transfer it electronically, but they lend it out in waves of credit bubbles with barely a fraction in reserve … With e-currency based on cryptographic proof, without the need to trust a third party middleman, money can be secure and transactions effortless.”***

**Satoshi Nakamoto 2009**


# Who is Satoshi Nakamoto?

**Satoshi Nakamoto is the name used by the *unknown person or people* who designed *bitcoin* (2008) and created its *original reference implementation* (2009). As part of the implementation, they also devised the *first blockchain database*. In the process they were the first to solve the *double-spending problem for digital currency*.**

**Nakamoto claimed to be a man living in Japan, born on 5 April 1975. Speculation about the true identity of Nakamoto has mostly focused on a *number of cryptography and computer science experts*. Thus, it reminds us of a well-known question in mathematics (but in this case, the mathematicians are well known):**

# Who is Niklas Bourbaki?

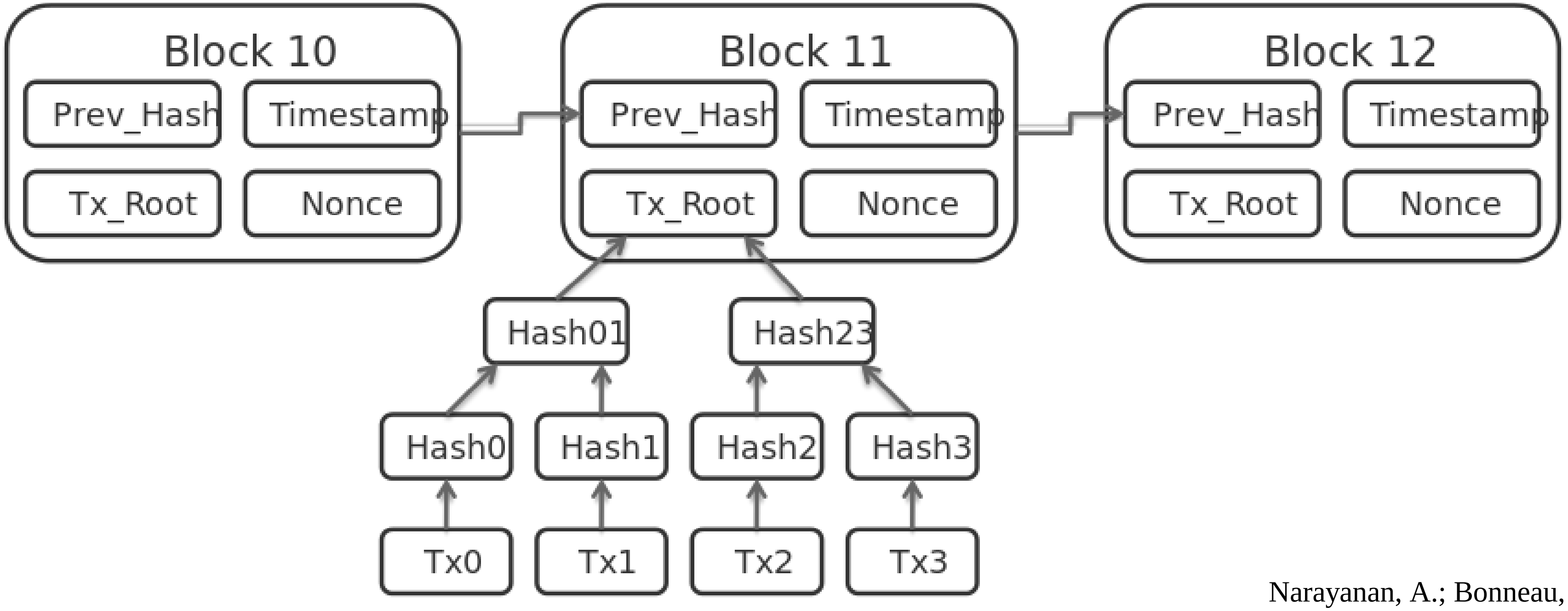
# Bitcoin as Decentral Currency

***The bitcoin-network*** **is founded on a *<u>decentral database</u>* which is administrated by *<u>all users</u>* with a bitcoin software which registers all transactions of currency.**

**The network is *peer-to-peer* and *transactions* take place between users *directly*, *without an intermediary* (e.g. central bank). The *<u>proof-of-work algorithm</u>* is *<u>computationally expensive</u>* and *<u>practically secure</u>*.**


**A *wallet* stores the *information necessary to transact bitcoins*. The *exchange rate* of bitcoin and other currencies depend on *supply and demand* (*speculative bubbles* !!!)**

# From Blockchain to the Internet of Value




***Blockchain*** **is an** ***expandable list*** **of records (blocks) which are linked with** ***cryptographic codes*****.**

**Each** ***block*** **contains a** ***cryptographic <u>hash pointer</u> to a previous block*****, a** ***<u>timestamp</u>*** **and** ***<u>transaction data</u>*****. New blocks are generated by a** ***consensus procedure*** **(e.g.** ***proof-of-work algorithm*****).**

Narayanan, A.; Bonneau, J.; Felten, E.; Miller, A.; Goldfeder, S. 2016

**A blockchain is a** ***decentralized*** **and** ***distributed digital ledger*** **that is used to** ***record transactions of digital values*** **across the nodes (computers) resp. users of a network (*****Internet of Values*****). The record** ***cannot be altered retroactively*** **without the** ***alteration of all subsequent blocks*** **and the collusion of the network.**

# Security of Hash Functions

It is *easy* *to compute hash value* $y = h(x)$ from $x$ but it's ***very hard*** to find $x$ given only $y$.

A *full hash inversion* has a known *computationally infeasible brute-force* running time, being O($2^k$) where $k$ is the *hash size* $k$=256 in SHA256.

If a *pre-image* was found anyone could *very efficiently verify* it by computing one hash.

There is a an *asymmetry* in full *pre-image mining* (computationally infeasible) vs *verification* (a single hash invocation).

Each *block* contains the *hash of the preceding block*, thus each block has a *chain of blocks* that together contain a *large amount of work*. *Changing a block* requires *regenerating all successors* and *redoing the work* they contain. This *protects* the *block chain* from *tampering*.

# Mining and Proof-of-Work

Generating a new valid block (mining) means solving a *cryptographic problem*: The *proof-of-work* requires *miners* to find a number called a *nonce*, such that when the block content is hashed along with the nonce, the result is numerically *smaller* than the *network's difficulty target*. This *proof* is *easy* for any node in the network to *verify*, but *extremely time-consuming to generate*, as for a *secure cryptographic hash*, miners must try *many different nonce values* (e.g. 0, 1, 2,…)

Proof-of-Work : (threshold inversely proportional to mining-difficulty)

1. initialize *block*, compute *root-hash* from *transactions*
2. *compute hash*: `h = SHA256(SHA256(block header))`
3. if `h >= threshold`, change nonce (*block header*) and *return* to step 2
4. otherwise (`h < threshold`): *valid block found, stop computing* and publish *block*.

# Nonce in Blockchain

**The *nonce* is a 32-bit field whose value is set so that the *hash of the block* will contain a *run of leading zeros**:**

- **Since it is *believed infeasible to predict* which *combination of bits* will result in the *right hash*, many different nonce values are tried: The *hash* is recomputed *for each value* until a hash containing the *required number of zero bits* is found.**

- **The *number of zero bits* required is set by the *mining difficulty*.**

- **The *resulting hash* has to be a *value less than the current mining difficulty*.**

- **As this *iterative calculation* requires *time* and *resources*, the presentation of the *block with the correct nonce value* constitutes *proof of work*.**

*** The rest of the fields (with defined meaning) may not be changed.**

# Mining Difficulty

*Mining difficulty* is a measure of how difficult it is to find a *hash below a given target* . The *Bitcoin network* has a *global block difficulty*:

```
difficulty = difficulty_1_target / current target
```

(target is a 256 bit number. )

The *difficulty* is *adjusted* every 2016 blocks based on the *time* it took to find them.

At the *desired rate of one block* each 10 minutes, 2016 blocks would take exactly *two weeks* to find.

time (2016 blocks) > 2 weeks ⇒ difficulty reduced
time (2016 blocks) < 2 weeks ⇒ difficulty increased

# Merkle Tree and Secure Verification of Large Data Structures

*Hash trees* can be used to *verify* any kind of *data stored, handled* and *transferred* in and between *computers*.

They can help ensure that *data blocks* received from *other peers* in a *peer-to-peer network* are *received undamaged* and *unaltered*, and even to *check* that the *other peers do not lie* and *send fake blocks* (*trusted computing*).

In some *proof-of-work* systems, users provide *hash collisions* as *proof* that they have performed a *certain amount of computation* to find them.

# Trust & Security by Computer Power Only?

**With *growing blockchain* (one block per ten minutes), the *difficulty of the proof-of-work* increases in proportion to *increasing computer power*:**

**Pros:**

- ***Increasing difficulty of proof-of-work* diminish the *probability of tampering* and improves *security & trust* in blockchain technology.**
- ***Security & trust* is completely based on *constructive algorithms* of proof-of-work with *finitely growing binary hash trees* („collision resistant“).**

**Cons:**

- ***Exponentially increasing computer power* means growing *power consumption* and (with that) *economic* and *environmental costs*: e.g., Bitcoin network Nov 2017 30,14 TWh p.a. > power consumption of Ireland.**
- **The *initially democratic idea* of a *decentralized cryptocurrency without central control* (*and failures*) *of banks* and *equal chances of clients* cannot be realized: Influence of members depends on *computer power* and *energy power* (monopolies *of states/concerns*).**

# 9. Bridging Logic, Mathematics, Computer Science, and Philosophy

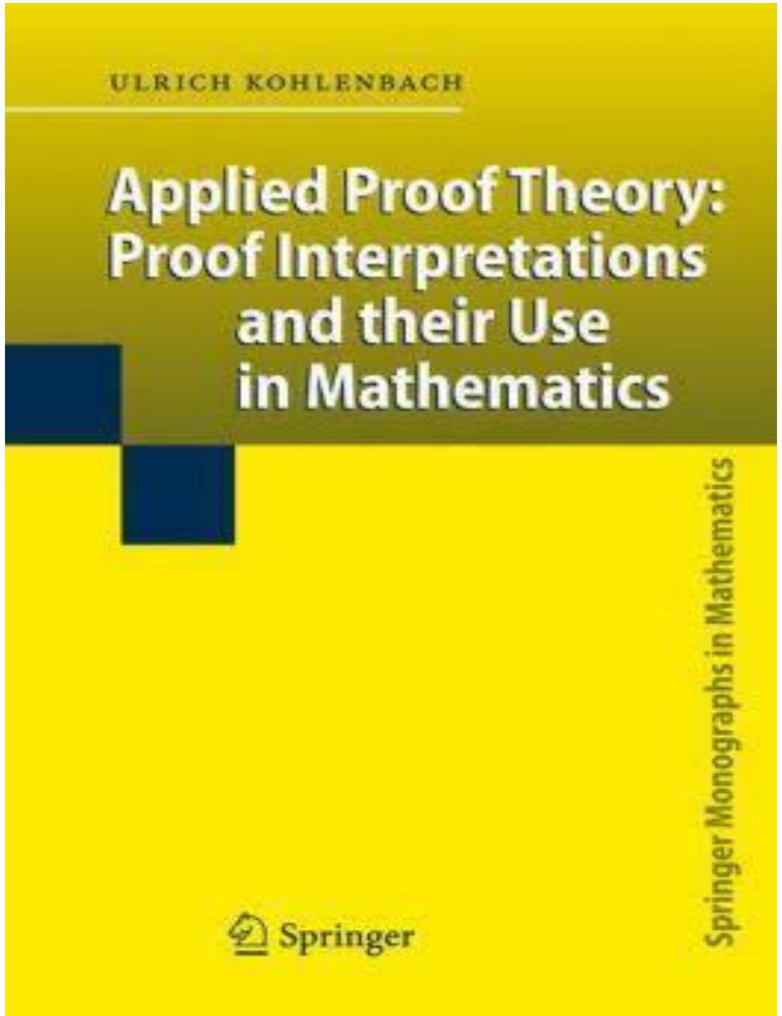
**CORE:**
*„Trust & Security by Constructive Methods in Mathematics and Operations Research!“*

***Proof Theory*:**
- intuitionistic/minimal logic
- functional interpretation
- information system

***Mathematics*:**
- numerical analysis
- functional analysis
- financial mathematics

***Computer Science*:**
- functional programing
- scientific computing
- program extraction

***Philosophy*:**
- intuitionism
- constructivism
- ethical evaluation


ULRICH KOHLENBACH
Applied Proof Theory: Proof Interpretations and their Use in Mathematics
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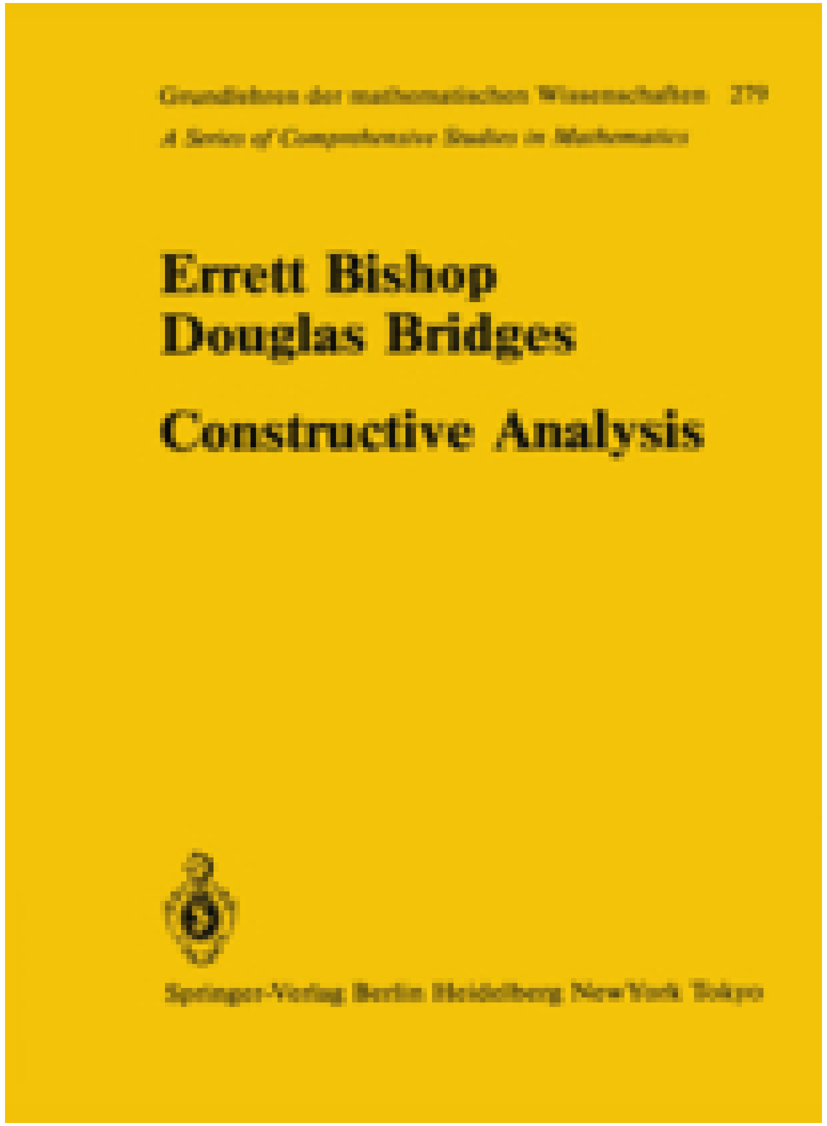
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A Series of Comprehensive Studies in Mathematics
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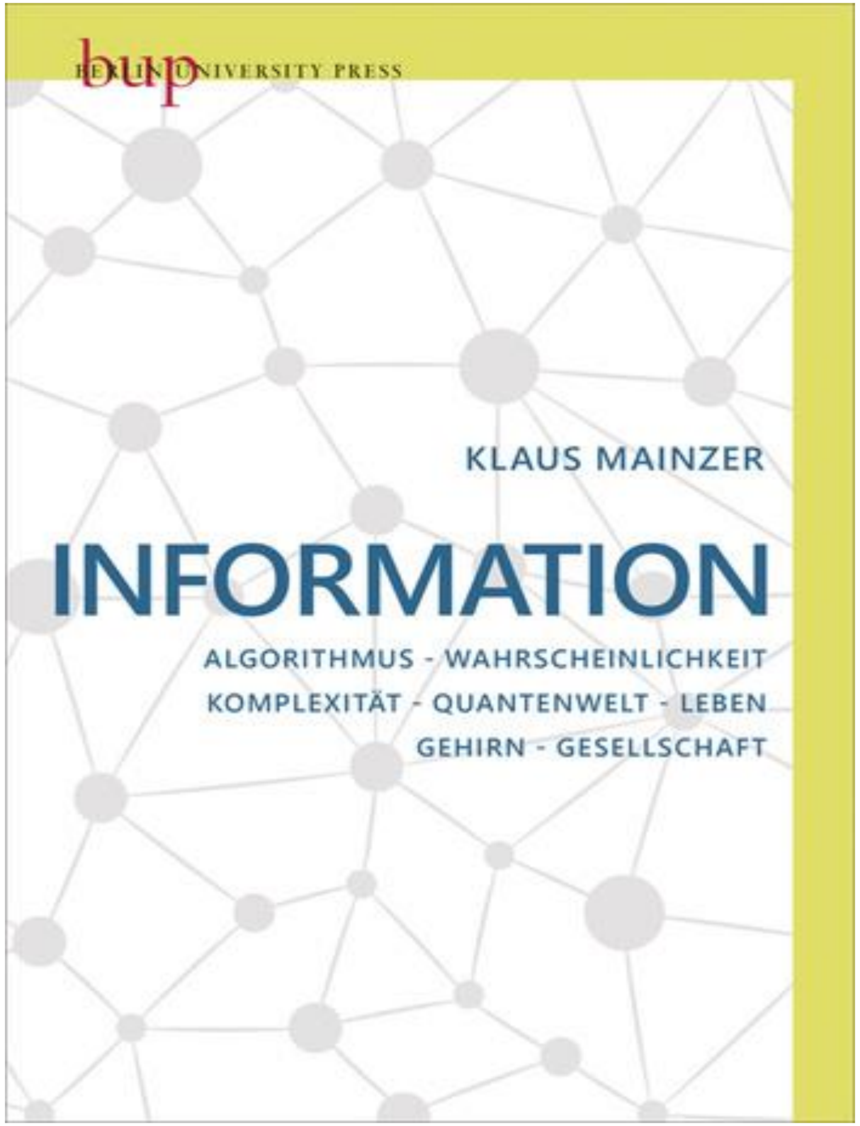
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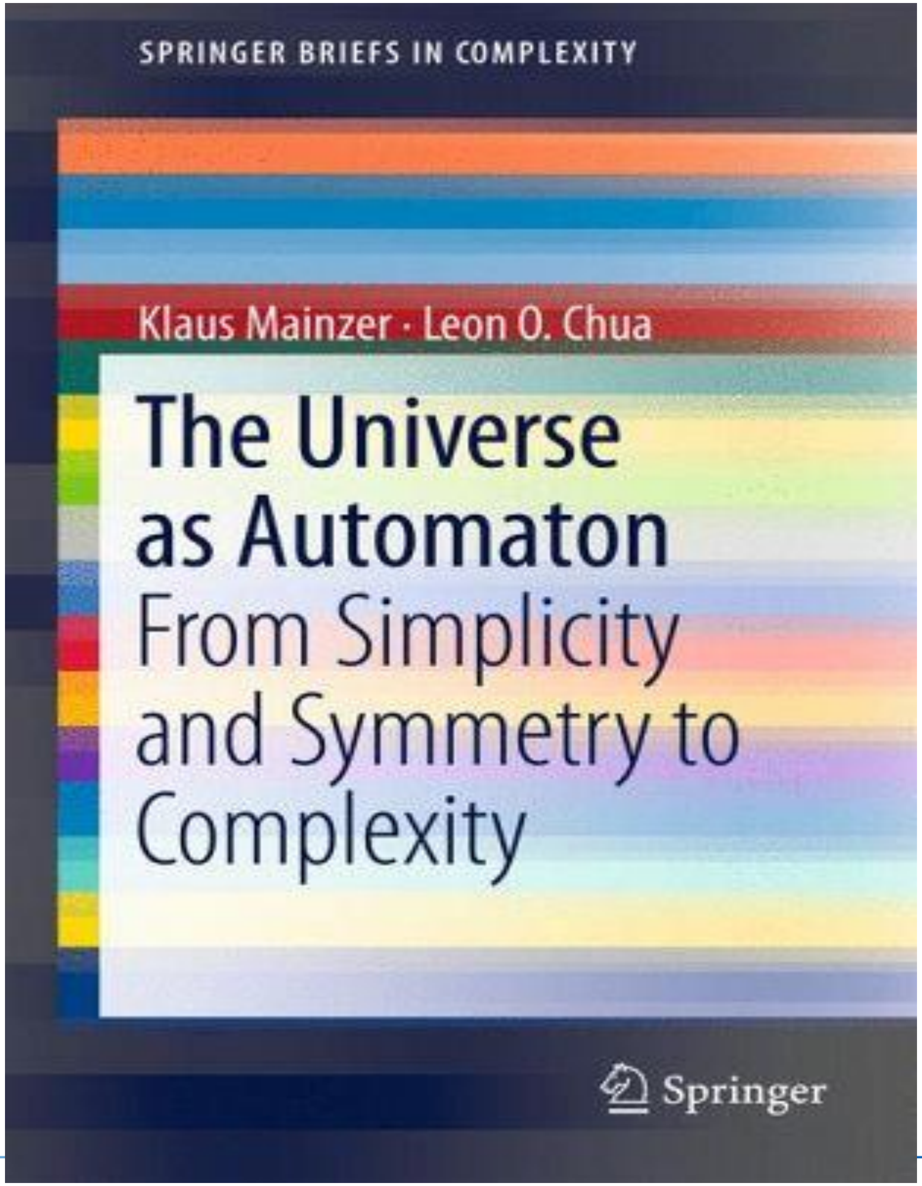
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**Künstliche Intelligenz – Wann übernehmen die Maschinen?**

Jeder kennt sie. Smartphones, die mit uns sprechen, Armbanduhren, die unsere Gesundheitsdaten aufzeichnen, Arbeitsabläufe, die sich automatisch organisieren, Autos, Flugzeuge und Drohnen, die sich selber steuern, Verkehrs- und Energiesysteme mit autonomer Logistik oder Roboter, die ferne Planeten erkunden, sind technische Beispiele einer vernetzten Welt intelligenter Systeme. Sie zeigen uns, dass unser Alltag bereits von KI-Funktionen bestimmt ist.

Auch biologische Organismen sind Beispiele von intelligenten Systemen, die in der Evolution entstanden und mehr oder weniger selbstständig Probleme effizient lösen können. Gelegentlich ist die Natur Vorbild für technische Entwicklungen. Häufig finden Informatik und Ingenieurwissenschaften jedoch Lösungen, die sogar besser und effizienter sind als in der Natur.

Seit ihrer Entstehung ist die KI-Forschung mit großen Visionen über die Zukunft der Menschheit verbunden. Löst die „künstliche Intelligenz“ also den Menschen ab? Dieses Buch ist ein Plädoyer für Technikgestaltung: KI muss sich als Dienstleistung in der Gesellschaft bewähren.

ISBN 978-3-662-48452-4

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