The fan theorem for uniform coconvex bars

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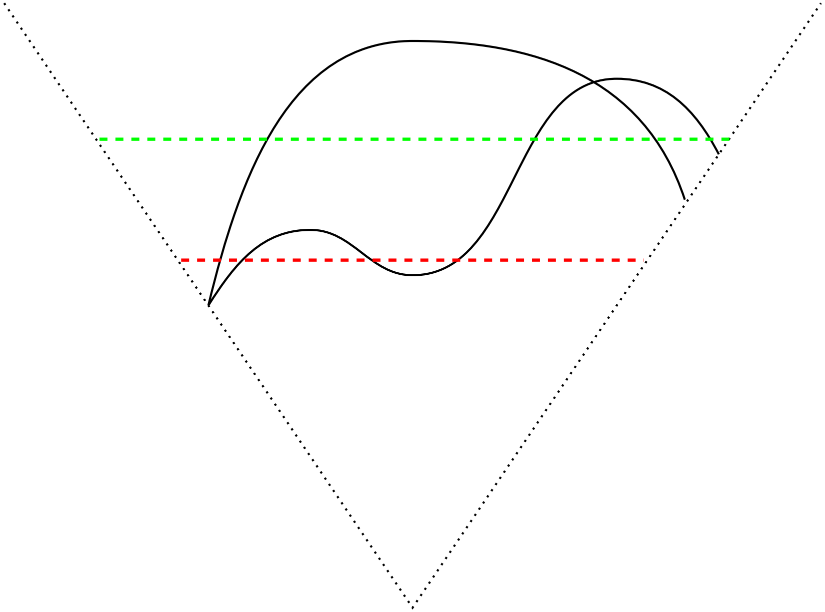
- Josef Berger and Gregor Svindland recently gave a constructive proof of the fan theorem for “coconvex” bars.
- They call a set $b \subseteq \{0,1\}^*$ coconvex if for every n and path s

$$\bar{s}(n) \in b \to \exists_m \bigl(\forall_{v \leq \bar{s}(m)}(v \in b) \vee \forall_{v \geq \bar{s}(m)}(v \in b)\bigr),$$

where $v \leq w$ means $|v| = |w|$ and v is left of w. Equivalently

$$\begin{aligned} \bar{s}(n) \in b \to \exists_{p,m}\bigl((p = 0 \to \forall_{v \leq \bar{s}(m)}(v \in b)) \wedge \\ (p = 1 \to \forall_{v \geq \bar{s}(m)}(v \in b))\bigr). \end{aligned}$$

Two “moduli” p and m, depending on s, n and b. Better name: finally coconvex.



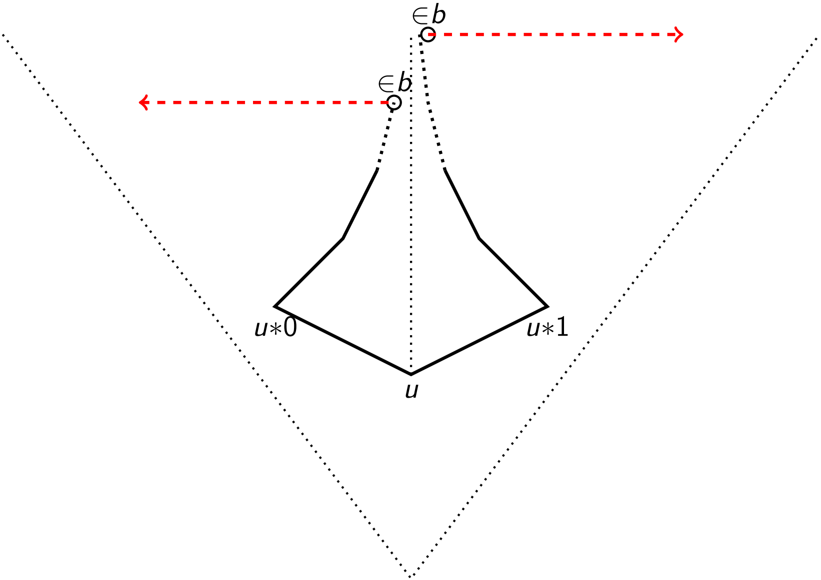
Uniform coconvexity with modulus d (direction)

- Simplification: p only, depending on node u (i.e., $p = d(u)$).
- Coconvex in the sense that the b-nodes at height n form the complement of a convex set.
- Special case of the B&S concept. Goal: better algorithm.

Definition

A set $b \subseteq \{0,1\}^*$ is uniformly coconvex with modulus d if for all u we have: if the innermost path from $u * p$ (where $p := d(u)$) hits b in some node $v \in b$, then

$$\begin{cases} \forall_w (w \leq v \to w \in b) & \text{if } p = 0, \\ \forall_w (w \geq v \to w \in b) & \text{if } p = 1. \end{cases}$$


∈b
∈b
u∗0
u∗1
u

Data

- Keep type levels low: paths are streams, not functions.
- Use corecursion instead of choice axioms or recursion.
- Free algebra $\mathbb{S}(\rho)$: given by one unary constructor C of type $\rho \to \mathbb{S}(\rho) \to \mathbb{S}(\rho)$. No nullary constructors: "cototal" objects, of the form $\mathrm{C}_x(s)$ with x of type ρ and s cototal. To construct such objects we use the corecursion operator ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbb{S}(\rho)}$, of type

$$\tau \to (\tau \to \rho \times (\mathbb{S}(\rho) + \tau)) \to \mathbb{S}(\rho).$$

It is defined by

$${}^{\mathrm{co}}\mathcal{R}xf = \begin{cases} y * z & \text{if } f(x) = \langle y, \mathrm{InL}(z)\rangle, \\ y * ({}^{\mathrm{co}}\mathcal{R}x'f) & \text{if } f(x) = \langle y, \mathrm{InR}(x')\rangle. \end{cases}$$

Lemma (Cototality of corecursion)

Let $f : \tau \to \rho \times (\mathbb{S}(\rho) + \tau)$ *be of* InR*-form, i.e.,* $f(x)$ *has the form* $\langle y, \mathrm{InR}(x') \rangle$ *for all* x*. Then* ${}^{\mathrm{co}}\mathcal{R}xf \in {}^{\mathrm{co}}T_{\mathbb{S}(\rho)}$ *for all* x*.*

Proof.

By coinduction with competitor predicate

$$X := \{\, z \mid \exists^{\mathrm{l}}_x (z = {}^{\mathrm{co}}\mathcal{R}xf) \,\}.$$

Need to prove that X satisfies the clause defining ${}^{\mathrm{co}}T_{\mathbb{S}(\rho)}$:

$$\forall_z (z \in X \to \exists^{\mathrm{d}}_y \exists^{\mathrm{r}}_{z'} (z' \in X \wedge z = y * z')).$$

Let $z = {}^{\mathrm{co}}\mathcal{R}xf$ for some x. Since f is assumed to be of InR-form we have y, x' such that $f(x) = \langle y, \mathrm{InR}(x') \rangle$. By the definition of ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbb{S}(\rho)}$ we obtain ${}^{\mathrm{co}}\mathcal{R}xf = y * ({}^{\mathrm{co}}\mathcal{R}x'f)$. Use ${}^{\mathrm{co}}\mathcal{R}x'f \in X$. □

- View trees as sets of nodes u, v, w of type $\mathbb{L}(\mathbb{B})$ (lists of booleans), which are downward closed.
- Paths are seen as cototal objects s of type $\mathbb{S}(\mathbb{B})$ (streams of booleans; no nullary constructor).
- Sets of nodes are given by (not necessarily total) functions b of type $\mathbb{L}(\mathbb{B}) \to \mathbb{B}$. To be or not to be in b is expressed by saying that $b(u)$ is defined with 1 or 0 as its value.
- A set b of nodes is a bar if every path s hits the bar, i.e., there is an n such that $\bar{s}(n) \in b$.

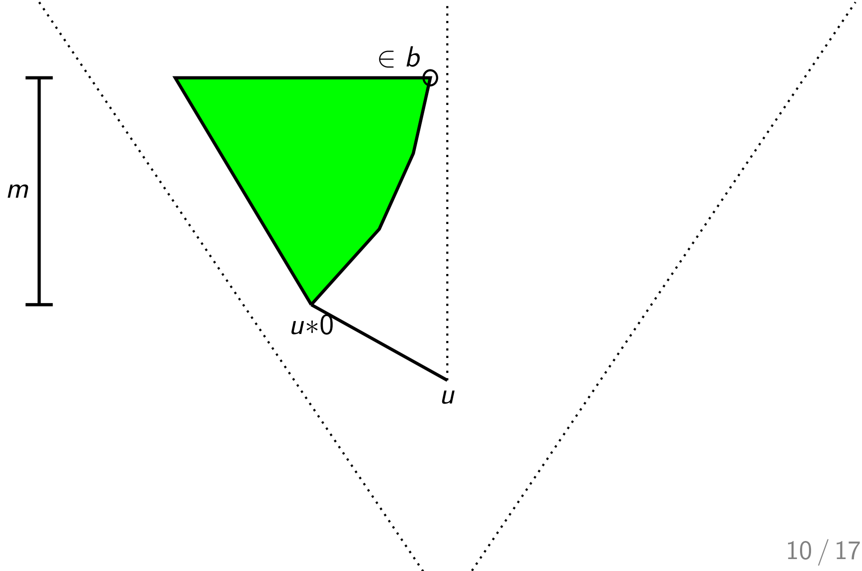
For simplicity assume: all bars b considered are upwards closed (i.e., $\forall_{u,p}(u \in b \to u * p \in b)$). This does not restrict generality.

Lemma (Distance)

Let b be a uniformly coconvex bar with modulus d. Then

$$\forall_u \exists_m \big(u * d(u) \in D_{b,m} := \{\, u \mid \forall_v (|v| = m \to u * v \in b) \,\} \big).$$

Proof. Given $u\colon \mathbb{L}(\mathbb{B})$, extend $u * d(u)$ by appending 1^∞ if $d(u) = 0$, and 0^∞ if $d(u) = 1$. Assume $d(u) = 0$. Since b is a bar, the path $u * 0 * 1^\infty$ hits b at $u * 0 * 1^m$ for some m. By uniform coconvexity, all $u*0*v$ with $|v| = m$ will be in b. Hence $u*0 \in D_{b,m}$.



The escape path $s_d \in \mathbb{S}(\mathbb{B})$ is constructed from d corecursively, as follows. Start with the root node. At any node u, take the opposite direction to what $d(u)$ says, and continue.

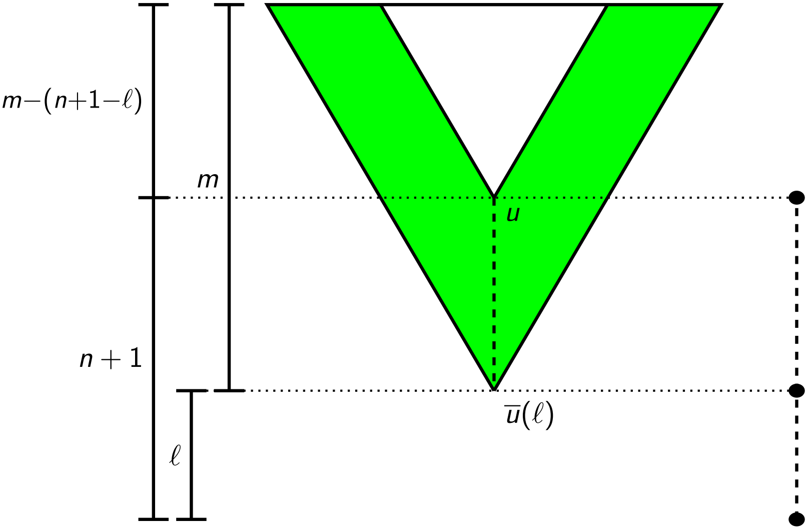
Lemma (Escape)

Let b be a uniformly coconvex bar with modulus d. Then

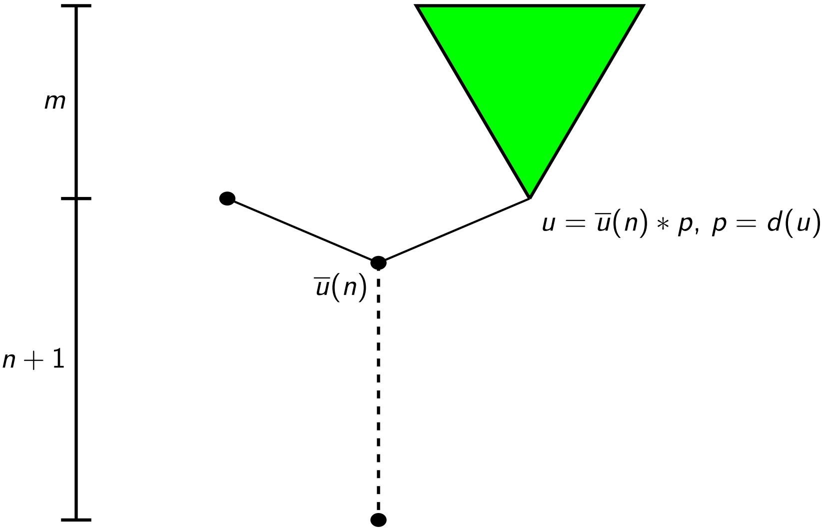
$$\forall_{n,u}(|u| = n \to u \neq \overline{s_d}(n) \to \exists_m(u \in D_{b,m}).$$

Proof. Induction on n. $n = 0$: false premise.
Step: let $|u * p| = n + 1$.

Case $\overline{u}(\ell) \neq \overline{s_d}(\ell)$ for some $\ell \leq n$. By IH $\overline{u}(\ell) \in D_{b,m}$ for some m, hence $u \in D_{b,m-(n+1-\ell)}$.



Case $\overline{u}(n) = \overline{s_d}(n)$ and $p \neq (s_d)_n$. Then $p = d(u)$ by definition of s_d. Hence $u * p \in D_{b,m}$ for some m, by the Distance lemma.

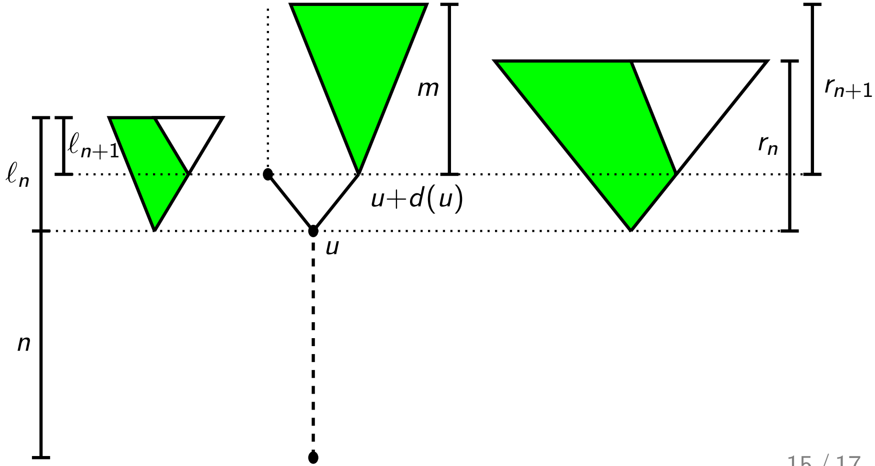


Lemma (Bounds)

Let b be a uniformly coconvex bar with modulus d. Then for every n there are bounds ℓ_n, r_n for the b-distances of all nodes of the same length n that are left / right of $\overline{s_d}(n)$.

Proof. For $n = 0$ there are no such nodes.

Consider $\overline{s_d}(n+1) = u * (s_d)_n$ of length $n+1$. Assume $(s_d)_n = 0$. Then every node to the left of $u * 0$ is a successor node of one to the left of u, and hence $\ell_{n+1} = \ell_n - 1$. The nodes to the right of $u * 0$ are $u * 1$ and then nodes which are all successor nodes of one to the right of u. Since $u * 1$ is $u * d(u)$, lemma Distance gives its b-distance m. Let $r_{n+1} = \max(m, r_n - 1)$.



Theorem

Let b be a uniformly coconvex bar with modulus d. Then b is a uniform bar, i.e.,

$$\exists_m \forall_u (|u| = m \to u \in b).$$

Let s_d be the escape path. Since b is a bar, the escape path s_d hits b at some length n. Use lemma Bounds: the uniform bound is $n + \max(\ell_n, r_n)$