Coequalisers in the category of basic pairs

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A short history

- Aczel (2006) introduced the notion of a set-generated class for dcpos using some terminology from domain theory.
- van den Berg (2013) introduced the principle NID on non-deterministic inductive definitions and set-generated classes in the constructive Zermelo-Frankel set theory **CZF**.
- Aczel et al. (2015) characterized set-generated classes using generalized geometric theories and a set generation axiom SGA in **CZF**.
- I-Kawai (2015) constructed coequalisers in the category of basic pairs in the extension of **CZF** with SGA.
- I-Nemoto (2016) introduced another NID principle, called nullary NID, and proved that nullary NID is equivalent to Fullness in a subsystem **ECST** of **CZF**.

The elementary constructive set theory

The language of a constructive set theory contains variables for sets and the binary predicates $=$ and $\in$. The axioms and rules are those of intuitionistic predicate logic with equality. In addition, **ECST** has the following set theoretic axioms:

Extensionality: $\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b]$.

Pairing: $\forall a \forall b \exists c \forall x (x \in c \leftrightarrow x = a \vee x = b)$.

Union: $\forall a \exists b \forall x [x \in b \leftrightarrow \exists y \in a (x \in y)]$.

Restricted Separation:

$$\forall a \exists b \forall x (x \in b \leftrightarrow x \in a \wedge \varphi(x))$$

for every *restricted* formula $\varphi(x)$. Here a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$.

The elementary constructive set theory

Replacement:

$$\forall a[\forall x \in a \exists! y \varphi(x,y) \rightarrow \exists b \forall y(y \in b \leftrightarrow \exists x \in a \varphi(x,y))]$$

for every formula $\varphi(x,y)$.

Strong Infinity:

$$\begin{aligned}&\exists a[0 \in a \wedge \forall x(x \in a \rightarrow x+1 \in a)\\ &\wedge \forall y(0 \in y \wedge \forall x(x \in y \rightarrow x+1 \in y) \rightarrow a \subseteq y)],\end{aligned}$$

where $x+1$ is $x \cup \{x\}$, and 0 is the empty set $\emptyset$.

The elementary constructive set theory

- Using Replacement and Union, the cartesian product $a \times b$ of sets a and b consisting of the ordered pairs $(x, y) = \{\{x\}, \{x, y\}\}$ with $x \in a$ and $y \in b$ can be introduced in **ECST**.
- A relation r between a and b is a subset of $a \times b$. A relation $r \subseteq a \times b$ is total (or is a multivalued function) if for every $x \in a$ there exists $y \in b$ such that $(x, y) \in r$.
- A function from a to b is a total relation $f \subseteq a \times b$ such that for every $x \in a$ there is exactly one $y \in b$ with $(x, y) \in f$.

The elementary constructive set theory

The class of total relations between a and b is denoted by $\mathrm{mv}(a, b)$:

$$r \in \mathrm{mv}(a, b) \Leftrightarrow r \subseteq a \times b \wedge \forall x \in a \exists y \in b((x, y) \in r).$$

The class of functions from a to b is denoted by b^a:

$$\begin{aligned} f \in b^a \Leftrightarrow & f \in \mathrm{mv}(a, b) \\ & \wedge \forall x \in a \forall y, z \in b((x, y) \in f \wedge (x, z) \in f \rightarrow y = z). \end{aligned}$$

The constructive set theory **CZF**

The constructive set theory **CZF** is obtained from **ECST** by replacing Replacement by

Strong Collection:

$$\forall a[\forall x \in a \exists y \varphi(x,y) \rightarrow \exists b(\forall x \in a \exists y \in b \varphi(x,y) \\ \wedge \forall y \in b \exists x \in a \varphi(x,y))]$$

for every formula $\varphi(x,y)$,

The constructive set theory **CZF**

and adding

Subset Collection:

$$\begin{aligned}\forall a \forall b \exists c \forall u [\forall x \in a \exists y \in b \varphi(x, y, u) \rightarrow \\ \exists d \in c (\forall x \in a \exists y \in d \varphi(x, y, u) \\ \wedge \forall y \in d \exists x \in a \varphi(x, y, u))]\end{aligned}$$

for every formula $\varphi(x, y, u)$, and

$\in$-Induction:

$$\forall a (\forall x \in a \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \varphi(a),$$

for every formula $\varphi(a)$.

The constructive set theory **CZF**

- In **ECST**, Subset Collection implies

  Fullness:

$$\forall a \forall b \exists c (c \subseteq \mathrm{mv}(a,b) \\ \wedge \forall r \in \mathrm{mv}(a,b) \exists s \in c (s \subseteq r)),$$

  and Fullness and Strong Collection imply Subset Collection.
- The notable consequence of Fullness is that b^a forms a set:
  Exponentiation: $\forall a \forall b \exists c \forall f (f \in c \leftrightarrow f \in b^a)$.
- For a set S, we write $\mathrm{Pow}(S)$ for the power class of S which is not a set in **ECST** nor in **CZF**:

$$a \in \mathrm{Pow}(S) \Leftrightarrow a \subseteq S.$$

Set-generated classes

Definition 1

Let S be a set, and let X be a subclass of $\mathrm{Pow}(S)$. Then X is set-generated if there exists a subset G, called a generating set, of X such that

$$\forall \alpha \in X \forall x \in \alpha \exists \beta \in G (x \in \beta \subseteq \alpha).$$

Remark 2

The power class $\mathrm{Pow}(S)$ of a set S is set-generated with a generating set

$$\{\{x\} \mid x \in S\}.$$

Rules

Definition 3

Let S be a set. Then a rule on S is a pair (a, b) of subsets a and b of S. A rule is called

- nullary if a is empty;
- elementary if a is a singleton;
- finitary if a is finitely enumerable.

A subset α of S is closed under a rule (a, b) if

$$a \subseteq \alpha \to b \between \alpha.$$

For a set R of rules on S, we call a subset α of S R-closed if it is closed under each rule in R.

Remark 4

Note that if a rule is nullary or elementary, then it is finitary.

NID principles

Definition 5

Let NID denote the principles that

- for each set S and set R of rules on S, the class of R-closed subsets of S is set-generated.

The principles obtained by restricting R in NID to a set of nullary, elementary and finitary rules are denoted by NID_0, NID_1 and $\mathrm{NID}_{<\omega}$, respectively.

Remark 6

Note that $\mathrm{NID}_{<\omega}$ implies NID_0 and NID_1.

The nullary NID

Theorem 7 (I-Nemoto 2015)

The following are equivalent over **ECST**.

1. NID_0.
2. *Fullness.*

Proposition 8 (I-Nemoto 2015)

NID_1 *implies* NID_0.

Remark 9

$$\mathrm{NID}_0 \longleftarrow \mathrm{NID}_1 \longleftarrow \mathrm{NID}_{<\omega}$$

Basic pairs

Definition 10

A basic pair is a triple $(X, \Vdash, S)$ of sets X and S, and a relation $\Vdash$ between X and S.

Relation pairs

Definition 11

A relation pair between basic pairs $\mathcal{X}_1 = (X_1, \Vdash_1, S_1)$ and $\mathcal{X}_2 = (X_2, \Vdash_2, S_2)$ is a pair (r, s) of relations $r \subseteq X_1 \times X_2$ and $s \subseteq S_1 \times S_2$ such that

$$\Vdash_2 \circ\, r = s \circ \Vdash_1,$$

that is, the following diagram commute.

$$\begin{tikzcd}
X_1 \arrow[r, "\Vdash_1"] \arrow[d, "r"'] & S_1 \arrow[d, "s"] \\
X_2 \arrow[r, "\Vdash_2"'] & S_2
\end{tikzcd}$$

Relation pairs

Definition 12

Two relation pairs (r_1, s_1) and (r_2, s_2) between basic pairs $\mathcal{X}_1$ and $\mathcal{X}_2$ are equivalent, denoted by $(r_1, s_1) \sim (r_2, s_2)$, if

$$\Vdash_2 \circ r_1 = \Vdash_2 \circ r_2,$$

or equivalently $s_1 \circ \Vdash_1 = s_2 \circ \Vdash_1$.

The category of basic pairs

Notation 13

For a basic pair $(X, \Vdash, S)$, we write

$$\Diamond D = \Vdash (D) \quad \text{and} \quad \mathrm{ext}\, U = \Vdash^{-1} (U)$$

for $D \in \mathrm{Pow}(X)$ and $U \in \mathrm{Pow}(S)$.

Proposition 14 (I-Kawai 2015)

Basic pairs and relation pairs form a category **BP**.

Coequalisers

Definition 15

A coequaliser of a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in a category $\mathbf{C}$ is a pair of an object C and a morphism $B \xrightarrow{e} C$ such that $e \circ f = e \circ g$, and it satisfies a universal property in the sense that for any morphism $B \xrightarrow{h} D$ with $h \circ f = h \circ g$, there exists a unique morphism $C \xrightarrow{k} D$ for which the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] \arrow[dr, "h"'] & C \arrow[d, dotted, "k"] \\
 & & D
\end{tikzcd}
$$

Coequalisers

Proposition 16 (I-Kawai 2015)

Let $\mathcal{X}_1 \underset{(r_2,s_2)}{\overset{(r_1,s_1)}{\rightrightarrows}} \mathcal{X}_2$ *be a parallel pair of relation pairs in* **BP**. *If a subclass*

$$Q = \{U \in \mathrm{Pow}(S_2) \mid \mathrm{ext}_1\, s_1^{-1}(U) = \mathrm{ext}_1\, s_2^{-1}(U)\}$$

of $\mathrm{Pow}(S_2)$ *is set-generated, then the parallel pair has a coequaliser.*

A NID principle

Definition 17

Let S be a set. Then a subset α of S is biclosed under a rule (a, b) if

$$a \between \alpha \leftrightarrow b \between \alpha.$$

For a set R of rules on S, we call a subset α of S R-biclosed if it is biclosed under each rule in R.

Definition 18

Let $\mathrm{NID}_{\mathrm{bi}}$ denotes the principles that

- for each set S and set R of rules on S, the class of R-biclosed subsets of S is set-generated.

A NID principle

Proposition 19

- NID_1 *implies* $\mathrm{NID}_{\mathrm{bi}}$.
- $\mathrm{NID}_{\mathrm{bi}}$ *implies* NID_0.

Remark 20

$$\mathrm{NID}_0 \longleftarrow \mathrm{NID}_{\mathrm{bi}} \longleftarrow \mathrm{NID}_1 \longleftarrow \mathrm{NID}_{<\omega}$$

Proof

Let R be a set of rules on a set S, and define a set R' of elementary rules on S by

$$R' = \{(\{x\}, b) \mid (a, b) \in R, x \in a\} \cup \{(\{y\}, a) \mid (a, b) \in R, y \in b\}.$$

Then it is straightforward to show that a subset α of S is R-biclosed if and only if it is R'-closed.
Therefore any generating set of the class of R'-closed subsets of S is a generating set of the class of R-biclosed subsets of S. Thus NID_1 implies $\mathrm{NID}_{\mathrm{bi}}$.

Proof

Let R be a set of nullary rules on a set S, and define a set R' of rules on $S \cup \{*_S\}$ by

$$R' = \{(\{*_S\}, b) \mid (\emptyset, b) \in R\},$$

where $*_S = \{x \in S \mid x \notin x\}$. Since α is R-closed if and only if $\alpha \cup \{*_S\}$ is R'-biclosed for each $\alpha \in \mathrm{Pow}(S)$, if G is a generating set of the class of R'-biclosed subsets of $S \cup \{*_S\}$, then the set

$$G' = \{\beta \cap S \mid \beta \in G, *_S \in \beta\}$$

is a generating set of the class of R-closed subsets of S. Therefore $\mathrm{NID}_{\mathrm{bi}}$ implies NID_0.

BP has coequalisers

Theorem 21

The following are equivalent over **ECST**.

1. $\mathrm{NID}_{\mathrm{bi}}$.
2. **BP** *has coequalisers.*

Remark 22

Since **BP** has small coproducts, in the presence of $\mathrm{NID}_{\mathrm{bi}}$, the category **BP** is cocomplete.

Proof

Suppose that $\mathrm{NID}_{\mathrm{bi}}$ holds, and let $\mathcal{X}_1 \underset{(r_2,s_2)}{\overset{(r_1,s_1)}{\rightrightarrows}} \mathcal{X}_2$ be a parallel pair of relation pairs in **BP**. Then, by Proposition 16, it suffices to show that the class

$$Q = \{U \in \mathrm{Pow}(S_2) \mid \mathrm{ext}_1\, s_1^{-1}(U) = \mathrm{ext}_1\, s_2^{-1}(U)\}$$

is set-generated. Since for each $U \in \mathrm{Pow}(S_2)$ and $x \in X_1$,

$$x \in \mathrm{ext}_1\, s_1^{-1}(U) \leftrightarrow \Diamond x \between s_1^{-1}(U) \leftrightarrow s_1(\Diamond x) \between U$$

and, similarly, $x \in \mathrm{ext}_1\, s_2^{-1}(U) \leftrightarrow s_2(\Diamond x) \between U$, we have

$$U \in Q \leftrightarrow \forall x \in X_1 [s_1(\Diamond x) \between U \leftrightarrow s_2(\Diamond x) \between U]$$

for each $U \in \mathrm{Pow}(S_2)$. Therefore Q is the class of subsets of S_2 biclosed under the set $\{(s_1(\Diamond x), s_2(\Diamond x)) \mid x \in X_1\}$ of rules on S_2, and so Q is set-generated by $\mathrm{NID}_{\mathrm{bi}}$.

Proof

Conversely, suppose that **BP** has coequalisers, and let R be a set of rules on a set S. Define basic pairs $\mathcal{X}_1$ and $\mathcal{X}_2$ by

$$\mathcal{X}_1 = (R, \Delta_R, R) \quad \text{and} \quad \mathcal{X}_2 = (S, \Delta_S, S),$$

and define relations r and s between R and S by

$$(a, b)\ r\ u \Leftrightarrow u \in a \quad \text{and} \quad (a, b)\ s\ u \Leftrightarrow u \in b$$

for each $(a, b) \in R$ and $u \in S$. Then (r, r) and (s, s) are relation pairs between $\mathcal{X}_1$ and $\mathcal{X}_2$, and hence $\mathcal{X}_1 \underset{(s,s)}{\overset{(r,r)}{\rightrightarrows}} \mathcal{X}_2$ is a parallel pair in **BP**. Since **BP** has coequalisers, there exist an object $\mathcal{Y} = (Y, \Vdash, T)$ and a relation pair $\mathcal{X}_2 \overset{(p,q)}{\to} \mathcal{Y}$ such that $(p, q) \circ (r, r) = (p, q) \circ (s, s)$, and satisfy the universal property.

Proof

For each $v \in T$, since

$$a \between q^{-1}(v) \leftrightarrow (a, b)\ (q \circ r)\ v \leftrightarrow (a, b)\ (q \circ s)\ v \leftrightarrow b \between q^{-1}(v)$$

for each $(a, b) \in R$, $q^{-1}(v)$ is an R-biclosed subset.
Let α be an R-biclosed subset of S, and consider a basic pair $\mathcal{Z} = (\{*\}, \Delta_{\{*\}}, \{*\})$ and a relation t between S and $\{*\}$ defined by

$$u\ t\ * \Leftrightarrow u \in \alpha.$$

Then (t, t) is a relation pair between $\mathcal{X}_2$ and $\mathcal{Z}$, and, since

$$(a, b)\ (t \circ r)\ * \leftrightarrow a \between \alpha \leftrightarrow b \between \alpha \leftrightarrow (a, b)\ (t \circ s)\ *$$

for each $(a, b) \in R$, we have $(t, t) \circ (r, r) = (t, t) \circ (s, s)$.
Therefore there exists a relation pair (j, k) between $\mathcal{Y}$ and $\mathcal{Z}$ such that $(t, t) = (j, k) \circ (p, q)$, by the universal property.

Proof

Since

$$y \in \alpha \leftrightarrow y\,t\,* \leftrightarrow y\,(k \circ q)\,* \leftrightarrow \exists v \in T(y\,q\,v \wedge v\,k\,*)$$

for each $y \in S$, if $x \in \alpha$, then $x \in q^{-1}(v) \subseteq \alpha$ for some $v \in T$. Therefore the subset G of $\mathrm{Pow}(S)$ defined by

$$G = \{q^{-1}(v) \mid v \in T\}$$

is a generating set of the class of R-biclosed subsets of S.

Work in progress

Definition 23

A rule (a, b) on a set S is called *n-ary* if there exists a surjection $n \to a$.

Remark 24

Note that if a rule is $n+1$-ary, then it is $n+2$-ary.

Definition 25

The principles obtained by restricting R in NID to a set of n-ary rules is denoted by NID_n.

Work in progress

Proposition 26

NID_2 *implies* $\mathrm{NID}_{<\omega}$.

Remark 27

$$\mathrm{NID}_0 \longleftarrow \mathrm{NID}_{\mathrm{bi}} \longleftarrow \mathrm{NID}_1 \longleftarrow \mathrm{NID}_2 \longleftrightarrow \cdots \longleftrightarrow \mathrm{NID}_{<\omega}$$

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