

Periods on the moduli space of genus 0 curves

SARAH CARR

(joint work with Francis Brown, Leila Schneps)

A recent theorem in the thesis of Francis Brown proves that any period over a connected component of the real part of $\mathfrak{M}_{0,n}(\mathbb{C})$ is a \mathbb{Q} linear combination of multizeta values. By studying the cohomology and geometry of $\mathfrak{M}_{0,n}(\mathbb{C}) = \mathfrak{M}_{0,n}$, we have found a method to formally represent these periods as linear combinations of pairs of n -polygons, one of which represents a connected component, or *cell*, of the real part of $\mathfrak{M}_{0,n}$, and the other a certain differential form which we call a *cell form*. These pairs of polygons form an algebra for the shuffle product. In this talk, we will outline the combinatorial structure of this algebra. As consequences, we obtain an explicit basis for the cohomology group, $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$, of differential forms converging on the boundary divisors which bound standard associahedron, δ , and hence a new method for studying multizeta values.

We denote by $(0, t_1, \dots, t_{n-3}, 1, \infty)$ a point in $\mathfrak{M}_{0,n}$ and by $\mathfrak{M}_{0,n}(\mathbb{R}) \subset \mathfrak{M}_{0,n}$ the points whose marked points are in \mathbb{R} . We can identify an oriented n -gon to a connected component, or *cell*, in $\mathfrak{M}_{0,n}(\mathbb{R})$ by labelling the n -gon with the marked points. This n -gon is associated to the cell given by the clockwise cyclic ordering of the labelled edges of the polygon. For example, a polygon cyclically labelled $(0, t_1, t_3, 1, t_2, \infty)$ is identified with the cell $0 < t_1 < t_3 < 1 < t_2 < \infty$ in $\mathfrak{M}_{0,n}(\mathbb{R})$.

Similarly we can associate an n -gon labelled by the marked points to a differential $(n-3)$ -form which we call a *cell form*. A cell form is defined as $dt_1 \wedge \dots \wedge dt_{n-3} / \Pi(s_i - s_{i-1})$, where the s_i are the cyclically labelled sides of the polygon. We leave the side labelled ∞ out of the product. For example, the polygon cyclically labelled $[0, 1, t_1, t_3, \infty, t_2]$ gives the cell form $dt_1 dt_2 dt_3 / ((t_1 - 1)(t_3 - t_1)(-t_2))$.

We consider a *period* on $\mathfrak{M}_{0,n}$ to be convergent integral, over a connected component in $\mathfrak{M}_{0,n}(\mathbb{R})$, of a differential form which is holomorphic on the interior of $\mathfrak{M}_{0,n}$ and which has at most logarithmic singularities on $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$. Up to a variable change corresponding to permuting the marked points, all periods can be written as integrals over the standard cell, $\delta := 0 < t_1 < \dots < t_{n-3} < 1$.

According to the above definitions, we can associate a pair of polygons to a cell and a cell form. Therefore, we have a map from pairs of n -gons to periods (and divergent integrals) given by mapping the pair to the integral of the cell form over the cell. This association and Brown's thesis have allowed us to prove some results and approach some conjectures about multizeta values and formal multizeta values.

Theorem 1. *The 01-cell forms given by polygons $[\dots, 0, 1, \dots, \infty]$ form a basis for $H^{n-3}(\mathfrak{M}_{0,n})$ of differential $(n-3)$ -forms convergent on the interior of $\mathfrak{M}_{0,n}$ and with at most logarithmic singularities on the boundary divisors, $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$.*

To prove this, it was enough to express Arnol'd's well-known basis in terms of 01-cell forms.

Definitions 2. Let \mathcal{P}_n be the vector space generated by oriented n -gons decorated by the marked points in $\mathfrak{M}_{0,n}$.

Recall that the shuffle product of lists A and B is defined as

$$A \text{ III } B = \sum_{\sigma \in \mathfrak{S}} \sigma(A \cdot B),$$

where σ runs over the permutations of the concatenation of A and B such that the orders of A and B are preserved.

Let $I_n \subset \mathcal{P}_n$ be the vector subspace generated by shuffle sums with respect to ∞ , in other words polygon sums of the form

$$\sum_{W \in A \text{ III } B} [W, \infty]$$

where A, B is a partition of $\{0, t_1, \dots, t_{n-3}, 1\}$.

Theorem 3. *\mathcal{P}_n/I_n is isomorphic to $H^{n-3}(\mathfrak{M}_{0,n})$.*

Proof. (Sketch) By the previous theorem, we have a natural surjective map

$$(1) \quad \mathcal{P}_n \twoheadrightarrow H^{n-3}(\mathfrak{M}_{0,n}),$$

which sends a polygon to its associated cell form. A calculation on rational functions shows that I_n is in the kernel of this map. A dimension count finishes the proof. \square

We would like to study cohomology of interesting subspaces of $H^{n-3}(\mathfrak{M}_{0,n})$ such as the space of differential forms which converge on the standard cell, δ . To do this we make use of the kernel I_n to create a basis of convergent 01-forms and what we refer to as *insertion forms*.

Some 01-forms naturally converge on δ . We define a *chord* on a cell form, ω , to be a set of marked points of a subsequence on ω of the length between 2 and $\lfloor \frac{n}{2} \rfloor$. The 01-forms which do not have any chords in common with δ converge on δ . However, some linear combinations of nonconvergent 01-forms converge on δ ; a certain generating set of these are *insertion forms*. Insertion forms are created according to a recursive procedure of inserting convergent shuffles (those whose shuffle factors have no chords in common with δ) into convergent 01-forms. For example,

$$(2) \quad \omega = [0, 1, t_1, t_2, \infty, t_3] + [0, 1, t_2, t_1, \infty, t_3]$$

$$(3) \quad = [0, 1, t_1 \text{ III } t_2, \infty, t_3]$$

is an insertion form obtained by inserting the convergent shuffle $t_1 \sharp t_2$ into the convergent 01-form $[0, 1, s_1, \infty, s_2]$. The shuffle factors are t_1 and t_2 , and are therefore too short to contain any chords.

Theorem 4. *The insertion forms and the convergent 01-cell forms form a basis for $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$.*

The proof of this theorem is the heart of our recent work and is given in [3]. It exploits the fact that I_n is the kernel of the map (1).

Now that we can explicitly describe the differential forms convergent on δ , we can define an algebra of periods, since by a variable change, all periods can be written as integrals of forms over δ . The algebra of periods has three known sets of relations:

- (1) invariance under the symmetric group action corresponding to a variable change;
- (2) forms given by shuffles with respect to one point are identically 0;
- (3) product map relations coming from the pullback of maps on moduli space (outlined in [2] and [3]).

The product map relations also allow us to define a multiplication law on periods.

We conjecture that these are the only relations on periods, but this question seems difficult to prove. A more strategic approach is to define a formal algebra on polygon pairs satisfying these and only these relations. Since the algebra of periods is isomorphic to the algebra of multizeta values ([2]), we conjecture that the formal algebra of pairs of polygons, which we call \mathcal{FC} , is isomorphic to the formal multizeta value algebra. With this association, we hope to approach some of the main conjectures on formal multizetas such as Zagier's dimension conjecture.

Conjecture 5 (Zagier). Let \mathcal{Z}_n be the vector space generated by weight n multizeta values. Then $d_n = \dim(\mathcal{Z}_n)$ is given by the recursive formula,

$$d_n = d_{n-2} + d_{n-3}.$$

This conjecture is true in small weight for \mathcal{FC} and we hope that its combinatorial recursive definition will allow us to make progress on this conjecture.

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