The algebra of cell-zeta values

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May 10, 2010

Abstract

A multizeta value or MZV is a real number,

$$\zeta(k_i, ..., k_d) = \sum_{n_1 > \cdots > n_d > 0} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}, k_i \in \mathbb{Z}, k_1 \geq 2. \quad (1)$$

We are interested in studying the $\mathbb{Q}$ algebra generated by these numbers. In this talk we would like to present a candidate for this algebra, by giving its generators and relations. The motivation for this work is the recent theorem by F. Brown:

Theorem 1. *All periods on $\mathcal{M}_{0,n+3}$ are $\mathbb{Q}$ linear combinations of $MZVs$.*

1 Background

1.1 $\mathcal{M}_{0,n+3}$

Definition 2. *$\mathcal{M}_{0,n+3}$ is the moduli space of genus 0 complex projective curves with $n+3$ distinct ordered marked points (punctures) modulo isomorphism.*

$\mathcal{M}_{0,n+3} \simeq (\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\})^n \setminus \Delta$ where $\Delta := \{t_i = t_j\}$. We denote a point in this space by an ordered $n+3$-tuple: $(0, t_i, ..., t_n, 1, \infty)$, where the t_i denote the punctures on the sphere, run through $\mathbb{C} \setminus \{0, 1\}$, are distinct and are different from 0, 1, and ∞.

1.2 Compactification and Boundary Divisors

We denote by $\overline{\mathcal{M}}_{0,n+3}$, the stable compactification of Deligne and Mumford of $\mathcal{M}_{0,n+3}$. The irreducible boundary divisors $\subset \overline{\mathcal{M}}_{0,n+3}$ of codimension 1 can be indexed by partitions of $\{0, t_1, ..., t_n, 1, \infty\} = S_n$.

In this talk the important spaces are $\mathcal{M}_{0,n+3}$ and $\overline{\mathcal{M}}_{0,n+3} \setminus \mathcal{M}_{0,n+3}$.

1.3 Periods

We denote by $\mathcal{M}_{0,n+3}(\mathbb{R})$ the set of points $(0, t_1, ..., t_n, 1, \infty)$ where $t_i \in \mathbb{R}$. The space is not connected, but is partitioned into cells. We call the cell $0 < t_1 < \cdots < t_n < 1$ δ_n, the standard cell.

The boundary of δ_n intersects the irreducible boundary divisors indexed by subsets of consecutive numbered marked points, $\{t_i, t_{i+1}, ..., t_{i_j}\}$. Let the collection of of boundary divisors sharing a boundary with δ_n be denoted by $\overline{\delta}_n$.

Example 3.

Definition 4. *A **period** on $\mathcal{M}_{0,n+3}$ is an integral,*

$$\int_{\delta_n} \omega,$$

where ω has rational function coefficient which is holomorphic on $\mathcal{M}_{0,n+3}$ and has at most simple poles on all the boundary divisors and of course has no poles on $\overline{\delta}_n$.

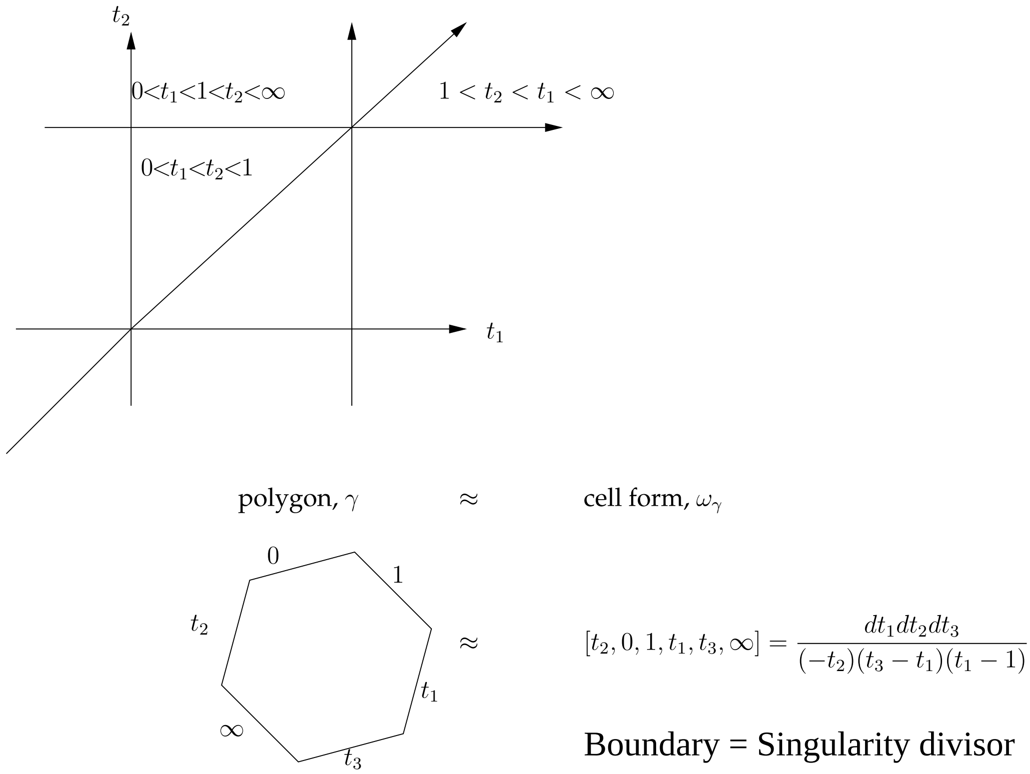

Figure 1: A cell-form in $\mathcal{M}_{0,6}$

2 Generators of the period algebra, cell-forms

I will start this section with an example from $\mathcal{M}_{0,6}$.

Example 5. *Consider the 6-gon with sides decorated by S_3. I may obtain a differential form by taking the 3-volume form and dividing by the product of successive differences of the sides, leaving any side containing ∞ out of the product as in figure 1.*

Definition 6. *A **cell-form**, ω_γ, on $\mathcal{M}_{0,n+3}$ is a differential n form associated to a cyclic order (or polygon) on S_n, $\gamma = [\infty, t_{i_0}, ..., t_{i_{n+1}}]$:*

$$\omega_\gamma := \frac{dt_1 \wedge \cdots \wedge dt_n}{\prod_{k=1}^{n+1}(t_{i_k} - t_{i_{k-1}})}.$$

We will refer to a **block** of a cyclic order as a sublist of that order.

Note: Let $I = \{t_i, t_{i+1}, ..., t_{i+j}\}$ denote a set of marked points with consecutive indices (as in the definition of Δ_{δ_n}). The integral, ω_γ converges if and only if γ contains no consecutive blocks which are orderings of such I.

Definition 7. *ω_γ is a **01-cell-form** if γ contains the block $(0, 1)$.*

Lemma 8. *The 01-cell-forms form a basis for $H^n_{dR}(\mathcal{M}_{0,n+3})$.*

Definition 9. *The **shuffle product** on lists, $A = (a_1, ..., a_j)$ and $B = (b_1, ..., b_k)$, is defined as*

$$A ш B = \sum_\sigma \sigma(A \cdot B),$$

where $\cdot$ denotes concatenation and the sum runs over all permutations $\sigma \in \mathfrak{S}_{j+k}$ where σ preserves the orders of both A and B.

Example 10.

$$(a,b)ш(c,d) = (a,b,c,d) + (a,c,b,d) + (a,c,d,b) + (c,a,b,d) + (c,a,d,b) + (c,d,a,b)$$

Definition 11. *A degree k shuffle product, $A_1ш \cdot шA_k$, is called a **convergent shuffle** in $I \subset S_n$ if*

1. *$\bigcup A_j = I$ where I is a set with consecutive indices, $I = \{t_i, t_{i+1}, ..., t_{i+j}\}$,*
2. *no factor, A_j, contains a block of consecutive indices.*

Example 12. *$t_2шt_3$ is a convergent shuffle. $t_3t_1t_4t_2шt_5$ is not a convergent shuffle because the first factor is a block whose associated set is a set of consecutive indices, $\{t_1, t_2, t_3, t_4\}$.*

Definition 13. *An **insertion form** is a linear combination of cell forms gotten from inserting a convergent shuffle on $I = \{t_i, ...\}$ into the place of t_i in a convergent cell-form and renumbering the indices so that there are no repeated marked points.*

Example 14. *$t_1шt_2$ is a convergent shuffle and $[0,1,t_1,\infty,t_2]$ is a convergent cell-form. From this we get the insertion form:*

$$[0,1,(t_1шt_2),\infty,t_3].$$

In the form we renamed t_2 as t_3 in order to make the variables in the form distinct.

Example 15. *All forms naturally associated to multizeta values are insertion forms gotten from inserting into a form on $\mathcal{M}_{0,5}$.*

$$\begin{aligned}
\zeta(3) &= \int_{\delta_3} [0,1,t_1,\infty,t_2шt_3] \\
\zeta(2,1) &= \int_{\delta_3} [0,1,t_1шt_2,\infty,t_3] \\
\zeta(2,1,1) &= \int_{\delta_4} [0,1,t_1шt_2шt_3,\infty,t_4] \\
\zeta(2,2) &= \int_{\delta_4} [0,1,t_1шt_3,\infty,t_2шt_4] \\
\zeta(4) &= \int_{\delta_4} [0,1,t_1,\infty,t_2шt_3шt_4]
\end{aligned}$$

Theorem 16. *The $\mathbb{Q}$ vector space of periods on $\mathcal{M}_{0,n+3}$ is generated by integrals over δ_n of insertion forms.*

Method of proof: Let $\mathcal{M}^{\delta}_{0,n+3} = \mathcal{M}_{0,n+3} \cup \Delta_{\delta_n}$, and let

$$Res_{d_I} : H^n_{dR}(\mathcal{M}_{0,n+3}) \to H^{n-1}_{dR}(\mathcal{M}_{0,n+2})$$

be the map which calculates the residue of a form along the irreducible boundary divisor, d_I. Then

$$H^n_{dR}(\mathcal{M}^{\delta}_{0,n+3}) \simeq \bigcap_{d_I \in \Delta_{\delta_n}} Ker(Res_{d_I}) \simeq \langle \omega : \omega \text{ is an insertion form} \rangle.$$

3 The formal cell number algebra

Lemma 17. *Let*

$$\begin{aligned}
\omega_{\gamma_1} &= [\infty, \sigma_1(0,1,t_1,...,t_j)] \\
&= [\infty, A_1, 0, A_2, 1, A_3] \\
\omega_{\gamma_2} &= [\infty, \sigma_2(0,1,t_{j+1},...,t_{j+k})] \\
&= [\infty, B_1, 0, B_2, 1, B_3].
\end{aligned}$$

Then

$$\int_{\delta_j} \omega_{\gamma_1} \int_{\delta_k} \omega_{\gamma_2} = \int_{\delta_{j+k}} \omega_{\gamma_1 ш \gamma_2},$$

where

$$\gamma_1 ш \gamma_2 = [\infty, A_1 ш B_1, 0, A_2 ш B_2, 1 A_3 ш B_3] \text{ and}$$
$$\delta_1 ш \delta_2 = \sqcup_{(i_1, \ldots i_{j+k})} 0 < t_{i_1} < \cdots t_{i_k} < 1,$$

where the lists, $(i_1, ..., i_{j+k})$ run over all terms in the shuffle product,

$$(t_1, ..., t_j) ш (t_{j+1}, ..., t_{j+k}).$$

Lemma 18. *Let $\mathcal{P}_n$ denote the $\mathbb{Q}$ vector space of $n+3$-gons decorated by S_n. Let*

$$\begin{aligned} \pi : \mathcal{P}_n &\to H^n_{dR}(\mathcal{M}_{0,n+3}) \\ \gamma &\mapsto \omega_\gamma. \end{aligned} \tag{2}$$

Then, $Ker(\pi) = \mathcal{I} = \langle [e, A ш B] : e \in S_n, A \cup B = S_n \setminus \{e\} \rangle$.

We call $\mathcal{I}$ the space of **shuffles with respect to one point**. This lemma is easy to prove using lemma 8.

Now, we will think of δ_n as an element of $\mathcal{P}_n$ by associating it with the order, $\delta_n = [0, t_1, ..., t_n, 1, \infty]$. This is natural if one just pictures this list replacing the commas by the symbol "$<$". Then we have the natural map,

$$\begin{aligned} \pi^2 : \mathcal{P}_n \times \mathcal{P}_n &\to \text{periods } \cup \infty \\ (\alpha, \gamma) &\mapsto \int_{\sigma(\alpha)} \sigma(\omega_\gamma), \end{aligned}$$

where $\sigma \in \mathfrak{S}_{n+3}$ such that $\sigma(\alpha) = \delta$.

Definition 19. ***The formal cell number algebra,*** *$\mathcal{FC}$. $\mathcal{FC}$ is a $\mathbb{Q}$ vector subspace of $\bigoplus_{n=0} \mathcal{P}_n \times \mathcal{P}_n$ generated by the pairs,*

$$\langle \big(\sigma(\delta_n), \sigma(\omega)\big) : \sigma \in \mathfrak{S}_{n+3},\ \omega \text{ is an insertion form} \rangle,$$

with the following relations,

1. *$\big(\sigma(\delta_n), \sigma(\omega)\big) = \big(\delta_n, \omega\big)$ (variable changes),*
2. *$\langle (\alpha, [e, A ш B]) \rangle = 0$ ($\mathcal{I} = Ker(\pi)$),*
3. $$(\alpha_1, \omega_1)(\alpha_2, \omega_2) = (\alpha_1 ш \alpha_2, \omega_1 ш \omega_2),$$

   according to the rules given in lemma 17.

There is a possibility that $\mathcal{FC}$ is isomorphic to the multizeta value algebra, Z. This conjecture seems out of reach at the moment because of questions of transcendence of MZVs. However, up to weight 9, $\mathcal{FC}$ verifies Zagier's dimension conjecture on the formal multizeta value algebra, $\mathcal{FZ}$. Therefore, we believe that the following conjecture is reasonable.

Conjecture 20. *$\mathcal{FC} \simeq \mathcal{FZ}$*

There is much work to be done. We are for the moment unable to find a possible candidate for such an isomorphism, because we don't know how to explicitly dig out the stuffle relation from the three relations on $\mathcal{FC}$, though we know that it can be done in low weights.

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