Cellular Multizeta Values

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Goals

The following is joint work with Francis Brown and Leila Schneps based on a combinatorial theory of *pairs of polygons*.

1. Objects: Periods on $\mathfrak{M}_{0,n}$

2. Method: Pairs of Polygons = Periods

3. Theorem: $H^\ell(\mathfrak{M}_{0,n}^\delta)$ = Insertion Polygons

4. Algebra: Polygon algebra

5. Theorem: Polygon algebra $\twoheadrightarrow$ Multizeta algebra

1. Periods on Moduli Space, $\mathfrak{M}_{0,n}$

Definition . *$\mathfrak{M}_{0,n} = \mathfrak{M}_{0,n}(\mathbb{C})$ is the space of Riemann spheres with n distinct, ordered marked points modulo isomorphism, i.e. modulo the action of $\mathrm{PSL}_2(\mathbb{C})$.*

We will denote a point in $\mathfrak{M}_{0,n}$ by $\overline{(z_1, z_2, ..., z_n)}$ or by the representative in its equivalence class $(0, t_1, ..., t_\ell, 1, \infty)$, $\ell = n - 3$.

$$\mathfrak{M}_{0,n} \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta,$$

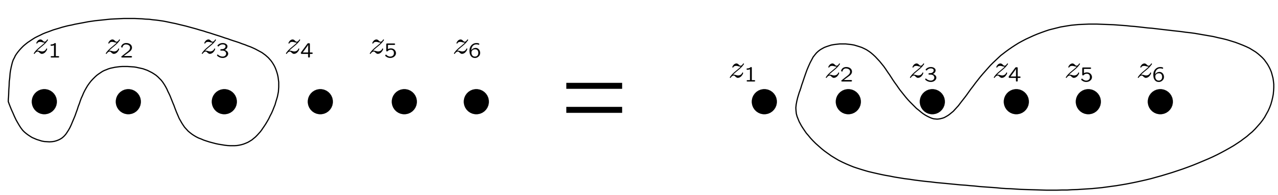
where Δ is the fat diagonal

$$\Delta = \{t_i = t_j\}.$$

Stable Compactification of $\mathfrak{M}_{0,n}$: $\overline{\mathfrak{M}}_{0,n}$

The boundary components added to $\mathfrak{M}_{0,n}$ to compactify correspond to loops around r points, $2 \le r \le \lfloor \frac{n}{2} \rfloor$. These boundary components (divisors) are blowups of the regions

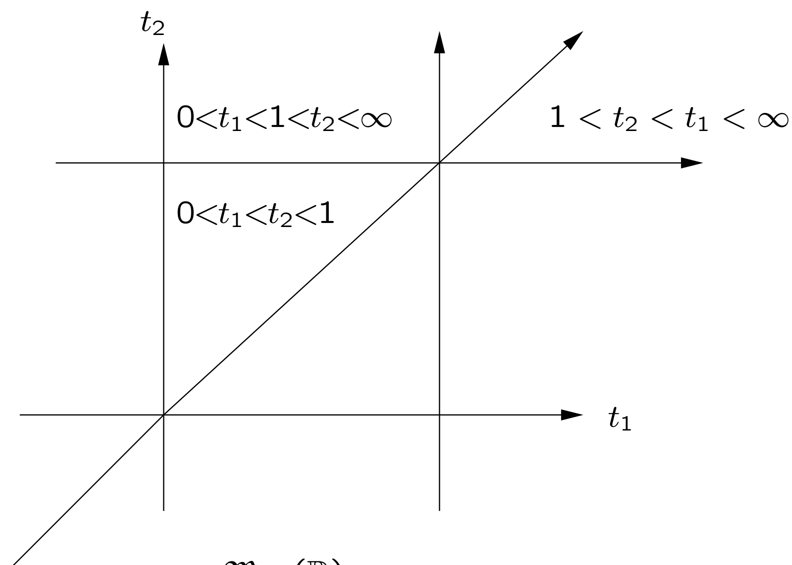
$$\{z_{i_1} = z_{i_2} = \cdots = z_{i_r} \mid 2 \le r \le \lfloor \frac{n}{2} \rfloor\} \subset (\mathbb{P}^1)^{n-3}.$$



$\mathfrak{M}_{0,n}(\mathbb{R})$: $\{(0, t_1, \ldots, t_\ell, 1, \infty)$ s.t. $t_i \in \mathbb{R}\}$.

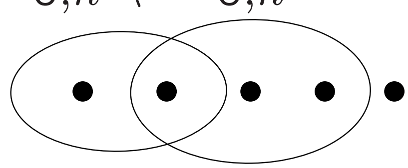
The connected components (cells) of $\mathfrak{M}_{0,n}(\mathbb{R})$ are given by fixed orderings of the t_i.

Example



$\mathfrak{M}_{0,5}(\mathbb{R})$

Standard Cell := $\delta = 0 < t_1 < ... < t_\ell < 1$

Boundary of δ in $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$ = loops around consecutive points 

Definition . *We will define a* **period** *on* $\mathfrak{M}_{0,n}$ *as a convergent integral,* $\int_\gamma \omega$, *over a cell* γ *in* $\mathfrak{M}_{0,n}(\mathbb{R})$, *of a form* ω *which is holomorphic on* $\mathfrak{M}_{0,n}$ *and has at most simple poles on* $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$.

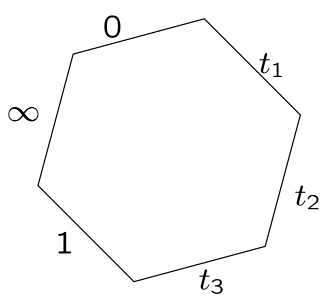
Up to a variable change corresponding to permuting $\{0, 1, t_1, \ldots, t_\ell\}$, all periods can be written as integrals over the standard cell, $0 < t_1 < \ldots < t_\ell < 1$.

Proposition . $H^\ell(\mathfrak{M}_{0,n}) \simeq$ *Vector space of differential* ℓ*-forms convergent on* $\mathfrak{M}_{0,n}$ *with at most simple poles on* $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$.

2. Polygons

We can associate an n-polygon decorated by the marked points, $\{0, t_1, ..., t_\ell, 1, \infty\}$, to the cell in $\mathfrak{M}_{0,n}(\mathbb{R})$ defined by the clockwise order of the marked points around the polygon.

polygon, γ $\approx$ cell, γ



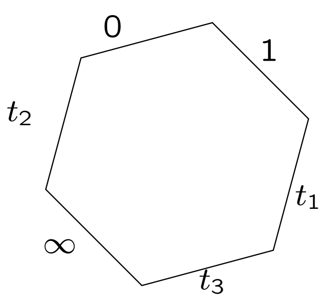
$\approx$ $(0, t_1, t_2, t_3, 1, \infty) = 0 < t_1 < t_2 < t_3 < 1 < \infty$

Definition . *Let γ be a polygon decorated by the marked points, $\{0, t_1, ..., t_\ell, 1, \infty\}$.*

The **cell form** *associated to γ, ω_γ, is the volume form, $dt_1 \cdots dt_\ell$, divided by the product of successive differences of marked points around the polygon.*

Example .

polygon, γ $\quad\approx\quad$ cell form, ω_γ



$\approx$

$$[t_2, 0, 1, t_1, t_3, \infty] = \frac{dt_1 dt_2 dt_3}{(-t_2)(t_3 - t_1)(t_1 - 1)}$$

Boundary = Singularity divisor

So we have a map from pairs of polygons to $\{\text{periods}\} \cup \{\infty\}$:

$$(\gamma, \omega) \mapsto \int_\gamma \omega.$$

Theorem . *(BCS)*

The cell forms $[0,1,...]$*, which we call 01-cell forms, form a basis for* $H^{\ell}(\mathfrak{M}_{0,n})$*.*

Proof. The proof of this theorem is based on results by Arnol'd who displayed $(n-2)!$ explicit linearly independent differential forms. These forms form a basis because $\dim(H^{\ell}(\mathfrak{M}_{0,n})) = (n-2)!$ by induction, since $\mathfrak{M}_{0,n} \to \mathfrak{M}_{0,n-1}$ is a fibration of fiber $\mathbb{P}^1 \setminus n-1$ points. It is easy to express Arnol'd's forms in terms of 01-cell forms. □

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Recall that the shuffle product of sequences, $\underline{a} = (a_1, ..., a_i)$ and $\underline{b} = (b_1, ..., b_j)$ is defined as

$$\underline{a} \sqcup\!\sqcup \underline{b} = \sum_{\sigma \in \mathfrak{S}_{i+j}} \sigma(a_1, ..., a_i, b_1, ...b_j)$$

over all σ that preserve the ordering of both $\underline{a}$ and $\underline{b}$.

Example . $(1,2) \sqcup\!\sqcup (3) = (1,2,3) + (1,3,2) + (3,1,2)$.

Shuffle product of polygons

$$\gamma_1 = [A_1, z_{i_1}, ..., A_r, z_{i_r}]$$
$$\gamma_2 = [B_1, z_{i_1}, ..., B_r, z_{i_r}].$$

We define the shuffle product of polygons with respect to the common points as

$$\gamma_1 \sqcup\!\sqcup \gamma_2 = [A_1 \sqcup\!\sqcup B_1, z_{i_1}, ..., A_r \sqcup\!\sqcup B_r, z_{i_r}].$$

Let $\mathcal{P}_n$ be the $\mathbb{Q}$-vector space generated by oriented n-gons with the sides indexed by

$$0, t_1, ..., t_\ell, 1, \infty.$$

So we have a natural map,

$$\phi : \mathcal{P}_n \to H^\ell(\mathfrak{M}_{0,n}).$$

Definition . *Let $I_n \subset \mathcal{P}_n$ be the vector subspace generated by shuffle products with respect to one point $\langle [A ш B, d] \rangle$, with $d \in \{z_1, \ldots, z_n\}$ and $A \cup B = \{z_1, \ldots, z_n\} \setminus \{d\}$.*

Lemma . $I_n \subset \ker(\phi)$.

Theorem . *(BCS)* $\mathcal{P}_n / I_n \simeq H^\ell(\mathfrak{M}_{0,n})$.

Proof.

- $\mathcal{P}_n \twoheadrightarrow H^\ell(\mathfrak{M}_{0,n})$
- $I_n \subset$ kernel
- $\dim(\mathcal{P}_n / I_n) = (n-2)!$ by Lyndon basis argument
- $\dim(H^\ell(\mathfrak{M}_{0,n})) = (n-2)!$

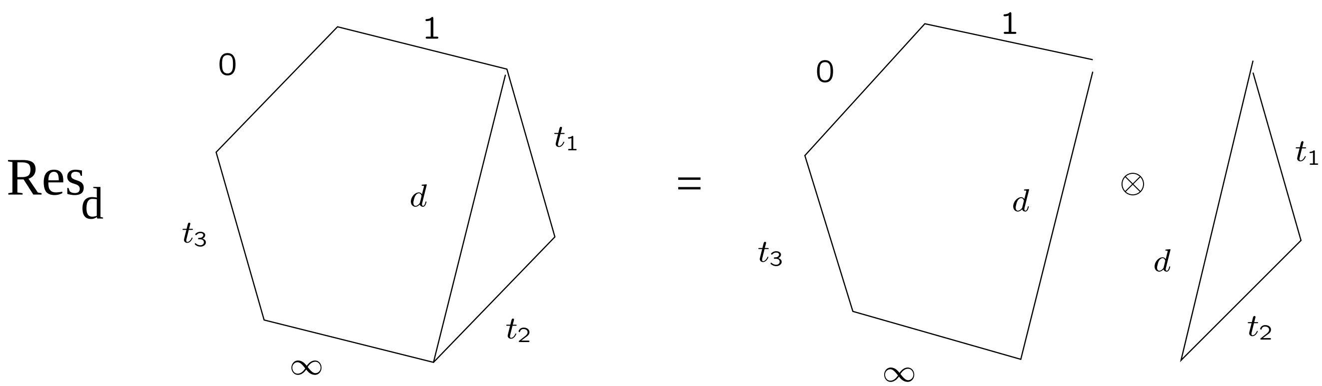
∎

3. Insertion basis for $H^\ell(\mathfrak{M}^\delta_{0,n})$

Definition . $\mathfrak{M}^\delta_{0,n} := \mathfrak{M}_{0,n} \cup \{$*the boundary divisors of* $\delta\}$.

To study $H^\ell(\mathfrak{M}^\delta_{0,n})$ we use the polygon structure to calculate the residues of linear combinations of 01-cell forms.

Example . The residue of a polygon (or cell form) along a divisor, d is a chord is given by



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Example . *Convergence of*

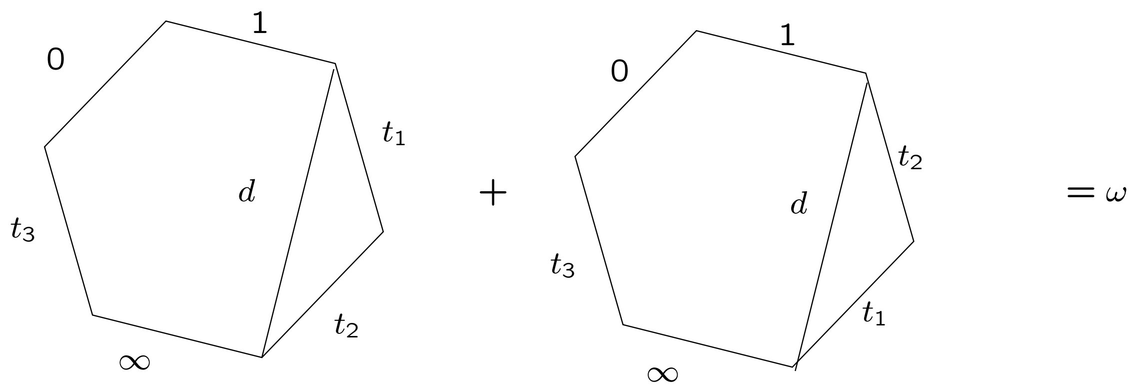
$$\omega = [0, 1, t_1, t_2, \infty, t_3] + [0, 1, t_2, t_1, \infty, t_3]$$

- *Identify bad divisors as chords,*

$$d := t_1 = t_2.$$

- $Res_d(\omega) = [0, 1, d, \infty, t_3] \otimes [t_1 ш t_2, d].$

- *Image in forms is 0, since the right hand factor is a shuffle w.r.t. one element.*



We write the form,

$$\omega = [0,1,t_1,t_2,\infty,t_3] + [0,1,t_2,t_1,\infty,t_3]$$
$$= [0,1,t_1 \sqcup\!\sqcup\, t_2,\infty,t_3].$$

It is obtained by inserting the shuffle $t_1 \sqcup\!\sqcup\, t_2$ into the smaller convergent cell form, $[0,1,t_1,\infty,t_2]$.

Insertion polygons are all obtained in this way and they map to **insertion forms**.

Theorem . *(BCS) The insertion forms form a basis for $H^{\ell}(\mathfrak{M}^{\delta}_{0,n})$ and the dimension can be counted using a recursion formula based on their construction.*

4. The algebra of periods, $\mathcal{C}$

We define a product map,

$$f : \mathfrak{M}_{0,n} \rightarrow \mathfrak{M}_{0,r} \times \mathfrak{M}_{0,s}$$

by specifying subsets T_1, T_2 of $T = \{z_1, ..., z_n\}$ such that if $T_1 = \{z_{i_1}, ..., z_{i_r}\}$, $T_2 = \{z_{j_1}, ..., z_{j_s}\}$, then $|T_1 \cap T_2| = 3$ and $T_1 \cup T_2 = T$. We define the product map as the product of two forgetful maps:

$$f : (z_1, ..., z_n) \mapsto (z_{i_1}, ..., z_{i_r}) \times (z_{j_1}, ..., z_{j_s})$$

such that the order of the original sequence is preserved ($i_{k-1}{<}i_k, j_{k-1}{<}j_k$).

Theorem . Given two periods, on $\mathfrak{M}_{0,r}$ and $\mathfrak{M}_{0,s}$ and a product map f, we have .

$$\int_{\gamma_1} \omega_1 \int_{\gamma_2} \omega_2 = \int_{f^{-1}(\gamma_1 \times \gamma_2)} f^*(\omega_1 \wedge \omega_2)$$
$$= \int_{\gamma_1 ш \gamma_2} \omega_1 ш \omega_2. \quad .$$

- (BCS) Product of cellforms equals their polygon shuffle product.

- Preimage of product domain is a shuffle.

Example .

$$f : (0, t_1, t_2, t_3, t_4, 1, \infty) \mapsto (0, t_1, t_2, 1, \infty) \times (0, t_3, t_4, 1, \infty)$$

$$(0, (t_1, t_2) ш (t_3, t_4), 1, \infty) = (0, t_1, t_2, t_3, t_4, 1, \infty) \sqcup (0, t_1, t_3, t_2, t_4, 1, \infty) \sqcup (0, t_3, t_1, t_4, t_2, 1, \infty) ... \mapsto (0, t_1, t_2, 1, \infty) \times (0, t_3, t_4, 1, \infty)$$

Definition . *We denote by $\mathcal{C}$ the algebra of periods on $\mathfrak{M}_{0,n}$ with multiplication given by the product maps.*

Conjecture . *All of the relations in $\mathcal{C}$ are given by variable changes and relations coming from the different product maps.*

Definition . *We denote by $\mathcal{FC}$ the formal algebra of pairs of polygons, (δ, ω), decorated by marked points where the ω is an insertion polygon modulo the following relations :*

1. $(\delta, \omega) = (\sigma(\delta), \sigma(\omega))$, *for all $\sigma \in \mathfrak{S}_n$ (variable changes)*
2. $(\gamma, [A ш B, z_i]) = 0$ *($I_n \mapsto 0$)*
3. *For each product map f, a corresponding shuffle product of polygons,*

$$(\gamma_1, \omega_1)(\gamma_2, \omega_2) = (\gamma_1 ш \gamma_2, \omega_1 ш \omega_2).$$

5. $\mathcal{FC} \twoheadrightarrow \mathcal{C} = \mathcal{Z}$

Definition . *Let* $n_1, \ldots, n_r \in \mathbb{N} \setminus \{0\}$ *such that* $n_1 \geq 2$*.* **Multizeta values** *are defined by nested sums*

$$\zeta(n_1, \ldots, n_r) = \sum_{k_1 > \ldots > k_r \geq 1} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \in \mathbb{R}$$

where the weight *is* $\ell = n_1 + \ldots + n_r$*.*

Definition . *We denote by* $\mathcal{Z}$ *the algebra of multizeta values.*

Theorem . *(Brown)* $\mathcal{Z} = \mathcal{C}$

Proof . ·

– (Kontsevich, Zagier) Any multizeta can be expressed as a period on $\mathfrak{M}_{o,n}$ (explicitly)
– Any convergent period on $\mathfrak{M}_{o,n}$ can be written as a linear combination of multizeta values (not explicitly)

Since periods satisfy the three defining relations of $\mathcal{FC}$, we have

$$\mathcal{FC} \twoheadrightarrow \mathcal{C} = \mathcal{Z}.$$

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Let $\mathcal{Z}_n$ be the $\mathbb{Q}$-vector space generated by weight n multizeta values and products of multizeta values of total weight n.

Conjecture . *(Zagier) Let* $d_n = \dim_{\mathbb{Q}} \mathcal{Z}_n$. *Then*

$$d_n = d_{n-2} + d_{n-3} \ ,$$

where $d_0 = 1, d_1 = 0, d_2 = 1$.

This conjecture is true for $\mathcal{FC}_{n+3}$, $n = 0, 1, 2, 3, 4, 5, 6$. We hope that the combinatorial structure will make this conjecture accessible for $\mathcal{FC}$.