Moulds and Multiple Zeta Values

1 Moulds

Definition 1. *The standard definition is that a mould is a function on "a variable number of variables". To flesh out this definition, in the general case, let $\mathcal{A}, \mathcal{B}$ be sets and K be an algebra. A* mould, *$M^\bullet = (M^\bullet, \mathcal{A}, K)$, is a map from the free monoid $\mathcal{A}^*$ into K and a bimould is defined as a function on the free monoid of the Cartesian product of two sets, $(\mathcal{A} \times \mathcal{B})^*$:*

$$\begin{aligned}
\textit{Mould} \qquad\qquad & M^\bullet : \mathcal{A}^* \to K \\
& \mathbf{w} = (w_1, ..., w_r) \mapsto M^{\mathbf{w}} \\
\textit{Bimould} \qquad\qquad & N^\bullet : (\mathcal{A} \times \mathcal{B})^* \to K \\
& \mathbf{w} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \cdots, \begin{pmatrix} u_r \\ v_r \end{pmatrix} \mapsto N^{\mathbf{w}}.
\end{aligned}$$

1.1 Examples

$Ze_*^\bullet$ is the bimould defined by

$$(Ze_*^\bullet, \mathbb{Q}/\mathbb{Z} \times \mathbb{N}^*, \mathbb{C}) := Ze_*^{\binom{\epsilon_1}{s_1}, \cdots, \binom{\epsilon_r}{s_r}} = \sum_{n_1 > n_2 > \cdots n_r > 0} \Pi n_k^{-s_k} e^{2\pi i \epsilon_k n_k}$$

with $s_1 \geq 2$. If we take $\epsilon_i = 0\ \forall i$ then we obtain the usual multiple zeta values. Sometimes people say that elements in the image of this mould are "colored multiple zeta values".

$Wa_*^\bullet$ is the mould defined by

$$(Wa_*^\bullet, \{e^{2\pi i k}; k \in \mathbb{Q}\} \cup \{0\}, \mathbb{C}) := Wa_*^{e^{2\pi i \epsilon_1} 0^{s_1 - 1} \cdots e^{2\pi i \epsilon_r} 0^{s_r - 1}} = Ze_*^{\substack{\epsilon_r \\ s_r}, \substack{\epsilon_{r-1} - \epsilon_r \\ s_{r-1}}, \cdots, \substack{\epsilon_1 - \epsilon_2 \\ s_1}}.$$

Hence we require that the first term be a root of unity and the last term be 0.

1.2 Operations on Moulds

Given two moulds (resp. bimoulds) $(M^\bullet, \mathcal{A}(\text{resp. } \times \mathcal{B}), K)$ and $(N^\bullet, \mathcal{A}(\text{resp. } \times \mathcal{B}), K)$ addition and multiplication are given by

$$M^\bullet + N^\bullet = C^\bullet: \; C^{\mathbf{w}} = M^{\mathbf{w}} + N^{\mathbf{w}}$$
$$M^\bullet \times N^\bullet = mu(M^\bullet, N^\bullet) = C^\bullet: \; C^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{w}^1\cdot\mathbf{w}^2} M^{\mathbf{w}^1} \cdot N^{\mathbf{w}^2}.$$

Swap

$$swap : (M^\bullet, \mathcal{A} \times \mathcal{B}, K) \to (M^\bullet, \mathcal{B} \times \mathcal{A}, K)$$
$$swap(M^\bullet)^{\binom{u_1}{v_1}, \dots, \binom{u_r}{v_r}} = M^{\left(\substack{v_r \\ u_1+\dots+u_r}, \substack{v_{r-1}-v_r \\ u_1+\dots+u_{r-1}}, \dots, \substack{v_1-v_2 \\ u_1}\right)}$$

Negation/Parity

$$nepar(M^\bullet)^{(w_1,\dots,w_r)} = (-1)^r M^{(-w_1,\dots,-w_r)}$$

Flexions
$\mathcal{A}$-semi-group, $\mathcal{B}$-abelian group
$\mathbf{w} = \dots\mathbf{w}^1 \cdot \mathbf{w}^2\dots$
$\mathbf{w}^1 = \begin{pmatrix} u_1 & \dots & u_r \\ v_1 & & v_r \end{pmatrix}, \mathbf{w}^2 = \begin{pmatrix} u_{r+1} & \dots & u_s \\ v_{r+1} & & v_s \end{pmatrix}$

$$\mathbf{w}^1\rceil := \begin{pmatrix} u_1 & \dots & u_{r-1} & \sum_{k=r}^{s} u_k \\ v_1 & & v_{r-1} & v_r \end{pmatrix} \qquad \lceil\mathbf{w^2} := \begin{pmatrix} \sum_{k=1}^{r+1} u_k & u_{r+2} & \dots & u_s \\ v_{r+1} & v_{r+2} & & v_s \end{pmatrix}$$
$$\mathbf{w}^1\rfloor := \begin{pmatrix} u_1 & \dots & u_r \\ v_1 - v_{r+1} & & v_r - v_{r+1} \end{pmatrix} \qquad \lfloor\mathbf{w}^2 := \begin{pmatrix} u_{r+1} & \dots & u_s \\ v_{r+1} - v_r & & v_s - v_r \end{pmatrix}$$

1.3 Symmetries

Definition 2. *A mould/bimould $A^\bullet$ is symmetral (resp. alternal) if*

$$\forall \mathbf{w}^1, \mathbf{w}^2, \sum_{\mathbf{w} \in sha(\mathbf{w}^1, \mathbf{w}^2)} A^{\mathbf{w}} = A^{\mathbf{w}^1} A^{\mathbf{w}^2} \ \textit{(resp.} \ = 0),$$

where $sha(\mathbf{w}^1, \mathbf{w}^2)$ denotes the shuffle product of sequences. We say such a mould is "as" (resp. "al"). $Wa_*^\bullet$ is as.

A mould/bimould $A^\bullet$ is symmetrel (resp. alternel) if

$$\forall \mathbf{w}^1, \mathbf{w}^2, \sum_{\mathbf{w} \in she(\mathbf{w}^1, \mathbf{w}^2)} A^{\mathbf{w}} = A^{\mathbf{w}^1} A^{\mathbf{w}^2} \ \textit{(resp.} \ = 0),$$

where $she(\mathbf{w}^1, \mathbf{w}^2)$ denotes the "contracting shuffle" or "stuffle" product of sequences, which is given by the recursion relation,

$\mathbf{w}^1 = (a_1, ..., a_r), \mathbf{w}^2 = (a_{r+1}, ..., a_{r+s})$
$she(\mathbf{w}^i, \emptyset) = \mathbf{w}^i$
$she(\mathbf{w}^1, \mathbf{w}^2) = a_1 \cdot she((a_2, ..., a_r), \mathbf{w}^2) \sqcup a_{r+1} \cdot she(\mathbf{w}^1, (a_{r+2}, ..., a_{r+s})) \sqcup (a_1 + a_{r+1}) \cdot she((a_2, ..., a_r), (a_{r+2}, ..., a_{r+s})).$

Such a mould is called es (resp. el). $Ze_*^\bullet$ is es.

1.4 More Examples

The following examples of moulds define two generating series for multiple zeta values, and provide a method of regularization of multiple zeta values.

Regularization

There exists a unique mould, $Ze^{\bullet}$, such that

- $Ze^{\bullet} = Ze^{\bullet}_{*}$ wherever $Ze^{\bullet}_{*}$ is defined,
- $Ze^{\bullet}$ is defined on all of $(\mathbb{Q}/\mathbb{Z} \times \mathbb{N}^{*})^{*}$
- $Ze^{\binom{0}{1}} = 0$,
- $Ze^{\bullet}$ is symmetrel

Likewise, there exists a unique mould, $Wa^{\bullet}$, such that

- $Wa^{\bullet} = Wa^{\bullet}_{*}$ wherever $Wa^{\bullet}_{*}$ is defined,
- $Wa^{\bullet}$ is defined on all $\{e^{2\pi ik}; k \in \mathbb{Q}\} \cup \{0\}$,
- $Wa^{(1)} = Wa^{(0)} = 0$,
- $Wa^{\bullet}$ is symmetral.

Generating Series

$$Zig^{\bullet} := Zig^{\binom{\epsilon_1}{v_1}, \cdots, \binom{\epsilon_r}{v_r}} = \sum_{s_i \geq 1} Ze^{\binom{\epsilon_1}{s_1}, \cdots, \binom{\epsilon_r}{s_r}} v_1^{s_1-1} \cdots v_r^{s_r-1}.$$

$$Zag^{\bullet} := Zag^{\binom{u_1}{\epsilon_1}, \cdots, \binom{u_r}{\epsilon_r}} = \sum_{s_i \geq 1} Wa^{\left(e^{2\pi i\epsilon_1} 0^{s_1-1}, \cdots, e^{2\pi i\epsilon_r} 0^{s_r-1}\right)} u_1^{s_1-1} (u_1+u_2)^{s_2-1} \cdots (u_1+u_2+\cdots+u_r)^{s_r-1}.$$

$Zag^{\bullet}$ is symmetral, whereas $Zig^{\bullet}$ satifies another symmetry, *symmetril*, closely related to symmetrel.

Conversion

$$(mono^\bullet, \mathbb{Q}/\mathbb{Z} \times \mathbb{N}^*, \mathbb{C}) := 1 + \sum_{k=1} mono^{\binom{0^k}{1^k}} t^k = exp\left(\sum_{k=2} (-1)^{k-1}\zeta(k)\frac{t^k}{k}\right)$$

$$:= 0 \text{ whenever } \begin{pmatrix} \epsilon_1 \\ s_1 \end{pmatrix}, \cdots, \begin{pmatrix} \epsilon_r \\ s_r \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(mini^\bullet, \mathbb{Q}[u_i] \times \mathbb{Q}/\mathbb{Z}, \mathbb{C}) := mini^{\left(\begin{smallmatrix} \epsilon_1 \\ v_1 \end{smallmatrix}, \dots, \begin{smallmatrix} \epsilon_r \\ v_r \end{smallmatrix}\right)} = mono^{\left(\begin{smallmatrix} \epsilon_1 \\ 1 \end{smallmatrix}, \dots, \begin{smallmatrix} \epsilon_r \\ 1 \end{smallmatrix}\right)}.$$

Proposition 3. $mini^\bullet \times swap(Zag^\bullet) = Zig^\bullet$

We say then that $Zag^\bullet$ is as/is, since it's symmetral and its swap is symmetril (up to multiplication by a commutative bimould).

2 Key Results

2.1 ARI/GARI

In order to keep simplicity for this talk, we take the following definition, which is more restricted than the usual general definition.

Definition 4. *Let $ARI_{al/il}$ be the Lie algebra with the following definition.*

- *As a vector space over $\mathbb{Q}$,*

$$(ARI_{al/il}, \mathbb{Q}[u_i] \times \mathbb{Q}/\mathbb{Z}, \mathbb{C}[[u_i]]) := \langle M^\bullet : M^\emptyset = 0, M \text{ is al}, swap(M^\bullet) \text{ is alternil}^{**} \rangle,$$

- *The Lie bracket is given by*

$$ari(M^\bullet, N^\bullet)^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{w}^1\mathbf{w}^2} A^{\mathbf{w}^1} B^{\mathbf{w}^2} - B^{\mathbf{w}^1} A^{\mathbf{w}^2} + \sum_{\mathbf{w}=\mathbf{w}^2\mathbf{w}^3\mathbf{w}^4} M^{\lfloor \mathbf{w}^3} N^{\mathbf{w}^2 \rceil \mathbf{w}^4} - N^{\lfloor \mathbf{w}^3} M^{\mathbf{w}^2 \rceil \mathbf{w}^4}$$
$$+ \sum_{\mathbf{w}=\mathbf{w}^1\mathbf{w}^2\mathbf{w}^3} M^{\mathbf{w}^1 \lceil \mathbf{w}^3} N^{\mathbf{w}^2 \rfloor} - N^{\mathbf{w}^1 \lceil \mathbf{w}^3} M^{\mathbf{w}^2 \rfloor}$$

*(** The alternil condition means "up to a multiplication by a commutative bimould".)*

Remarks

- The *ari* bracket is equal to the Lie-Poisson bracket $\{,\}$, up to a variable change, on the usual Lie algebra of multizeta values ($\mathfrak{dm}$). However, the *ari* bracket can be defined on a more general set of vector spaces, which is a tool Ecalle uses in his proofs.
- The flexions $(\lfloor, \rceil, \rfloor, \lceil)$ in the definition of the "*ari*" bracket correspond to the derivations, $D_f(x) = 0, D_f(y) = [y, f]$, which give the definition of the above mentioned Poisson bracket.

Definition 5. *By taking the ari-exponential of the Lie algebra, $ARI_{al/il}$, we obtain a Lie group, $GARI_{as/is}$ which has the following presentation.*

- *As a set,*

$$(GARI_{as/is}, \mathbb{Q}[u_i] \times \mathbb{Q}/\mathbb{Z}, \mathbb{C}[[u_i]]) := \{M^\bullet : M^\emptyset = 1, M \textit{ is as}, swap(M^\bullet) \textit{ is symmetril}^{**}\},$$

- *The group law is given by*

$$gari(A^\bullet, B^\bullet)^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{a^1 b^1 c^1 \cdots b^s c^s a^{s+1}}} A^{\lceil \mathbf{b^1} \rceil} \cdots A^{\lceil \mathbf{b^s} \rceil} B^{\mathbf{a^1} \rfloor} \cdots B^{\mathbf{a^{s+1}} \rfloor} (B^{-1})^{\lfloor \mathbf{c^1}} \cdots (B^{-1})^{\lfloor \mathbf{c^s}},$$

  where $\mathbf{s} \geq 0, \mathbf{b^i} \neq \emptyset$ *(*$\forall 1 \leq \mathbf{i} \leq \mathbf{s}$*),* $\mathbf{c^i} \cdot \mathbf{a^{i+1}} \neq \emptyset$ *(*$\forall 1 \leq \mathbf{i} \leq \mathbf{s-1}$*) and* (B^{-1}) *denotes the inverse for standard mould multiplication.*

- *$invgari(M^\bullet)$ is inverse of a mould $M^\bullet$ for the gari product,*

$$gari(invgari(A^\bullet), A^\bullet) = gari(A^\bullet, invgari(A)^\bullet) = \mathbf{1}^\bullet, \textit{ where } \mathbf{1}^\emptyset = 1, \mathbf{1}^{\mathbf{w}} = 0.$$

IMPORTANT FACT

The mould $Zag^\bullet$ is an element of the Lie group, GARI.

2.2 Canonical Decomposition into Irreductibles

Theorem 6. *The mould $Zag^\bullet$ may be decomposed into three factors,*

$$Zag^\bullet = gari(Zag^\bullet_I, Zag^\bullet_{II}, Zag^\bullet_{III})$$

such that:

- *The even/odd length components of $Zag^\bullet_{I,II}$ are even/odd functions of $\mathbf{w}$, while the even/odd legth components of $Zag^\bullet_{III}$ are odd/even functions of $\mathbf{w}$;*
- *Each component is decomposed as a series in a basis of $ARI_{al/il}$, which when evaluated at $\epsilon_i = 0$ are irreducible elements of the $\mathbb{Q}$ algebra of multiple zeta values, $\mathbb{Z}eta$;*
- *The irreducibles appearing as coefficients in the factors give us a factorization for the multiple zeta value algebra,*
$$\mathbb{Z}eta := \mathbb{Z}eta_I \otimes \mathbb{Z}eta_{II} \otimes \mathbb{Z}eta_{III}.$$

2.3 $Zag^{\bullet}_{III}$

- The factor $Zag^{\bullet}_{III}$ is the most simple to express explicitly,

$$gari(Zag^{\bullet}_{III}, Zag^{\bullet}_{III}) = gari(nepar(invgari(Zag^{\bullet})), Zag^{\bullet}).$$

  By linearizing, you can see that indeed this provides an odd/even function on components of even/odd length.

- The length 1 component is given by

$$Zag_{III}^{\binom{u_1}{0}} = \sum_{s\geq 3,\ odd} \zeta(s) u_1^{s-1}.$$

- The associated factor in the multiple zeta value algebra, $\mathbb{Z}eta_{III}$, is generated by irreducibles of odd depth, i.e. linear combinations of $\zeta(s_1, ..., s_r)$ where r is odd. The mould of such irreducibles is denoted by $Irr^{\bullet}_{III}$.

- We get an explicit expression for the set of irreducible multiple zeta values in factor $Zag^{\bullet}_{III}$ in terms of a mould $loma^{\bullet}$, which is a generating mould which (vaguely speaking) forms a basis of rational polynomials for $ARI_{al/il}$ (the explicit construction is out of the scope of this talk). We have

$$Zag^{\bullet}_{III} := \sum_{s_i\geq 1,\ r\ odd} Irr_{III}^{s_1,...,s_r} loma^{\bullet}_{s_1} \cdots loma^{\bullet}_{s_r},$$

  where $loma^{\bullet}_{s_i}$ is the restriction of $loma^{\bullet}$ to the weight s_i terms.

2.4 $Zag_I^\bullet$

· $Zag_I^{\mathbf{w}} \in \mathbb{Q}[[u_i, \pi^2]]$ (when $\epsilon_1 = 0$), which in the language of zetas, means that the corresponding factor in the $\mathbb{Z}eta$ algebra, $\mathbb{Z}eta_I$, is generated by $6\zeta(2) = \pi^2$.

· The explicit factorization of $Zag_I^\bullet$ from $Zag^\bullet$ is a very costly analytic contruction, whose difficulty comes from getting rid of unwanted singularities. The formula is the following:

$$Zag_I^\bullet = gari(tal^\bullet, invgari(pal^\bullet), expari(roma^\bullet)),$$

and again the definition of *pal* and *roma* go out of the scope, since they are very long.

2.5 $Zag_{II}^\bullet$

· $Zag_{II}^\bullet$ is explicitly calculated by factoring $Zag^\bullet$ by $Zag_I^\bullet$ and $Zag_{III}^\bullet$.

· $Zag_{II}^\bullet$ may be factored as a generating series for the irreducible multiple zeta values of even depth in the same manner as $Zag_{III}^\bullet$, providing a set of "canonical" irreducibles for $\mathbb{Z}eta_{II}$,

$$Zag_{II}^\bullet := \sum_{s_i \geq 1,\ r\ even} Irr_{II}^{s_1,\dots,s_r} loma_{s_1}^\bullet \cdots loma_{s_r}^\bullet.$$