Broadhurst's Dimensions

May 10, 2010

Abstract

Broadhurst conjectured a dimension for the graded pieces of the $\mathbb{Q}$ vector space of multiple zeta values based on large scale experimental computations. In this talk, I will give a new meaning to these dimensions by showing that his generating series is in fact the generating series for a vector space which is conjecturally isomorphic to the $\mathbb{Q}$ v.s. of multizeta values.

1 Background

Definition 1. MZV

$$\zeta(\underline{k}) = \zeta(k_1, ..., k_d) := \sum_{n_1 > n_2 > \cdots > n_d > 0} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}, \ k_i \in \mathbb{Z}^+ \ k_1 \geq 2$$
$$wt(\underline{k}) = \sum k_i$$
$$dp(\underline{k}) = d.$$

Definition 2.

$$\mathcal{Z}_n := \langle \zeta(\underline{k}) : \ wt(\underline{k}) = n \rangle_{\mathbb{Q}}$$
$$\mathcal{Z}_n^{\leq d} := \langle \zeta(\underline{k}) : dp(\underline{k}) \leq d \rangle_{\mathbb{Q}} \subseteq \mathcal{Z}_n$$

Property 1.

$$\mathcal{Z}_l^{\leq b} \cdot \mathcal{Z}_m^{\leq c} \subseteq \mathcal{Z}_{l+m}^{\leq b+c}$$

Definition 3. *The vector space of* ***even period polynomials*** *of weight n, $\mathcal{P}_n \subset \mathbb{Q}[X]$, is the space of even polynomials P satisfying*

$$degree(P) \leq n-4$$
$$P(X) + X^{n-2} P(\frac{-1}{X}) = 0,$$
$$P(X) + X^{n-2} P(1 - \frac{1}{X}) + (X-1)^{n-2} P(\frac{1}{1-X}) = 0.$$

Don says that this is the wrong definition of the period polynomial structure.

Let $(S|w)$ to denote the coefficient of the word or monomial w in a polynomial (or power series) S.

Lemma 1. *From the first relation, one may easily deduce that a weight n period polynomial satisfies*

$$(P|X^{2i}) = -(P|X^{n-2-2i}).$$

2 The Broadhurst Kreimer conjecture

Broadhurst and Kreimer conjectured the following formula for Poincaré series for the algebra of multiple zeta values.

Definition 4. $\mathcal{D}(X,Y)$

$$\begin{aligned} \mathcal{O}(X) &:= \frac{X^3}{1-X^2} \\ \mathcal{S}(X) &:= \frac{X^{12}}{(1-X^4)(1-X^6)} = \sum_{n\geq 12,\ even} dim(\mathcal{P}_n)X^n \\ \mathcal{E}(X) &:= \frac{X^2}{1-X^2} \\ \mathcal{D}^0(X,Y) &:= \frac{1}{1-\mathcal{O}Y+\mathcal{S}Y^2-\mathcal{S}Y^4} \\ \mathcal{D}(X,Y) &:= (1+\mathcal{E}Y)\,\mathcal{D}^0(X,Y). \end{aligned}$$

Conjecture 2. *(BK)*

$$\mathcal{D}(X,Y) = \sum_{n,d} dim(\mathcal{Z}_n^{\leq d}/\mathcal{Z}_n^{\leq d-1})X^nY^d.$$

3 Coefficients in the B-K series and multiple zeta values

Remarks 1.

1. The coefficient, $\mathcal{E}$, corresponds to following the well-known conjecture.

   Conjecture 3.
   $$\mathcal{Z}_\bullet := \mathcal{Z}^0 \otimes \mathbb{Q}[\pi^2].$$

   Note that $\pi^2 = k\zeta(2)$ where $k \in \mathbb{Q}$ and that $\mathcal{E}$ is the Poincaré series for the for $\mathbb{Q}[X^2] \simeq \mathbb{Q}[\pi^2]$.

2. The coefficient, $\mathcal{O}$, corresponds to another well-known conjecture.

   Conjecture 4. *(Zagier)*
   $$(\mathcal{Z}^0)^\vee \simeq \mathbb{Q}\langle f_i; i\geq 3, odd\rangle.$$

   Note that $\mathcal{O}$ is the Poincaré series for $\mathbb{Q}\langle f_i\rangle$ and that if we ignore the depth filtration, i.e. set $Y=1$, then $\mathcal{D}$ is the Poincaré series for the conjectured multizeta value algebra (from conjectures 3 and 4) graded by weight.

3. To explain the appearance of $\mathcal{S}$, we introduce the following algebras.

Definition 5. *Let*
$$\mathcal{NZ}_\bullet = \mathcal{Z}_\bullet/\langle \mathcal{Z}^2_{>0} \oplus \mathbb{Q}\cdot\pi^2 \oplus \mathbb{Q}\rangle,$$
which we know by the work of Ecalle and Racinet to be a Lie coalgebra.

We define $\mathcal{FNZ}$ to be the Lie coalgebra satisfying all known relations between MZVs, and we have that
$$\mathcal{FNZ} \twoheadrightarrow \mathcal{NZ}.$$

The dual of $\mathcal{FNZ}$ is a Lie algebra, denoted $\mathfrak{ds}$. The Lie algebra, $\mathfrak{ds}$ can be seen as a subvector space of $\mathbb{Q}\langle\langle x,y\rangle\rangle$ and it is graded by weight, which is the length of the words. In $\mathbb{Q}\langle\langle x,y\rangle\rangle$, the depth of a monomial is the number of y in it. The elements in $\mathfrak{ds}$ are $\mathbb{Q}$ linear combinations of words in varying depth. We say that an element $f \in \mathfrak{ds}$ has depth d if $d = min(\text{depth}(w); w$ has non-zero coefficient in $f)$.

The Lie algebra $\mathfrak{ds}$, is not graded by depth but it possesses a descending depth filtration:

$$\mathfrak{ds}^1 \supseteq \mathfrak{ds}^2 \supseteq \mathfrak{ds}^3 \supseteq \cdots .$$

We know at least that $\mathfrak{ds}$ possesses at least one element of depth 1 in each odd weight $n \geq 3$. Let $\{s_n = x^{n-1}y+$ terms of higher depth; $n \geq 3$, odd$\}$ be a set of depth one generators of $\mathfrak{ds}$.

In [S], Schneps identifies a Zagier period polynomial with a Poisson bracket of depth 1 elements of $\mathfrak{ds}$ by

$$\pi_n : \mathcal{P}_n \to \mathcal{F}$$
$$\sum_{i+j=n} a_{ij}(X^{i-1} - X^{j-1}) \mapsto \sum_{i+j=n} a_{ij}\{s_i, s_j\}.$$

Theorem 5. *(Schneps) For all n, Im$(\pi_n) \subset \mathfrak{ds}^4$. More precisely, the coefficient of any monomial of depth 3 or less in any element in the image of π_n is 0.*

(This result was known to Eichler and Shimura in a different context).

4 Construction of an algebra satisfying the B-K conjecture

Definition 6. *Let $\mathcal{F}$ be the non-commutative polynomial algebra over $\mathbb{Q}$ generated by elements of odd weight:*

$$\mathcal{F} = \mathbb{Q}\langle\langle s_3, s_5, s_7, ...\rangle\rangle.$$

*The **weight** of a word in $\mathcal{F}$ is the sum of the indices. We therefore have a grading of $\mathcal{F}$ by the weight:*

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n.$$

Don says that we should define this as a tensor algebra, not a concatenation algebra.

We define a depth filtration on $\mathcal{F}$ as follows. First, let $\mathcal{F}_{\mathcal{P}}$ be the image of the linear map

$$\bigoplus_{n=12, n \ even} \mathcal{P}_n \to \mathcal{F}$$
$$\sum_{i+j=n} a_{ij}(X^{i-1} - X^{j-1}) \mapsto \sum_{i+j=n} a_{ij}[f_i, f_j].$$

We call the image of a period polynomial by this map a "Lie period polynomial".

Definition 7. ***Special depth filtration***

$$\begin{aligned}
V^1 &:= \langle s_i \rangle_{\mathbb{Q}} \subset \mathcal{F} \\
V^2 &:= \langle s_i s_j \rangle_{\mathbb{Q}} \\
V^3 &:= \langle s_i s_j s_k \rangle_{\mathbb{Q}} \\
V^4 &:= \langle s_i s_j s_k s_l \cup \mathcal{F}_{\mathcal{P}} \rangle_{\mathbb{Q}} \\
\forall d \geq 5,\ V^d &:= \langle ST : S \in V^i, T \in V^j, i + j = d \rangle_{\mathbb{Q}} \\
\mathcal{F}^d &:= \langle \cup_{i \geq d} V^i \rangle_{\mathbb{Q}}.
\end{aligned}$$

Don says that we should not use unions of spaces. We cannot generate a space with a space. Nobody does that.

Since $\mathcal{F}$ is graded by weight, this definition gives us a filtration on the weight graded pieces:

$$\mathcal{F}_n = \mathcal{F}_n^1 \supseteq \mathcal{F}_n^2 \supseteq \cdots \supseteq \mathcal{F}_n^{\lfloor n/3 \rfloor}.$$

Notice that the degree and the depth of the polynomials in $\mathcal{F}$ correspond up to a factor of a (or many) period polynomial(s). This definition is inspired by Broadhurst and Kreimer's series, because the denominator is telling us that we should take $\mathcal{F}_{\mathcal{P}}$ out of depth 2 and put it in depth 4.

Theorem 6. *(CGS) If conjecture 7 holds, the dimensions of the graded parts associated to the depth filtration of $\mathcal{F}$ are given by the Broadhurst-Kreimer series, $\mathcal{D}^0$:*

$$dim(\mathcal{F}_n^d/\mathcal{F}_n^{d+1}) = \big(\mathcal{D}^0|x^n y^d\big).$$

Conjecture 7.

$$\langle \mathcal{F}_\mathcal{P} \cdot V^1 \rangle_\mathbb{Q} \cap \langle V^1 \cdot \mathcal{F}_\mathcal{P} \rangle_\mathbb{Q} = 0.$$

More precisely, we want to show that for any given odd n, and any set of period polymomials, each of weight $j < n$, $\{P_j\}$, then the following polynomials are always linearly independent,

$$\{X^{n-j-1}P_j(Y) + Y^{n-j-1}P_j(X) \ : \ j < n\}.$$

Proof. From the expression for $\mathcal{D}^0$, we have that

$$(\mathcal{D}^0|y^d) = \sum_{4a+2b+c=d} \binom{a+b+c}{c}\binom{a+b}{b}\mathcal{S}^{a+b}\mathcal{O}^c,$$

so that

$$(\mathcal{D}^0|x^n y^d) = \Big(\sum_{4a+2b+c=d} (-1)^b \binom{a+b+c}{c}\binom{a+b}{b}\mathcal{S}^{a+b}\mathcal{O}^c|x^n\Big).$$

Now, all we have to do is show that this is the same as the dimension of $\mathcal{F}_n^d/\mathcal{F}_n^{d+1}$.

To show that this is indeed the generating series, we first identify subspaces of $\mathcal{F}_{n,k}^d \subseteq \mathcal{F}_n^d$, which are generated by polynomials of degree k. This provides a bi-grading on $\mathcal{F}^d$ by the degree and the weight.

Remarks 2. *When $d < k$, then $\mathcal{F}_{n,k}^d = \mathcal{F}_{n,k}^{d+1}$.*

This is because each V^i is generated by monomials of degree $\leq i$. So we have:

$$\mathcal{F}_{n.k}^d \subseteq \langle \cup_{i\geq k} V_n^i \rangle_\mathbb{Q} \subseteq \langle \cup_{i\geq d+1} V_n^i \rangle_\mathbb{Q} = \mathcal{F}_{n,k}^{d+1} \subseteq \mathcal{F}_{n,k}^d.$$

Remarks 3. *When $d-k$ is odd, $k < d$, then $\mathcal{F}_{n,k}^d = \mathcal{F}_{n,k}^{d+1}$.*

By definition, the degree and the depth only vary by factors of $\mathcal{F}_\mathcal{P}$ which is of degree 2 and of maximal depth 4. Therefore the appearance of a factor of a Lie period polynomial in an element of $\mathcal{F}$ increases the depth of that element by 2.

From remarks 1 and 2, we have the expression,

$$\mathcal{F}_n^d/\mathcal{F}_n^{d+1} = \mathcal{F}_{n,d}^d/\mathcal{F}_{n,d}^{d+1} \oplus \mathcal{F}_{n,d-2}^d/\mathcal{F}_{n,d-2}^{d+1} \oplus \cdots \oplus \mathcal{F}_{n,d-2\lfloor d/4\rfloor}^d/\mathcal{F}_{n,d-2\lfloor d/4\rfloor}^{d+1}. \tag{1}$$

The key to the proof is to write (for a positive):

$$\mathcal{F}_{n,d-2a}^d \simeq \sum_{(j_0,\ldots,j_a)} \big((\mathcal{F}^1)^{j_0} \cdot \mathcal{F}_\mathcal{P} \cdot (\mathcal{F}^1)^{j_1} \cdot \cdots \cdot \mathcal{F}_\mathcal{P} \cdot (\mathcal{F}^1)^{j_a}\big)_n = \sum_{i=1}^N (E^{J_i})_n,$$
$$\mathcal{J} := \{(j_0,...,j_a); \Sigma j_i = d-4a, \ j_i \geq 0\} = \{J_1,...,J_N\}. \tag{2}$$

This is an isomorphism because this vector space contains all elements with degree $d-2a$ and special depth d, since there are a factors of $\mathcal{F}_\mathcal{P}$ in there. In order to count the dimension of $\mathcal{F}_{n,d-2a}^d$, then we need to determine the dimension of the intersection of the vector spaces in the sum (2).

The Möbius inversion formula for the dimension of vector spaces is analogous to the inclusion/exclusion principle for sets:

$$dim(\mathcal{F}_{n,d-2a}^d) = \left|\Big(\sum_{i=0}^N E^{J_i}\Big)_n\right| = \sum_{T\subseteq[1,N],|T|\geq 1} (-1)^{|T|-1}\left|\Big(\bigcap_{i\in T} E^{J_i}\Big)_n\right|. \tag{3}$$

We need to establish the following lemmas.

Lemma 8.

$$(\mathcal{F}^1)^2 \cdot \mathcal{F}_{\mathcal{P}} \cap \mathcal{F}_{\mathcal{P}} \cdot (\mathcal{F}^1)^2 = \mathcal{F}_{\mathcal{P}} \cdot \mathcal{F}_{\mathcal{P}}.$$

This is true because $\mathcal{F}_{\mathcal{P}} \subset (\mathcal{F}^1)^2$. It implies that

$$|\cap_{i\in T} E^{J_i}|_n = |\mathcal{F}_{\mathcal{P}}^{a+b}(\mathcal{F}^1)^c|_n.$$

Conjecture 7 tells us that

$$E^{1,\cdots} \bigcap E^{0,\cdots} = (\mathcal{F}^1 \cdot \mathcal{F}_{\mathcal{P}} \otimes \cdots) \bigcap (\mathcal{F}_{\mathcal{P}} \otimes \cdots) = 0$$

because the first factor of $\mathcal{F}^1$ on the left hand side "crosses" the first factor of $\mathcal{F}_{\mathcal{P}}$ on the right hand side. In the general case, the dimension count reduces to counting non-crossing partitions of $d-4a$.

We proved that the terms on the right hand side of equation (3) regroup to give the dimension,

$$\sum_{2b+c=d-4a} (-1)^b \binom{a+b+c}{c} \binom{a+b-1}{b} \left| \left(\mathcal{F}_{\mathcal{P}}^{a+b} \cdot (\mathcal{F}^1)^c\right)_n \right|.$$

Since the generating series for $\mathcal{F}_{\mathcal{P}}$ is $\mathcal{S}$ and the generating series for $\mathcal{F}^1$ is $\mathcal{O}$, by equation (3) we have

$$dim(\mathcal{F}^d_{n,d-2a}) = \sum_{2b+c=d-4a} (-1)^b \binom{a+b+c}{c} \binom{a+b-1}{b} (\mathcal{S}^{a+b}\mathcal{O}^c | x^n y^d).$$

Then

$$dim(\mathcal{F}^d_{n,d-2a}/\mathcal{F}^{d+1}_{n,d-2a}) = \sum_{2b+c=d-4a} (-1)^b \binom{a+b+c}{c} \binom{a+b}{b} (\mathcal{S}^{a+b}\mathcal{O}^c | x^n y^d).$$

By taking the sum now over a from equation (1), we have that

$$dim(\mathcal{F}^d_n/\mathcal{F}^{d+1}_n) = \sum_{2b+c+4a=d} (-1)^b \binom{a+b+c}{c} \binom{a+b}{b} (\mathcal{S}^{a+b}\mathcal{O}^c | x^n y^d).$$

□

a list of positive integers:

References

[IKZ] K. Ihara, M. Kaneko, D. Zagier, *Derivation and double shuffle relations for muliple zeta values*, Compositio Math. 142 (2006), pp. 307-338

[R] G. Racinet, *Séries génératrices non-commutatives de polyzêtas et associateurs de Drinfel'd*, Thèse de doctorat, Université de Picardie-Jules Verne (2000)

[S] L. Schneps, *On the Poisson bracket on the free Lie algebra in two generators*, J. Lie Theory, Vol. 16 (2006), pp. 19-37

[Z] D. Zagier, *Quelques conséquences surprenantes de la cohomologie de* $\mathrm{SL}_2(\mathcal{Z})$, Leçons de Mathématiques d'Aujourd'hui, E. Charpentier and N. Nikolski, eds., Cassini, Paris (2000), pp. 99-123