Axiomatic treatment of variable binding and substitution

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Function symbols: $\boldsymbol{\lambda}$ (binary), a (binary), $\boldsymbol{\sigma}$ (ternary);

Relation symbols: V (unary), FV (binary), red;

Object variables: $p, q, r, s, t, u, v, w, x, y, z, ...$

Abbreviations.

$t \in \mathrm{V} := \mathrm{V}(t)$, $x \in \mathrm{FV}(t) := \mathrm{FV}(x, t)$, $t \in \mathsf{Abst} := \exists x \exists r(\mathrm{V}(x) \wedge t = \boldsymbol{\lambda} xr)$.

Terms are written in the form $f\mathbf{t}_1...\mathbf{t}_n$ or $f(\mathbf{t}_1, ..., \mathbf{t}_n)$.

Axioms

(A1) $\mathsf{a}\, rs \notin \mathrm{V} \wedge \boldsymbol{\lambda} xr \notin \mathrm{V} \wedge \mathsf{a}\, rs \neq \boldsymbol{\lambda} xr \wedge (\mathsf{a}\, rs = \mathsf{a}\, r's' \to r = r' \wedge s = s') \wedge (x \in \mathrm{V} \wedge \boldsymbol{\lambda} xr = \boldsymbol{\lambda} xr' \to r = r')$.

(A2.1) $t \in \mathrm{V} \to (y \in \mathrm{FV}(t) \leftrightarrow y = t)$.

(A2.2) $y \in \mathrm{FV}(\mathsf{a}\, rs) \leftrightarrow y \in \mathrm{FV}(r) \vee y \in \mathrm{FV}(s)$.

(A2.3) $x \in \mathrm{V} \to (y \in \mathrm{FV}(\boldsymbol{\lambda} xr) \leftrightarrow y \neq x \wedge y \in \mathrm{FV}(r))$.

(A3) Induction:

$$\forall x{\in}\mathrm{V}\mathcal{F}(x) \wedge \forall r \forall s[\mathcal{F}(r) \wedge \mathcal{F}(s) \to \mathcal{F}(\mathsf{a}\, rs)] \wedge \forall t \in \mathsf{Abst}[\forall r(\exists x{\in}V(t = \boldsymbol{\lambda} xr) \to \mathcal{F}(r)) \to \mathcal{F}(t)] \;\to\; \forall t \mathcal{F}(t).$$

(A4.1) $t \in \mathrm{V} \wedge x = t \to \boldsymbol{\sigma}(t, x, q) = q$.

(A4.2) $t \in \mathrm{V} \wedge x \neq t \to \boldsymbol{\sigma}(t, x, q) = t$.

(A4.3) $\boldsymbol{\sigma}(\mathsf{a}\, rs, x, q) = \mathsf{a}\, \boldsymbol{\sigma}(r, x, q)\boldsymbol{\sigma}(s, x, q)$.

(A4.4) $x, y \in \mathrm{V} \wedge y \neq x \wedge y \notin \mathrm{FV}(q) \to \boldsymbol{\sigma}(\boldsymbol{\lambda} yr, x, q) = \boldsymbol{\lambda} y\, \boldsymbol{\sigma}(r, x, q)$.

(A4.5) $x \in \mathrm{V} \to \boldsymbol{\sigma}(\boldsymbol{\lambda} xr, x, t) = \boldsymbol{\lambda} xr$.

(A5) $x, y \in \mathrm{V} \to (\boldsymbol{\lambda} xr = \boldsymbol{\lambda} yr' \;\leftrightarrow\; y \notin \mathrm{FV}(\boldsymbol{\lambda} xr) \wedge r' = \boldsymbol{\sigma}(r, x, y))$.

(A6) $\forall t \exists y \in \mathrm{V}(y \notin \mathrm{FV}(t))$.

Convention. Informally, the variables x, y, z are considered to range over V. This means that in the following we tacitly assume $x \in \mathrm{V}, y \in \mathrm{V}, z \in \mathrm{V}$, and read $\forall x\mathcal{F}(x)$, $\exists x\mathcal{F}(x)$ as abbreviations for $\forall x(\mathrm{V}(x) \to \mathcal{F}(x))$, $\exists x(\mathrm{V}(x) \wedge \mathcal{F}(x))$, resp.

Lemma 0.

$t \in \mathsf{Abst} \to \exists x \exists r(t = \boldsymbol{\lambda} xr \wedge x \notin \mathrm{FV}(s_1) \wedge ... \wedge x \notin \mathrm{FV}(s_n))$.

Proof :

Let $s := \mathsf{a}\, t\, \mathsf{a}\, s_1\, \mathsf{a}\, s_2 \ldots \mathsf{a}\, s_{n-1}s_n$.

1. $\exists x(x \notin \mathrm{FV}(s))$ [(A6)],
2. $x \notin \mathrm{FV}(s) \to x \notin \mathrm{FV}(t) \wedge x \notin \mathrm{FV}(s_1) \wedge \ldots \wedge x \notin \mathrm{FV}(s_n)$ [(A2.2)]
3. $t \in \mathsf{Abst} \to \exists y \exists r'(t = \boldsymbol{\lambda} yr')$ [Def]
4. $x \notin \mathrm{FV}(t) \wedge t = \boldsymbol{\lambda} yr' \;\to\; t = \boldsymbol{\lambda} yr' = \boldsymbol{\lambda} x\sigma(r', y, x)$ [(A5)].

Lemma 1.

(a) $x \in \mathrm{FV}(t) \to (z \in \mathrm{FV}(\boldsymbol{\sigma}(t,x,s)) \leftrightarrow (x \neq z \wedge z \in \mathrm{FV}(t)) \vee z \in \mathrm{FV}(s))$.

(b) $x \notin \mathrm{FV}(t) \to \boldsymbol{\sigma}(t,x,s) = t$.

(c) $\boldsymbol{\sigma}(t,x,x) = t$.

(d) $t \in \mathsf{Abst} \ \to \ \sigma(t,x,s) \in \mathsf{Abst}$.

Proof :

(a) Induction: Assume $t = \boldsymbol{\lambda} yr \wedge y \notin \mathrm{FV}(s) \wedge y \neq x$.

1. $x \in \mathrm{FV}(r)$ [from $x \in \mathrm{FV}(t)$ by (A2.3)]

2. $z \in \mathrm{FV}(\boldsymbol{\sigma}(t,x,s)) \overset{(A4.4)}{\leftrightarrow}$

$z \in \mathrm{FV}(\boldsymbol{\lambda} y \boldsymbol{\sigma}(r,x,s)) \overset{(A2.3)}{\leftrightarrow}$

$y \neq z \wedge z \in \mathrm{FV}(\boldsymbol{\sigma}(r,x,s)) \overset{\mathrm{IH},1.}{\leftrightarrow}$

$y \neq z \wedge [(x \neq z \wedge z \in \mathrm{FV}(r)) \vee z \in \mathrm{FV}(s)] \overset{y \notin \mathrm{FV}(s)}{\leftrightarrow}$

$(x \neq z \wedge y \neq z \wedge z \in \mathrm{FV}(r)) \vee z \in \mathrm{FV}(s) \overset{(A2.3)}{\leftrightarrow}$

$(x \neq z \wedge z \in \mathrm{FV}(t)) \vee z \in \mathrm{FV}(s)$.

(b) Induction: Assume $t = \boldsymbol{\lambda} yr \wedge y \neq x \wedge y \notin \mathrm{FV}(s)$.

$x \notin \mathrm{FV}(\boldsymbol{\lambda} yr) \wedge y \neq x \to x \notin \mathrm{FV}(r) \overset{\mathrm{IH}}{\to} \boldsymbol{\sigma}(r,x,s) = r \overset{(A4.4)}{\to} \boldsymbol{\sigma}(t,x,s) = \boldsymbol{\lambda} y \boldsymbol{\sigma}(r,x,s) = \boldsymbol{\lambda} yr = t$.

(c) Induction: Assume $t = \boldsymbol{\lambda} yr \wedge y \neq x$. Then also $y \notin \mathrm{FV}(x)$ by (A2.1).

$\sigma(t,x,x) = \sigma(\boldsymbol{\lambda} yr,x,x) \overset{(A4.4)}{=} \boldsymbol{\lambda} y \sigma(r,x,x) \overset{\mathrm{IH}}{=} \boldsymbol{\lambda} yr = t$.

(d) $t \in \mathsf{Abst} \overset{\mathrm{L.0}}{\to} \exists y, r(t = \boldsymbol{\lambda} yr \wedge x \neq y \wedge y \notin \mathrm{FV}(s)) \overset{(A4.4)}{\to} \exists y, r(\sigma(t,x,s) = \boldsymbol{\lambda} y \sigma(r,x,s))$.

Lemma 2.

(a) $z \neq x \wedge (x \notin \mathrm{FV}(t) \vee z \notin \mathrm{FV}(q)) \ \to \ \boldsymbol{\sigma}(\boldsymbol{\sigma}(t,z,s),x,q) = \boldsymbol{\sigma}(\boldsymbol{\sigma}(t,x,q),z,\boldsymbol{\sigma}(s,x,q))$.

(b) $x \notin \mathrm{FV}(t) \ \to \ \boldsymbol{\sigma}(\boldsymbol{\sigma}(t,z,x),x,q) = \boldsymbol{\sigma}(t,z,q)$.

(c) $\boldsymbol{\lambda} xr = \boldsymbol{\lambda} yr' \ \to \ \boldsymbol{\sigma}(r,x,s) = \boldsymbol{\sigma}(r',y,s)$.

Proof :

(a) Induction:

1. $t \in \mathrm{V}$:

1.1. $t = z$: $LHS \overset{(A4.1)}{=} \boldsymbol{\sigma}(s,x,q) \overset{(A4.1)}{=} \boldsymbol{\sigma}(z,z,\boldsymbol{\sigma}(s,x,q)) \overset{(A4.2)}{=} RHS$.

1.2. $t = x$: Then $z \notin \mathrm{FV}(q)$ and thus $LHS \overset{(A4.2)}{=} \boldsymbol{\sigma}(t,x,q) \overset{(A4.1)}{=} q \overset{\mathrm{L.1b}}{=} \boldsymbol{\sigma}(q,z,\boldsymbol{\sigma}(s,x,q)) \overset{(A4.1)}{=} RHS$.

1.3. $t \neq z \wedge t \neq x$: $LHS \overset{(A4.2)}{=} \boldsymbol{\sigma}(t,x,q) \overset{(A4.2)}{=} t \overset{(A4.2)}{=} \boldsymbol{\sigma}(t,z,\boldsymbol{\sigma}(s,x,q)) \overset{(A4.2)}{=} RHS$.

2. $t \in \mathsf{Abst}$: Assume $t = \boldsymbol{\lambda} yr \wedge y \neq x \wedge y \neq z \wedge y \notin \mathrm{FV}(s) \wedge y \notin \mathrm{FV}(q) \wedge y \notin \mathrm{FV}(\boldsymbol{\sigma}(s,x,q))$.

By (A2.3) from the premise and $y \neq x$ we obtain $x \notin \mathrm{FV}(r) \vee z \notin \mathrm{FV}(q)$.

$LHS \overset{(A4.4)}{=} \boldsymbol{\sigma}(\boldsymbol{\lambda} y \boldsymbol{\sigma}(r,z,s),x,q) \overset{(A4.4)}{=} \boldsymbol{\lambda} y \boldsymbol{\sigma}(\boldsymbol{\sigma}(r,z,s),x,q) \overset{\mathrm{IH}}{=} \boldsymbol{\lambda} y \boldsymbol{\sigma}(\boldsymbol{\sigma}(r,x,q),z,\boldsymbol{\sigma}(s,x,q)) \overset{(A4.4)}{=}$

$= \boldsymbol{\sigma}(\boldsymbol{\lambda} y \boldsymbol{\sigma}(r,x,q),z,\boldsymbol{\sigma}(s,x,q)) \overset{(A4.4)}{=} RHS$.

(b) 1. $x = z$: L.1c.

2. $x \neq z$: $\boldsymbol{\sigma}(\boldsymbol{\sigma}(t,z,x),x,q) \overset{(a)}{=} \boldsymbol{\sigma}(\boldsymbol{\sigma}(t,x,q),z,\boldsymbol{\sigma}(x,x,q)) \overset{\mathrm{L.1b},(A4.1)}{=} \boldsymbol{\sigma}(t,z,q)$.

(c) 1. $x = y$: $\boldsymbol{\lambda} xr = \boldsymbol{\lambda} xr' \overset{(A1)}{\to} r = r' \ \to \ \boldsymbol{\sigma}(r,x,s) = \boldsymbol{\sigma}(r',x,s)$.

2. $x \neq y$: $\boldsymbol{\lambda} xr = \boldsymbol{\lambda} yr' \overset{(A5)}{\to} y \notin \mathrm{FV}(r) \wedge r' = \boldsymbol{\sigma}(r,x,y) \ \to \ \boldsymbol{\sigma}(r',y,s) = \boldsymbol{\sigma}(\boldsymbol{\sigma}(r,x,y),y,s) \overset{(b)}{=} \boldsymbol{\sigma}(r,x,s)$.

According to Lemma 2c in an *definitional extension* of the present axiom system we can introduce a new function symbol β together with the *axiom*

(A7) $x \in \mathrm{V} \to \beta(\boldsymbol{\lambda} xr, s) = \boldsymbol{\sigma}(r, x, s)$.

Lemma 3.

(a) $t = \boldsymbol{\lambda} xr \;\leftrightarrow\; x \notin \mathrm{FV}(t) \wedge t \in \mathsf{Abst} \wedge r = \beta(t, x)$.

(b) $t \in \mathsf{Abst} \wedge z \in \mathrm{FV}(\beta(t, s)) \;\to\; z \in \mathrm{FV}(t) \vee z \in \mathrm{FV}(s)$.

Proof :

(a) "$\to$": $t = \boldsymbol{\lambda} xr \to \beta(t, x) \overset{(A7)}{=} \boldsymbol{\sigma}(r, x, x) \overset{\text{L.1c}}{=} r$.

"$\leftarrow$": $x \notin \mathrm{FV}(t) \wedge t \in \mathsf{Abst} \wedge r = \beta(t, x) \;\to\; \exists y, r'(x \notin \mathrm{FV}(t) \wedge t = \boldsymbol{\lambda} yr' \wedge r \overset{(A7)}{=} \boldsymbol{\sigma}(r', y, x)) \overset{(A5)}{\to} t = \boldsymbol{\lambda} xr$.

(b) $z{\in}\mathrm{FV}(\beta(\boldsymbol{\lambda} xr, s)) \to z{\in}\mathrm{FV}(\boldsymbol{\sigma}(r, x, s)) \overset{\text{L.1a,b}}{\to} (x \neq z \wedge z{\in}\mathrm{FV}(r)) \vee z{\in}\mathrm{FV}(s) \overset{(A2.3)}{\to} z{\in}\mathrm{FV}(\boldsymbol{\lambda} xr) \vee z{\in}\mathrm{FV}(s)$.

Lemma 4.

$t \in \mathsf{Abst} \to \boldsymbol{\sigma}(\beta(t, s), z, q) = \beta(\boldsymbol{\sigma}(t, z, q), \boldsymbol{\sigma}(s, z, q))$.

Proof :

By Lemma 0 we can assume $t = \boldsymbol{\lambda} xr \wedge z \neq x \wedge x \notin \mathrm{FV}(q)$. Now $\boldsymbol{\sigma}(\beta(t, s), z, q) \overset{(A7)}{=} \boldsymbol{\sigma}(\boldsymbol{\sigma}(r, x, s), z, q) \overset{\text{L.2a}}{=}$
$= \boldsymbol{\sigma}(\boldsymbol{\sigma}(r, z, q), x, \boldsymbol{\sigma}(s, z, q)) \overset{(A7)}{=} \beta(\boldsymbol{\lambda} x \boldsymbol{\sigma}(r, z, q), \boldsymbol{\sigma}(s, z, q)) \overset{(A4.4)}{=} \beta(\boldsymbol{\sigma}(t, z, q), \boldsymbol{\sigma}(s, z, q))$.

Now we introduce a new binary relation symbol red together with the *axiom*

(A8) $\mathsf{red}(t, t') \leftrightarrow$

$t = t' \;\vee$

$\exists r, r', s, s'\big(t = \mathbf{a}\, rs \wedge t' = \beta(r', s') \wedge \mathsf{red}(r, r') \wedge \mathsf{red}(s, s') \wedge r' \in \mathsf{Abst}\big) \;\vee$

$\exists r, r', s, s'\big(t = \mathbf{a}\, rs \wedge t' = \mathbf{a}\, r's' \wedge \mathsf{red}(r, r') \;\wedge \mathsf{red}(s, s')\big) \;\vee$

$\exists r, r', x\big(x \in \mathrm{V} \wedge t = \boldsymbol{\lambda} xr \wedge t' = \boldsymbol{\lambda} xr' \wedge \mathsf{red}(r, r')\big)$.

Lemma 5.

(a) $\mathsf{red}(t, t') \wedge z \in \mathrm{FV}(t') \to z \in \mathrm{FV}(t)$.

(b) $\mathsf{red}(t, t') \wedge \mathsf{red}(q, q') \to \mathsf{red}(\boldsymbol{\sigma}(t, x, q), \boldsymbol{\sigma}(t', x, q'))$.

(c) $\mathsf{red}(\boldsymbol{\lambda} xr, t') \to \exists r'(t' = \boldsymbol{\lambda} xr' \wedge \mathsf{red}(r, r'))$.

(d) $\mathsf{red}(t, t') \wedge \mathsf{red}(s, s') \wedge t \in \mathsf{Abst} \to \mathsf{red}(\beta(t, s), \beta(t', s'))$.

Proof :

(d) $t \in \mathsf{Abst} \to \exists x, r(t = \boldsymbol{\lambda} xr)$.

$t = \boldsymbol{\lambda} xr \wedge \mathsf{red}(t, t') \overset{(c)}{\to} \exists r'(t' = \boldsymbol{\lambda} xr' \wedge \mathsf{red}(r, r'))$.

$t = \boldsymbol{\lambda} xr \wedge t' = \boldsymbol{\lambda} xr' \wedge \mathsf{red}(r, r') \wedge \mathsf{red}(s, s') \overset{(b)}{\to}$

$\beta(t, s) = \boldsymbol{\sigma}(r, x, s) \wedge \beta(t', s') = \boldsymbol{\sigma}(r', x, s') \wedge \mathsf{red}(\boldsymbol{\sigma}(r, x, s), \boldsymbol{\sigma}(r', x, s'))$.

(b) Abb.: $\boldsymbol{\sigma}(t) := \boldsymbol{\sigma}(t, x, q)$ and $\boldsymbol{\sigma}'(t) := \boldsymbol{\sigma}(t, x, q')$.

1. $t = \mathbf{a}\, rs \wedge t' = \beta(r', s') \wedge \mathsf{red}(r, r') \wedge \mathsf{red}(s, s') \wedge r' \in \mathsf{Abst}$: $\boldsymbol{\sigma}(t) \overset{(A4.3)}{=} \mathbf{a}\, \boldsymbol{\sigma}(r)\boldsymbol{\sigma}(s) \wedge \boldsymbol{\sigma}(t') \overset{\text{L.4}}{=} \beta(\boldsymbol{\sigma}(r'), \boldsymbol{\sigma}(s')) \wedge$ $\boldsymbol{\sigma}(r') \overset{\text{L.1d}}{\in} \mathsf{Abst}$ and $\mathsf{red}(\boldsymbol{\sigma}(r), \boldsymbol{\sigma}'(r')) \wedge \mathsf{red}(\boldsymbol{\sigma}(s), \boldsymbol{\sigma}'(s'))$ by IH.

Further $\boldsymbol{\sigma}(t) = \mathbf{a}\, \boldsymbol{\sigma}(r)\boldsymbol{\sigma}(s)$ and $\boldsymbol{\sigma}'(t') = \beta(\boldsymbol{\sigma}'(r'), \boldsymbol{\sigma}'(s'))$. Hence $\mathsf{red}(\boldsymbol{\sigma}(t), \boldsymbol{\sigma}'(t'))$ by (A8).

2. $t = \boldsymbol{\lambda} yr \wedge t' = \boldsymbol{\lambda} yr' \wedge \mathsf{red}(r, r')$:

By L.0 there exist z with $z \neq x \wedge z \notin \mathrm{FV}(t) \wedge z \notin \mathrm{FV}(t') \wedge z \notin \mathrm{FV}(q)$.

(A5) $\to$ $t = \boldsymbol{\lambda} z \boldsymbol{\sigma}(r, y, z) \wedge t' = \boldsymbol{\lambda} z \boldsymbol{\sigma}(r', y, z)$.

$t = \boldsymbol{\lambda} yr = \boldsymbol{\lambda} z \boldsymbol{\sigma}(r, y, z) \wedge \mathsf{red}(r, r') \overset{\text{IH}}{\to} \mathsf{red}(\boldsymbol{\sigma}(r, y, z), \boldsymbol{\sigma}(r', y, z))$.

$t = \boldsymbol{\lambda} z \boldsymbol{\sigma}(r, y, z) \wedge \mathsf{red}(\boldsymbol{\sigma}(r, y, z), \boldsymbol{\sigma}(r', y, z)) \overset{\text{IH}}{\to} \mathsf{red}(\boldsymbol{\sigma}(\boldsymbol{\sigma}(r, y, z), x, q), \boldsymbol{\sigma}(\boldsymbol{\sigma}(r', y, z), x, q)) \to$

$\to \mathsf{red}(\boldsymbol{\lambda} z \boldsymbol{\sigma}(\boldsymbol{\sigma}(r, y, z), x, q), \boldsymbol{\lambda} z \boldsymbol{\sigma}(\boldsymbol{\sigma}(r', y, z), x, q))$.

$t = \boldsymbol{\lambda} z \boldsymbol{\sigma}(r, y, z) \wedge t' = \boldsymbol{\lambda} z \boldsymbol{\sigma}(r', y, z) \wedge x \neq z \wedge z \notin \mathrm{FV}(q) \to$

$\to \boldsymbol{\sigma}(t, x, q) = \boldsymbol{\lambda} z \boldsymbol{\sigma}(\boldsymbol{\sigma}(r, y, z), x, q) \wedge \boldsymbol{\sigma}(t', x, q) = \boldsymbol{\lambda} z \boldsymbol{\sigma}(\boldsymbol{\sigma}(r', y, z), x, q)$.

(a) 1. $z \in \mathrm{FV}(\beta(r', s')) \overset{\text{L.3b}}{\to} z \in \mathrm{FV}(r') \vee z \in \mathrm{FV}(s') \overset{\text{IH}}{\to} z \in \mathrm{FV}(r) \vee z \in \mathrm{FV}(s) \to z \in \mathrm{FV}(\mathbf{a}\, rs)$.

2. $z \in \mathrm{FV}(\boldsymbol{\lambda} xr') \to z \neq x \wedge z \in \mathrm{FV}(r') \overset{\text{IH}}{\to} z \neq x \wedge z \in \mathrm{FV}(r) \to z \in \mathrm{FV}(\boldsymbol{\lambda} xr)$.

(c) $\mathsf{red}(\boldsymbol{\lambda} xr, t') \to \exists p, p', y(\boldsymbol{\lambda} xr = \boldsymbol{\lambda} yp \wedge t' = \boldsymbol{\lambda} yp' \wedge \mathsf{red}(p, p'))$.

$\boldsymbol{\lambda} xr = \boldsymbol{\lambda} yp \wedge t' = \boldsymbol{\lambda} yp' \wedge \mathsf{red}(p, p') \to x \notin \mathrm{FV}(\boldsymbol{\lambda} yp) \wedge t' = \boldsymbol{\lambda} yp' \wedge \mathsf{red}(p, p') \to x \notin \mathrm{FV}(t')$;

$t' = \boldsymbol{\lambda} yp' \wedge x \notin \mathrm{FV}(t') \to t' = \boldsymbol{\lambda} x \boldsymbol{\sigma}(p', y, x)$;

$\mathsf{red}(p, p') \wedge r = \boldsymbol{\sigma}(p, y, x) \to \mathsf{red}(r, \boldsymbol{\sigma}(p', y, x))$.

Theorem. $\mathsf{red}(t, t') \wedge \mathsf{red}(t, t'') \to \exists \tilde{t}(\mathsf{red}(t', \tilde{t}) \wedge \mathsf{red}(t'', \tilde{t}))$.

Proof by induction on t:

1. $t = \mathbf{a}\, rs \wedge t' = \beta(r', s') \wedge \mathsf{red}(r, r') \wedge \mathsf{red}(s, s') \wedge r' \in \mathsf{Abst}$:

1.1. $t'' = \beta(r'', s'') \wedge \mathsf{red}(r, r'') \wedge \mathsf{red}(s, s'') \wedge r'' \in \mathsf{Abst}$:

By I.H. there are $\tilde{r}, \tilde{s}$ such that $\mathsf{red}(r', \tilde{r})$, $\mathsf{red}(r'', \tilde{r})$, $\mathsf{red}(s', \tilde{s})$, $\mathsf{red}(s'', \tilde{s})$.

Hence by Lemma 5d $\mathsf{red}(\beta(r', s'), \beta(\tilde{r}, \tilde{s}))$ and $\mathsf{red}(\beta(r'', s''), \beta(\tilde{r}, \tilde{s}))$.

1.2. $t'' = \mathbf{a}\, r''s'' \wedge \mathsf{red}(r, r'') \wedge \mathsf{red}(s, s'')$:

By I.H. there are $\tilde{r}, \tilde{s}$ such that $\mathsf{red}(r', \tilde{r})$, $\mathsf{red}(r'', \tilde{r})$, $\mathsf{red}(s', \tilde{s})$, $\mathsf{red}(s'', \tilde{s})$.

Hence by Lemma 5d $\mathsf{red}(\beta(r', s'), \beta(\tilde{r}, \tilde{s}))$ and $\mathsf{red}(\mathbf{a}\, r''s'', \beta(\tilde{r}, \tilde{s}))$.

2. $t = \mathbf{a}\, rs \wedge t' = \mathbf{a}\, r's' \wedge t'' = \mathbf{a}\, r''s''$: immediate by I.H.

3. $t = \boldsymbol{\lambda} xr$, $t' = \boldsymbol{\lambda} xr' \wedge \mathsf{red}(r, r')$:

By Lemma 5c we then have $t'' = \boldsymbol{\lambda} xr''$ with $\mathsf{red}(r, r'')$, and the claim follows immediately by the I.H.

4. $t \in V$: trivial.