A survey on ordinal notations around the Bachmann-Howard ordinal

Dedicated to Gerhard Jäger on the occasion of his 60th birthday

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Introduction

In recent years a renewed interest in ordinal notations around the Bachmann-Howard ordinal $\phi_{\varepsilon_{\Omega+1}}(0)$ has evolved, amongst others caused by Gerhard Jäger's metapredicativity program. Therefore it seems worthwile to review some important results of this area and to present detailed and streamlined proofs for them. The results in question are mainly comparisons of various functions which in the past have been used for describing ordinals not much larger than the Bachmann-Howard ordinal. We start with a treatment of the Bachmann hierarchy $\big(\phi_\alpha\big)_{\alpha\le\Gamma_{\Omega+1}}$ from [Ba50]. This hierarchy consists of normal functions $\phi_\alpha : \Omega \to \Omega$ ($\alpha \le \Gamma_{\Omega+1}$) which are defined by transfinite recursion on α referring to previously defined fundamental sequences $(\alpha[\xi])_{\xi<\tau_\alpha}$ (with $\tau_\alpha \le \Omega$). The most important new concept in Bachmann's approach is the systematic use of ordinals $\alpha > \Omega$ as indices for functions from Ω into Ω. Bachmann describes his approach as a generalization of a method introduced by Veblen in [Veb08]; according to him the initial segment $(\phi_\alpha)_{\alpha<\Omega^\Omega}$ is just a modified presentation of a system of normal functions defined by Veblen. But actually this connection is not so easy to see. At the end of §1 we will establish the connection between $(\phi_\alpha)_{\alpha<\Omega^\Omega}$ and Schütte's Klammersymbols [Sch54] for which the relation to [Veb08] is clear (cf. [Sch54, footnote 4]). In §2 we give an alternative characterization of the Bachmann hierarchy which instead of fundamental sequences $\big(\alpha[\xi]\big)_{\xi<\tau_\alpha}$ uses finite sets $K\alpha \subseteq \Omega$ of *coefficients* ("Koeffizienten"). For $\alpha < \varepsilon_{\Omega+1}$, $K\alpha$ is almost identical to the set $C(\alpha)$ of *constituents* (i.e., ordinals $< \Omega$ which occur in the complete base Ω Cantor normal form of α) in [Ge67], where it was shown how to construct a recursive system of ordinal notations on the basis of Bachmann's functions.

In the 1960s, the Bachmann method for generating hierarchies of normal functions on Ω was extended by Pfeiffer [1964] and, much further, by Isles [1970]. These extensions were highly complex; especially the Isles approach was so complicated that it was practically unusable for proof-theoretic applications. Therefore Feferman, in unpublished work around 1970, proposed an entirely different and much simpler method for generating hierarchies of normal functions θ_α ($\alpha \in On$) (see e.g. [Fef87]). Aczel (in [Acz??]) showed how the θ_α ($\alpha < \Gamma_{\Omega+1}$) correspond to Bachmann's ϕ_α. (Independently, Weyhrauch [Wey76] established the same results for $\alpha < \varepsilon_{\Omega+1}$.) In addition, Aczel generalized Feferman's definition and conjectured that the generalized hierarchy (θ_α) matches up with the Isles functions. This conjecture was proved by Bridge in [Bri72], [Bri75]. In §3 of the present paper we show how Feferman's functions θ_α ($\alpha < \Gamma_{\Omega+1}$) can also be defined by use of the $K\alpha$'s. Together with the content of §2 this leads to an easy comparison of the hierarchies $(\phi_\alpha)_{\alpha<\Gamma_{\Omega+1}}$ and $(\theta_\alpha)_{\alpha<\Gamma_{\Omega+1}}$ which becomes particularly simple if one switches to the fixed-point-free versions: $\overline{\phi}_\alpha(\beta) = \overline{\theta}_\alpha(\beta)$ for all $\alpha < \Gamma_{\Omega+1}$, $\beta < \Omega$ (Theorem 3.7).

In §§4,5 we deal with the unary functions $\vartheta : \varepsilon_{\Omega+1} \to \Omega$ and $\psi : \varepsilon_{\Omega+1} \to \Omega$ which play an important rôle in [RW93]. We show that $\overline{\theta}_{1+\alpha}(\beta) = \vartheta(\Omega\alpha + \beta)$ (for $\alpha < \varepsilon_{\Omega+1}$, $\beta < \Omega$) and refine a result from [RW93] on the relationship between ϑ and ψ. In §6, largely following [Wey76], we show how the Bachmann hierarchy below $\varepsilon_{\Omega+1}$ can be defined by means of functionals of finite higher types.

Preliminaries. The letters $\alpha, \beta, \gamma, \delta, \xi, \eta, \zeta$ always denote ordinals. *On* denotes the class of all ordinals and *Lim* the class of all limit ordinals. We are working in ZFC. So, every ordinal α is identical to the set $\{\xi \in On : \xi < \alpha\}$, and we have $\beta < \alpha \Leftrightarrow \beta \in \alpha$ and $\beta \leq \alpha \Leftrightarrow \beta \subseteq \alpha$. For $X \subseteq On$ we define: $X <(\leq)\, \alpha :\Leftrightarrow \forall x \in X(x <(\leq)\, \alpha)$ and $\alpha \leq X :\Leftrightarrow \exists x \in X(\alpha \leq x)$, i.e., $X < \alpha \Leftrightarrow X \subseteq \alpha$ and $\alpha \leq X \Leftrightarrow \neg(X < \alpha)$. By $\mathbb{H}$ we denote the class $\{\gamma \in On : \forall \alpha, \beta < \gamma(\alpha+\beta < \gamma)\} = \{\omega^\alpha : \alpha \in On\}$ of all *additive principal numbers* (*Hauptzahlen*), and by $\mathbb{E}$ the class $\{\alpha \in On : \omega^\alpha = \alpha\} = \{\varepsilon_\alpha : \alpha \in On\}$ of all *epsilon-numbers*. A *normal function* is a strictly increasing continuous function $F : On \to On$. The normal functions $\varphi_\alpha : On \to On$ $(\alpha \in On)$ are defined by: $\varphi_0(\beta) := \omega^\beta$, and $\varphi_\alpha :=$ ordering (or enumerating) function of $\{\beta : \forall \xi < \alpha(\varphi_\xi(\beta) = \beta)\}$, if $\alpha > 0$. The family $(\varphi_\alpha)_{\alpha \in On}$ is called *the Veblen hierarchy over* $\lambda\xi.\omega^\xi$. An ordinal α is called *strongly critical* iff $\varphi_\alpha(0) = \alpha$. The class of all strongly critical ordinals is denoted by SC, and its enumerating function by $\lambda\alpha.\Gamma_\alpha$. It is well-known that $\lambda\alpha.\Gamma_\alpha$ is again a normal function, and that $\Gamma_\Omega = \Omega$, where Ω is the least regular ordinal $> \omega$.

§1 Fundamental sequences and the Bachmann hierarchy

The following stems from Bachmann's seminal paper [Ba50], but in some minor details we deviate from that paper. We start by assigning to each limit number $\alpha \leq \Gamma_{\Omega+1}$ a fundamental sequence $(\alpha[\xi])_{\xi<\tau_\alpha}$ with $\tau_\alpha \leq \Omega$. The definition of $\alpha[\xi]$ is based on the normal form representation of α in terms of $0, +, \cdot, F$, where $(F_\alpha)_{\alpha \in On}$ is the Veblen hierarchy over $\lambda x.\Omega^x$, i.e., $F_0(\beta) := \Omega^\beta$, and $F_\alpha :=$ ordering function of $\{\beta : \forall \xi < \alpha(F_\xi(\beta) = \beta)\}$, if $\alpha > 0$. The relationship between F_α and φ_α for $\alpha > 0$ is given by

$$F_\alpha(\beta) = \varphi_\alpha(\widetilde{\alpha} + \beta) \text{ with } \widetilde{\alpha} := \begin{cases} \Omega{+}1 & \text{if } 0 < \alpha < \Omega \\ 1 & \text{if } \alpha = \Omega \\ 0 & \text{if } \Omega < \alpha \end{cases}.$$

From this it follows that $\Gamma_{\Omega+1}$ is the least fixed point of $\lambda\alpha.F_\alpha(0)$.

For completeness note, that $F_0(\beta) = \varphi_0(\Omega\beta)$.

Abbreviations.

1. $\Lambda := \Gamma_{\Omega+1} \ = \min\{\alpha : F_\alpha(0) = \alpha\}$.

2. $\alpha|\gamma \ :\Leftrightarrow \ \exists\xi(\gamma = \alpha{\cdot}\xi)$

3. $\alpha =_{\rm NF} \gamma + \Omega^\beta\eta \ :\Leftrightarrow \ \alpha = \gamma + \Omega^\beta\eta \ \& \ 0 < \eta < \Omega \ \& \ \Omega^{\beta+1}|\gamma$.

4. $\gamma =_{\rm NF} F_\alpha(\beta) \ :\Leftrightarrow \ \alpha, \beta < \gamma = F_\alpha(\beta)$.

Propositions.

(a) For each $0 < \delta < \Lambda$ there are unique γ, β, η such that $\delta =_{\rm NF} \gamma + \Omega^\beta\eta$.

(b) For each $\delta \in {\rm ran}(F_0) \cap \Lambda$ there are unique α, β such that $\delta =_{\rm NF} F_\alpha(\beta)$.

(c) $\delta < \Lambda \ \Rightarrow \ (\delta =_{\rm NF} F_\alpha(\beta) \Leftrightarrow \beta < \delta = F_\alpha(\beta))$.

Definition of a fundamental sequence $(\lambda[\xi])_{\xi<\tau_\lambda}$ **for each limit number** $\lambda \leq \Lambda$

1. $\lambda =_{\rm NF} \gamma + \Omega^\beta\eta \notin {\rm ran}(F_0)$:

1.1. $\eta \in \mathit{Lim}$: $\tau_\lambda := \eta$ and $\lambda[\xi] := \gamma + \Omega^\beta{\cdot}(1{+}\xi)$.

1.2. $\eta = \eta_0{+}1$: $\tau_\lambda := \tau_{\Omega^\beta}$ and $\lambda[\xi] := \gamma + \Omega^\beta\eta_0 + \Omega^\beta[\xi]$.

2. $\lambda =_{\rm NF} F_\alpha(\beta)$:

2.1. $\beta \in \mathit{Lim}$: $\tau_\lambda := \tau_\beta$ and $\lambda[\xi] := F_\alpha(\beta[\xi])$

2.2. $\beta \notin \mathit{Lim}$: Let $\lambda^- := \begin{cases} 0 & \text{if } \beta = 0 \\ F_\alpha(\beta_0){+}1 & \text{if } \beta = \beta_0{+}1 \end{cases}$

2.2.0. $\alpha = 0$: Then $\beta = \beta_0{+}1$. $\tau_\lambda := \Omega$ and $\lambda[\xi] := \Omega^{\beta_0}{\cdot}(1{+}\xi)$.

2.2.1. $\alpha = \alpha_0{+}1$: $\tau_\lambda := \omega$ and $\lambda[n] := F_{\alpha_0}^{(n+1)}(\lambda^-)$.

2.2.2. $\alpha \in \mathit{Lim}$: $\tau_\lambda := \tau_\alpha$ and $\lambda[\xi] := F_{\alpha[\xi]}(\lambda^-)$.

3. $\tau_\Lambda := \omega$ and $\Lambda[0] := 1$, $\Lambda[n{+}1] := F_{\Lambda[n]}(0)$.

Definition.

For each limit $\lambda \leq \Lambda$ we set $\lambda[\tau_\lambda] := \lambda$.

Further $\tau_0 := 0$, $0[\xi] := 0$ and $\tau_{\alpha+1} := 1$, $(\alpha{+}1)[\xi] := \alpha$.

Lemma 1.1.

$\lambda =_{\rm NF} F_\alpha(\beta) < \Lambda \ \& \ \beta \in \mathit{Lim} \ \& \ 1 \leq \xi < \tau_\beta \ \Rightarrow \ \lambda[\xi] =_{\rm NF} F_\alpha(\beta[\xi])$.

Proof: Cf. Appendix.

Lemma 1.2. Let $\lambda \in \mathit{Lim} \cap (\Lambda{+}1)$.

(a) $\xi < \eta \leq \tau_\lambda \ \Rightarrow \ \lambda[\xi] < \lambda[\eta]$.

(b) $\lambda = \sup_{\xi<\tau_\lambda} \lambda[\xi]$.

(c) $\eta \in \mathit{Lim} \cap (\tau_\lambda + 1) \ \Rightarrow \ \lambda[\eta] \in \mathit{Lim} \ \& \ \tau_{\lambda[\eta]} = \eta \ \& \ \forall\xi < \eta(\lambda[\eta][\xi] = \lambda[\xi])$.

(d) $\xi < \tau_\lambda \ \& \ \lambda[\xi] < \delta \leq \lambda[\xi{+}1] \implies \lambda[\xi] \leq \delta[1]$.

The proof of (a),(b),(c) is left to the reader. The proof of (d) will be given in the Appendix.

We now introduce a binary relation $\lll$ which corresponds to Bachmann's $\to$ (cf. [Ba50] p.123,130) and is essential for proving the basic properties of the Bachmann hierarchy. The advantage of $\ll$ over $\to$ is that its definition does not refer to the functions ϕ_α but only to the fundamental sequences $(\alpha[\xi])_{\xi<\tau_\alpha}$.

Definition of $\ll^1$, $\ll$ and $\underline{\ll}$

1. $\beta \ll^1 \alpha \;:\Leftrightarrow\; \alpha \le \Lambda \;\&\; \beta \in \{\alpha[\xi] : \xi < \tau_\alpha^\circ\}$, where $\tau_\alpha^\circ := \begin{cases} \omega & \text{if } \tau_\alpha = \Omega \\ \tau_\alpha & \text{otherwise} \end{cases}$.
2. $\ll$ ($\underline{\ll}$) is the transitive (transitive and reflexive) closure of $\ll^1$.

Lemma 1.3. Let $\alpha \le \Lambda$.
(a) $\alpha \in \mathit{Lim} \;\&\; \xi{+}1 < \tau_\alpha \;\Rightarrow\; \alpha[\xi]{+}1 \,\underline{\ll}\, \alpha[\xi{+}1]$.
(b) $\alpha \in \mathit{Lim} \;\&\; \xi < \eta < (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; \alpha[\xi] \ll \alpha[\eta]$.
(c) $\beta \ll \alpha \;\Rightarrow\; \beta{+}1 \,\underline{\ll}\, \alpha$.
(d) $n < \omega \;\&\; n \le \alpha \;\Rightarrow\; n \,\underline{\ll}\, \alpha$.

Proof :
(a) By induction on δ we prove: $\alpha[\xi] < \delta \le \alpha[\xi{+}1] \Rightarrow \alpha[\xi] + 1 \,\underline{\ll}\, \delta$.

1. $\delta = \delta_0{+}1$ with $\alpha[\xi] \le \delta_0$: Then either $\alpha[\xi]{+}1 = \delta$ or $\alpha[\xi]{+}1 \overset{\text{IH}}{\underline{\ll}} \delta_0 \ll^1 \delta$.

2. $\delta \in \mathit{Lim}$:
By Lemma 1.2a,d, $\alpha[\xi] < \delta[2] < \alpha[\xi{+}1]$. Hence $\alpha[\xi]{+}1 \overset{\text{IH}}{\underline{\ll}} \delta[2] \ll^1 \delta$.

(b) Induction on η:
1. $\eta = \eta_0{+}1 < \tau_\alpha$: $\alpha[\xi] \overset{\text{IH}}{\underline{\ll}} \alpha[\eta_0] \ll^1 \alpha[\eta_0] + 1 \overset{\text{(a)}}{\underline{\ll}} \alpha[\eta]$.
2. $\eta \in \mathit{Lim}$: Then $\tau_{\alpha[\eta]} = \eta$ and $\alpha[\xi] = \alpha[\eta][\xi] \ll^1 \alpha[\eta]$.

(c) We may assume $\beta \ll^1 \alpha$, i.e. $\beta = \alpha[\xi]$ with $\xi < \tau_\alpha^\circ$.
Then either $\tau_\alpha^\circ = 1 \;\&\; \beta{+}1 = \alpha$ or $\tau_\alpha^\circ \in \mathit{Lim} \;\&\; \alpha[\xi] + 1 \overset{\text{(a)}}{\underline{\ll}} \alpha[\xi{+}1] \ll^1 \alpha$.

(d) Induction on n:
1. Using Lemma 1.2a we get $0 \,\underline{\ll}\, \alpha$ by transfinite induction on α.
2. $n{+}1 \le \alpha \;\Rightarrow\; n < \alpha \;\&\; n \overset{\text{IH}}{\underline{\ll}} \alpha \;\Rightarrow\; n \ll \alpha \overset{\text{(c)}}{\Rightarrow} n{+}1 \,\underline{\ll}\, \alpha$.

Definition.
An Ω-*normal function* is a strictly increasing continuous function $f : \Omega \to \Omega$.
A set $M \subseteq \Omega$ is Ω-*club* (*closed and unbounded in* Ω) iff
$\forall X \subseteq M(X \neq \emptyset \;\&\; \sup(X) < \Omega \Rightarrow \sup(X) \in M)$ and $\forall \alpha < \Omega \exists \beta \in M(\alpha < \beta)$.

It is well-known that $M \subseteq \Omega$ is Ω-club if, and only if, M is the range of some Ω-normal function. Hence the ordering function of any Ω-club set is Ω-normal.

The collection of Ω-club sets has the following closure properties:
1. If f is Ω-normal then $\{\beta \in \Omega : f(\beta) = \beta\}$ is Ω-club.
2. If $(M_\xi)_{\xi<\alpha}$ is a sequence of Ω-club sets with $0 < \alpha < \Omega$ then $\bigcap_{\xi<\alpha} M_\xi$ is Ω-club.
3. If $(M_\xi)_{\xi<\Omega}$ is a sequence of Ω-club sets then also $\{\alpha \in \Omega : \alpha \in \bigcap_{\xi<\alpha} M_\xi\}$ is Ω-club.

Drawing upon 1.-3. and upon the above assignment of fundamental sequences we now define Bachmann's hierarchy of Ω-normal functions ϕ_α $(\alpha \le \Lambda)$.

Definition. $\phi_\alpha : \Omega \to \Omega$ is the ordering function of the Ω-club set R_α, where R_α is defined by recursion on α as follows:

$R_0 := \mathbb{H} \cap \Omega$,

$R_{\alpha+1} := \{\beta \in \Omega : \phi_\alpha(\beta) = \beta\}$,

$$R_\alpha := \begin{cases} \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} & \text{if } \tau_\alpha \in \Omega \cap \mathit{Lim} \\ \{\beta \in \Omega \cap \mathit{Lim} : \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}\} & \text{if } \tau_\alpha = \Omega \end{cases}$$

Notes.

1. In Lemma 1.5d we will show that $R_\alpha = \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$ if $\tau_\alpha = \Omega$.

2. As mentioned above, our definition of the Bachmann hierarchy (and of F_α) diverges in some minor points from [Ba50]. As a consequence of this, Bachmann's ordinals $H(1) = \varphi_{F_{\Omega(1)+1}}(1)$ and $\varphi_{F_{\omega_2+1}(1)}(1)$ are $\phi_{F_\Omega(0)}(0)$ and $\phi_\Lambda(0)$, respectively, in the present paper. For more details cf. [Acz72, Note on p.35].

Lemma 1.4.

(a) $\alpha_0 \lll \alpha \;\Rightarrow\; R_\alpha \subseteq R_{\alpha_0}$.

(b) $\alpha_0 \ll \alpha \;\Rightarrow\; \phi_{\alpha_0}(0) < \phi_\alpha(0)$.

(c) $n < \alpha \cap \omega \;\&\; \beta \in R_\alpha \;\Rightarrow\; \omega{\cdot}n < \beta \in \mathit{Lim}$.

Proof :

(a) It suffices to prove $R_\alpha \subseteq R_{\alpha_0}$ for $\alpha_0 \ll^1 \alpha$.

1. $\alpha = \alpha_0{+}1$: Then $R_\alpha = \{\beta \in \Omega : \phi_{\alpha_0}(\beta) = \beta\} \subseteq R_{\alpha_0}$.

2. $\tau_\alpha \in \Omega \cap \mathit{Lim}$: Then $\alpha_0 \in \{\alpha[\xi] : \xi < \tau_\alpha\}$ and thus $R_\alpha = \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} \subseteq R_{\alpha_0}$.

3. $\tau_\alpha = \Omega$: $\beta \in R_\alpha \Rightarrow \omega \le \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]} \Rightarrow \beta \in \bigcap_{\xi<\omega} R_{\alpha[\xi]} \subseteq R_{\alpha_0}$, since $\alpha_0 \in \{\alpha[\xi] : \xi < \omega\}$.

(b) 1. $\alpha = \alpha_0{+}1$: $\beta := \phi_\alpha(0) \in R_\alpha \;\Rightarrow\; \phi_{\alpha_0}(0) < \phi_{\alpha_0}(\beta) = \beta$.

2. $\alpha_0{+}1 < \alpha$: $\alpha_0 \ll \alpha \overset{1.3c}{\Rightarrow} \alpha_0{+}1 \lll \alpha \overset{(a)}{\Rightarrow} R_\alpha \subseteq R_{\alpha_0+1} \;\Rightarrow$
$\phi_{\alpha_0}(0) \overset{1.}{<} \phi_{\alpha_0+1}(0) \le \phi_\alpha(0)$.

(c) We have $1 \le \phi_0(0) < \phi_1(0) < \ldots$ and $\phi_{k+1}(0) \in \mathit{Lim}$. Hence $\omega{\cdot}n < \phi_{n+1}(0)$.
Further: $n < \alpha \overset{1.3d}{\Rightarrow} n{+}1 \lll \alpha \overset{(a)}{\Rightarrow} R_\alpha \subseteq R_{n+1} \subseteq \{\beta : \phi_{n+1}(0) \le \beta \in \mathit{Lim}\}$.

Lemma 1.5. For each $\alpha \in \mathit{Lim} \cap (\Lambda{+}1)$ the following holds:

(a) $\xi < \eta < (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; R_{\alpha[\eta]} \subseteq R_{\alpha[\xi]} \;\&\; \phi_{\alpha[\xi]}(0) < \phi_{\alpha[\eta]}(0)$.

(b) $\xi < (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; \xi \le \phi_{\alpha[\xi]}(0)$.

(c) $\lambda \in \mathit{Lim} \cap (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; R_{\alpha[\lambda]} = \bigcap_{\xi<\lambda} R_{\alpha[\xi]}$.

(d) $\tau_\alpha = \Omega \;\Rightarrow\; R_\alpha = \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$.

(e) $n < \omega \;\Rightarrow\; \phi_{\alpha[n]}(0) < \phi_\alpha(0)$.

Proof :

(a) follows from Lemmata 1.3b, 1.4a,b.

(b) follows from (a).

(c) By Lemma 1.2c we have $\tau_{\alpha[\lambda]} = \lambda$ and $\alpha[\lambda][\xi] = \alpha[\xi]$. Hence, by definition, $R_{\alpha[\lambda]} = \bigcap_{\xi<\lambda} R_{\alpha[\xi]}$.

(d) $R_\alpha = \{\beta \in \Omega \cap Lim : \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}\} \overset{\text{(c)}}{=} \{\beta \in \Omega : \beta \in R_{\alpha[\beta]}\} \overset{\text{(b)}}{=}$
$= \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$.

(e) follows from Lemma 1.4b.

Schütte's Klammersymbols

In [Sch54], building on [Veb08], Schütte introduced a system of ordinal notations based on so-called 'Klammersymbols'. A Klammersymbol is a matrix $\begin{pmatrix}\xi_0 \dots \xi_n \\ \alpha_0 \dots \alpha_n\end{pmatrix}$ with $0 \le \alpha_0 < \alpha_1 < \dots < \alpha_n < \Omega$ and $\xi_0, \dots, \xi_n < \Omega$. Two Klammersymbols are defined to be equal if they are identical after deleting all columns of the form $\binom{0}{\alpha_i}$. This means that one can identify the Klammersymbol $\begin{pmatrix}\xi_0 \dots \xi_n \\ \alpha_0 \dots \alpha_n\end{pmatrix}$ with the ordinal $\Omega^{\alpha_n}\xi_n + \dots + \Omega^{\alpha_0}\xi_0$. Under this identification the $<$-relation between ordinals induces a well-ordering $\prec$ on the Klammersymbols. To each Ω-normal function f and each Klammersymbol A an ordinal $fA < \Omega$ is assigned by $\prec$-recursion: $f\binom{\xi}{0} := f(\xi)$, and for $\xi_1 > 0$, the function $\lambda x.f\begin{pmatrix}x & \xi_1 \dots \xi_n \\ 0 & \alpha_1 \dots \alpha_n\end{pmatrix}$ is the ordering function of the set $\{\beta \in \Omega : \forall\xi < \xi_1 \forall\alpha_0 < \alpha_1 [f\begin{pmatrix}\beta & \xi & \xi_2 & \dots & \xi_n \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n\end{pmatrix} = \beta]\}$. In this subsection we will locate the values $\phi_0 A$ within the Bachmann hierarchy, i.e., we will prove $\phi_0\begin{pmatrix}\beta & \xi_0 & \dots & \xi_n \\ 0 & 1{+}\alpha_0 & \dots & 1{+}\alpha_n\end{pmatrix} = \phi_{\Omega^{\alpha_n}\xi_n+\dots+\Omega^{\alpha_0}\xi_0}(\beta)$.

Lemma 1.6. Assume $\alpha =_{\mathrm{NF}} \gamma + \Omega^{\delta_1}\xi_1$ with $\delta_1 < \Omega$.

(a) $\xi < \xi_1 \;\Rightarrow\; \gamma + \Omega^{\delta_1}\xi + 1 \,\underline{\ll}\, \gamma + \Omega^{\delta_1}(\xi{+}1) \,\underline{\ll}\, \alpha$.

(b) $\xi < \xi_1 \;\&\; \delta_0 < \delta_1 \;\Rightarrow\; \gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_0+1} \;\underline{\ll}\; \alpha$.

(c) $\beta \in R_\alpha \;\Leftrightarrow\; \forall\xi < \xi_1 [\,\phi_{\gamma+\Omega^{\delta_1}\xi}(\beta) = \beta \;\&\; \forall\delta_0 < \delta_1(\phi_{\gamma+\Omega^{\delta_1}\xi+\Omega^{\delta_0}\beta}(0) = \beta)\,]$.

Proof :

(a) Let $\hat\alpha := \gamma + \Omega^{\delta_1+1}$, $\eta := -1 + (\xi + 1)$, and $\eta_1 := -1 + \xi_1$. Then $\hat\alpha[\eta] = \gamma + \Omega^{\delta_1}(\xi{+}1)$, $\hat\alpha[\eta_1] = \gamma + \Omega^{\delta_1}\xi_1 = \alpha$, and $\eta \le \eta_1 < \tau_{\hat\alpha}$. Hence $\gamma + \Omega^{\delta_1}(\xi{+}1) \,\underline{\ll}\, \alpha$ by Lemma 1.3b. For the first inequality one needs the following auxiliary lemma (to be proved by induction on δ_1): $\Omega^{\delta_1}|\gamma_1 \;\Rightarrow\; \gamma_1 + 1 \,\underline{\ll}\, \gamma_1 + \Omega^{\delta_1}$.

(b) $\gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_0+1} \overset{(*)}{\underline{\ll}} \gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_1} = \gamma + \Omega^{\delta_1}(\xi{+}1) \overset{\text{(a)}}{\underline{\ll}} \gamma + \Omega^{\delta_1}\xi_1 = \alpha$.

$(*)$ Let $\gamma_1 := \gamma + \Omega^{\delta_1}\xi$. We have $\delta_1 = \delta + n$ with ($\delta_0 < \delta \in Lim$ or $\delta = \delta_0 + 1$). Further, $\gamma_1 + \Omega^{\delta_0+1} \,\underline{\ll}\, \gamma_1 + \Omega^\delta \ll \gamma_1 + \Omega^{\delta+1} \ll \dots \ll \gamma_1 + \Omega^{\delta+n}$.

(c) We have to show:

$\beta \in R_\alpha \;\Leftrightarrow\; \forall\xi < \xi_1[\,\beta \in R_{\gamma+\Omega^{\delta_1}\xi+1} \;\&\; \forall\delta_0 < \delta_1(\beta \in R_{\gamma+\Omega^{\delta_1}\xi+\Omega^{\delta_0+1}})\,]$.

"$\Rightarrow$": Cf. Lemma 1.4a and (a), (b).

"$\Leftarrow$": We distinguish the following cases:

1. $\xi_1 \in Lim$: $\beta \in \bigcap_{\xi<\xi_1} R_{\gamma+\Omega^{\delta_1}(1+\xi)} = R_\alpha$.

2. $\xi_1 = \xi_0{+}1$:

2.1. $\delta_1 = 0$: Then $\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+1} = R_\alpha$.

2.2. $\delta_1 = \delta_0{+}1$: $\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+\Omega^{\delta_0+1}} = R_\alpha$.

2.3. $\delta_1 \in Lim$: Since $\delta_1 < \Omega$, we then have $\tau_\alpha = \delta_1$ and $\alpha[\xi] = \gamma+\Omega^{\delta_1}\xi_0+\Omega^{1+\xi}$. From $\forall\xi < \delta_1(\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+\Omega^{\xi+1}})$ we get $\beta \in \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi+1]} \overset{1.5a}{\subseteq} \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} = R_\alpha$.

Definition. Due to the fact that every ordinal can be uniquely represented in the form $\Omega\alpha + \beta$ with $\beta < \Omega$ it is possible to code the binary function $(\alpha,\beta) \mapsto \phi_\alpha(\beta)$ $(\alpha \le \Lambda,\ \beta < \Omega)$ into a unary one by

$\phi\langle\Omega\alpha + \beta\rangle := \phi_\alpha(\beta)$ $(\alpha \le \Lambda,\ \beta < \Omega)$.

Using $\phi\langle\cdot\rangle$, the values of the Klammersymbols can be presented in a particularly nice way (cf. Theorem 1.8a below).

Lemma 1.7. Assume $\tilde\alpha =_{\rm NF} \gamma_1 + \Omega^{\alpha_1}\xi_1$ with $0 < \alpha_1 < \Omega$.

(a) $\lambda x.\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi_1 + x\rangle$ enumerates

$Q := \{\beta \in \Omega : \forall\xi < \xi_1\forall\alpha_0 < \alpha_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$.

(b) If $\alpha_1 = \alpha_0 + 1$ then $Q = \{\beta \in \Omega : \forall\xi < \xi_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$.

Proof :

There are δ_1 and γ such that $\alpha_1 = 1 + \delta_1$ and $\gamma_1 = \Omega\gamma$. Let $\alpha := \gamma + \Omega^{\delta_1}\xi_1$.

From (the proof of) Lemma 1.6c we get

$$\begin{aligned} R_\alpha &= \{\beta \in \Omega : \forall\xi < \xi_1[\,\phi\langle\Omega\gamma + \Omega^{1+\delta_1}\xi + \beta\rangle = \beta \;\& \\ &\qquad\qquad \forall\delta_0 < \delta_1(\phi\langle\Omega\gamma + \Omega^{1+\delta_1}\xi + \Omega^{1+\delta_0}\beta\rangle = \beta)\,]\} \\ &= \{\beta \in \Omega : \forall\xi < \xi_1\forall\alpha_0 < \alpha_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}, \text{ and} \end{aligned}$$

$R_\alpha = \{\beta \in \Omega : \forall\xi < \xi_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$, if $\alpha_1 = \alpha_0{+}1$.

On the other side, $\lambda x.\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi_1 + x\rangle = \lambda x.\phi\langle\Omega\alpha + x\rangle$ enumerates R_α.

Theorem 1.8. For $\alpha_0 < \ldots < \alpha_n < \Omega$ and $\xi_0, \ldots, \xi_n < \Omega$:

(a) $\phi_0\begin{pmatrix}\xi_0 & \cdots & \xi_n \\ \alpha_0 & \ldots & \alpha_n\end{pmatrix} = \phi\langle\Omega^{\alpha_n}\xi_n + \ldots + \Omega^{\alpha_0}\xi_0\rangle$.

(b) $\phi_0\begin{pmatrix}\beta & \xi_0 & \cdots & \xi_n \\ 0 & 1{+}\alpha_0 & \ldots & 1{+}\alpha_n\end{pmatrix} = \phi_{\Omega^{\alpha_n}\xi_n+\ldots+\Omega^{\alpha_0}\xi_0}(\beta)$.

Proof :

(a) W.l.o.g. $\alpha_0 = 0$.

1. $n=0$: $\phi\langle\Omega^0\xi_0\rangle = \phi\langle\Omega{\cdot}0+\xi_0\rangle = \phi_0(\xi_0) = \phi_0\begin{pmatrix}\xi_0\\0\end{pmatrix}$.

2. $n>0$: W.l.o.g. $\xi_1>0$.

By Lemma 1.7a, $\lambda x.\phi\langle\Omega^{\alpha_n}\xi_n+\ldots+\Omega^{\alpha_1}\xi_1+x\rangle$ is the ordering function of $\{\beta\in\Omega : \forall\xi<\xi_1\forall\alpha_0<\alpha_1[\phi\langle\Omega^{\alpha_n}\xi_n+\ldots+\Omega^{\alpha_1}\xi+\Omega^{\alpha_0}\beta\rangle=\beta]\}$.

Combining this with the above given definition of $\phi_0 A$ (for Klammersymbols A) the assertion is established by induction on $\Omega^{\alpha_n}\xi_n+\ldots+\Omega^{\alpha_0}\xi_0$.

(b) $\phi_0\begin{pmatrix}\beta & \xi_0 & \cdots & \xi_n\\ 0 & 1{+}\alpha_0 & \ldots & 1{+}\alpha_n\end{pmatrix} \overset{\text{(a)}}{=} \phi\langle\Omega^{1+\alpha_n}\xi_n+\ldots+\Omega^{1+\alpha_0}\xi_0+\Omega^0\beta\rangle =$

$$= \phi\langle\Omega{\cdot}(\Omega^{\alpha_n}\xi_n+\ldots+\Omega^{\alpha_0}\xi_0)+\beta\rangle.$$

Lemma 1.9. For $\xi_0,\ldots,\xi_n<\Omega$ let $\varphi^{n+1}(\xi_n,\ldots,\xi_0) := \phi\langle\Omega^n\xi_n+\ldots+\Omega^0\xi_0\rangle$. Then the following holds:

(i) $\varphi^{n+1}(0,\ldots,0,\beta) = \phi_0(\beta)$.

(ii) If $0<k\le n$ and $\xi_k>0$, then $\lambda x.\varphi^{n+1}(\xi_n,\ldots,\xi_k,0,\ldots,0,x)$ enumerates $\{\beta\in\Omega : \forall\xi<\xi_k(\varphi^{n+1}(\xi_n,\ldots,\xi_{k+1},\xi,\beta,0,\ldots,0)=\beta)\}$.

Proof of (ii):
By definition, $\varphi^{n+1}(\xi_n,\ldots,\xi_k,\vec{0},x) = \phi\langle\gamma+\Omega^k\xi_k+\Omega^0 x\rangle$ with $\gamma := \Omega^n\xi_n+\ldots+\Omega^{k+1}\xi_{k+1}$.

Therefore by Lemma 1.7a,b, $\lambda x.\varphi^{n+1}(\xi_n,\ldots,\xi_k,\vec{0},x)$ enumerates $\{\beta\in\Omega : \forall\xi<\xi_k[\phi\langle\gamma+\Omega^k\xi+\Omega^{k-1}\beta\rangle=\beta]\}$.

Note.
φ^{n+1} ($n\ge 1$) is known as the $n{+}1$-ary Veblen function.
Usually it is *defined* by (i), (ii).

§2 Characterization of ϕ_α via $K\alpha$

In [Ge67] the Bachmann hierarchy (ϕ_α) restricted to $\alpha<\varepsilon_{\Omega+1}$ is studied, and thereby, as a technical tool, the sets $C(\alpha)$ and $ND(\alpha)$ (of *constituents* and *nondistinguished constituents* of α) are defined. From Lemmata 3.1, 3.2 and Theorems 3.1, 3.3 of this paper one can derive the following interesting result which provides an alternative definition of the Bachmann hierarchy not referring to fundamental sequences:

(G) $R_\alpha = \{\gamma\in R_0 : C(\alpha)\le\gamma \;\&\; ND(\alpha)<\gamma \;\&$

$$\forall\xi<\alpha(C(\xi)<\gamma\Rightarrow\phi_\xi(\gamma)=\gamma)\}\quad(\alpha<\varepsilon_{\Omega+1}).$$

In the following we will directly prove an analogue of (G), namely Theorem 2.4, and then exemplarily derive Gerber's Theorems 4.1, 4.3 (our 2.7, 2.8) from that.

Definition of $K\alpha$ for $\alpha \le \Lambda$

1. $K\alpha := \begin{cases} \emptyset & \text{if } \alpha \in \{0,\Omega\} \\ \{\alpha\} & \text{if } \alpha \in \mathit{Lim} \cap \Omega \\ K\alpha_0 & \text{if } \alpha = \alpha_0{+}1 < \Omega \end{cases}$

2. $\Omega < \alpha =_{\mathrm{NF}} \gamma + \Omega^\beta \eta \notin \mathrm{ran}(F_0)$: $K\alpha := K\gamma \cup K\beta \cup K\eta$.

3. $\Omega < \alpha =_{\mathrm{NF}} F_\xi(\eta) < \Lambda$: $K\alpha := K'\xi \cup K\eta$ with $K'\xi := \begin{cases} \emptyset & \text{if } \xi = 0 \\ \{\omega\} \cup K\xi & \text{if } \xi > 0 \end{cases}$

4. $K\Lambda := \{\omega\}$.

Remark. $K(\alpha_0{+}1) = K\alpha_0$.

Lemma 2.1. $\lambda \in \mathit{Lim}$ & $1 \le \xi \le \tau_\lambda \;\Rightarrow\; K\lambda[\xi] = K\lambda[1] \cup K\xi$.

Proof :

1. $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta \eta \notin \mathrm{ran}(F_0)$:

1.1. $\eta \in \mathit{Lim}$: $\tau_\lambda = \eta$ and $\lambda[\xi] = \gamma + \Omega^\beta(1{+}\xi)$.

$\xi \le \eta \;\Rightarrow\; K\lambda[\xi] = K\gamma \cup K\beta \cup K\xi$.

1.2. $\eta = \eta_0{+}1$: $\tau_\lambda = \tau_{\Omega^\beta}$ and $\lambda[\xi] = \gamma + \Omega^\beta \eta_0 + \Omega^\beta[\xi]$.

$K\lambda[\xi] = K\gamma \cup K(\Omega^\beta\eta_0) \cup K(\Omega^\beta[\xi]) \overset{\mathrm{IH}}{=} K\gamma \cup K(\Omega^\beta\eta_0) \cup K(\Omega^\beta[1]) \cup K\xi$.

2. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$:

2.1. $\beta \in \mathit{Lim}$: Then by Lemma 1.1, $\lambda[\xi] =_{\mathrm{NF}} F_\alpha(\beta[\xi])$ and thus $K\lambda[\xi] = K'\alpha \cup K(\beta[\xi]) \overset{\mathrm{IH}}{=} K'\alpha \cup K\beta[1] \cup K\xi = K\lambda[1] \cup K\xi$.

2.2. $\beta \notin \mathit{Lim}$: Then $K\lambda^- = \begin{cases} K'\alpha \cup K\beta & \text{if } \beta = \beta_0{+}1 \;\&\; \beta_0 < F_\alpha(\beta_0) \\ K\beta & \text{otherwise} \end{cases}$.

Hence $K\lambda = K'\alpha \cup K\beta = K'\alpha \cup K\lambda^-$.

2.2.0. $\alpha = 0$: Then $\lambda = \Omega^{\beta_0+1}$, $\tau_\lambda = \Omega$ and $\lambda[\xi] = \Omega^{\beta_0}(1{+}\xi)$.

Hence $K\lambda[\xi] = K\beta_0 \cup K\xi$.

2.2.1. $\alpha = \alpha_0{+}1$: Then $\tau_\lambda = \omega$ and, for $\xi < \omega$, $K\lambda[\xi] = K(F_{\alpha_0}^{(\xi+1)}(\lambda^-)) = K'\alpha \cup K\lambda^-$ and $K\xi = \emptyset$.

Further $K\lambda[\omega] = K\lambda = K'\alpha \cup K\lambda^- = K'\alpha \cup K\lambda^- \cup K\omega$.

2.2.2. $\alpha \in \mathit{Lim}$: For $\xi < \tau_\lambda = \tau_\alpha$ we have

$K\lambda[\xi] = KF_{\alpha[\xi]}(\lambda^-) = K\alpha[\xi] \cup \{\omega\} \cup K\lambda^- \overset{\mathrm{IH}}{=} K\alpha[1] \cup \{\omega\} \cup K\lambda^- \cup K\xi$.

Further $K\lambda = K\alpha \cup \{\omega\} \cup K\lambda^- \overset{\mathrm{IH}}{=} K\alpha[1] \cup \{\omega\} \cup K\lambda^- \cup K\tau_\alpha$.

3. $\lambda = \Lambda$: For $1 \le \xi \le \omega$ we have $K\Lambda[\xi] = \{\omega\}$, whence $K\Lambda[\xi] = K\Lambda[1] \cup K\xi$.

Lemma 2.2.

(a) $\alpha \in \mathit{Lim}$ & $\alpha[\xi] \le \delta \le \alpha[\xi{+}1] \;\Rightarrow\; K\alpha[\xi] \subseteq K\delta$.

(b) $\delta < \alpha$ & $K\delta < \xi \in \mathit{Lim} \cap \tau_\alpha \;\Rightarrow\; \delta < \alpha[\xi]$.

Proof :

(a) Induction on δ:

1. $\delta = \alpha[\xi]$: trivial.

2. $\delta = \delta_0 + 1$ with $\alpha[\xi] \le \delta_0$: Then $K\alpha[\xi] \overset{\text{IH}}{\subseteq} K\delta_0 = K\delta$.

3. $\alpha[\xi] < \delta \in \mathit{Lim}$: Then, by L.1.2d, $\alpha[\xi] \le \delta[1]$. Hence $K\alpha[\xi] \overset{\text{IH}}{\subseteq} K\delta[1] \overset{2.1}{\subseteq} K\delta$.

(b) Assume $\alpha[0] \le \delta$. Then by Lemma 1.2a,b,c there exists $\zeta < \tau_\alpha$ such that $\alpha[\zeta] \le \delta < \alpha[\zeta{+}1]$. By (a) and Lemma 2.1 we get $K\zeta \subseteq K\alpha[\zeta] \subseteq K\delta < \xi \in \mathit{Lim}$. Hence $\delta < \alpha[\zeta{+}1] < \alpha[\xi]$.

Definition.

$\mathsf{k}(\alpha) := \max(K\alpha \cup \{0\})$. $\quad \mathsf{k}^+(\alpha) := \max\{\mathsf{k}(\alpha[1]){+}1, \mathsf{k}(\alpha)\}$.

Lemma 2.3.

(a) $\mathsf{k}(\alpha) \le \mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1$;

(b) $\mathsf{k}^+(\alpha{+}1) = \mathsf{k}(\alpha) + 1$;

(c) $\mathsf{k}^+(\alpha) \le \phi_\alpha(0)$.

Proof :

(a) By Lemma 2.1, $\mathsf{k}(\alpha) = \max\{\mathsf{k}(\alpha[1]), \mathsf{k}(\tau_\alpha)\}$ and thus

$\mathsf{k}^+(\alpha) = \max\{\mathsf{k}(\alpha[1]) + 1, \mathsf{k}(\tau_\alpha)\} \quad (*)$.

(b) $\mathsf{k}^+(\alpha{+}1) = \max\{\mathsf{k}(\alpha){+}1, \mathsf{k}(\alpha{+}1)\} = \mathsf{k}(\alpha){+}1$.

(c) Induction on α:

1. $\mathsf{k}^+(0) = 1 \le \phi_0(0)$.

2. $\alpha > 0$: By IH and Lemma 1.5e, $\mathsf{k}(\alpha[1]) \le \phi_{\alpha[1]}(0) < \phi_\alpha(0)$. By Lemma 1.5b, $\mathsf{k}(\tau_\alpha) \le \phi_\alpha(0)$. Hence $\mathsf{k}^+(\alpha) \overset{(*)}{=} \max\{\mathsf{k}(\alpha[1]) + 1, \mathsf{k}(\tau_\alpha)\} \le \phi_\alpha(0)$.

Theorem 2.4. $R_\alpha = \{\beta \in R_0 : \mathsf{k}^+(\alpha) \le \beta \;\&\; \forall \xi < \alpha (K\xi < \beta \Rightarrow \phi_\xi(\beta) = \beta)\}$.

Proof :

"$\subseteq$": Assume $\beta \in R_\alpha$. By Lemmata 1.4a, 2.3a,c we get $\mathsf{k}^+(\alpha) \le \beta \in R_0$. The second part is proved by induction on α. So let $\delta < \alpha \;\&\; K\delta < \beta \in R_\alpha$.

1. $\alpha = \delta + 1$: $\beta \in R_{\delta+1}$ implies $\phi_\delta(\beta) = \beta$.

2. $\alpha = \alpha_0 + 1 \;\&\; \delta < \alpha_0$: From $\delta < \alpha_0 \;\&\; K\delta < \beta \in R_\alpha \subseteq R_{\alpha_0}$ we obtain $\phi_\delta(\beta) = \beta$ by IH.

3. $\alpha \in \mathit{Lim} \;\&\; \tau_\alpha < \Omega$: Then $\beta \in \bigcap_{\xi < \tau_\alpha} R_{\alpha[\xi]}$ and $\delta < \alpha$.
From this we get $\exists \xi < \tau_\alpha (\beta \in R_{\alpha[\xi]} \;\&\; \delta < \alpha[\xi])$ and then $\phi_\delta(\beta) = \beta$ by IH.

4. $\tau_\alpha = \Omega$: By Lemmata 1.4c, 1.5c we get $\beta \in \mathit{Lim} \cap R_{\alpha[\beta]}$. From $\delta < \alpha$ and $K\delta < \beta \in \mathit{Lim} \cap \tau_\alpha$ we get $\delta < \alpha[\beta]$ by Lemma 2.2b. Now we have $\beta \in R_{\alpha[\beta]}$ and $\delta < \alpha[\beta] < \alpha \;\&\; K\delta < \beta$ which by IH yields $\phi_\delta(\beta) = \beta$.

"$\supseteq$": Assume (1) $\mathsf{k}^+(\alpha) \le \beta \in R_0$, and (2) $\forall \delta < \alpha (K\delta < \beta \Rightarrow \beta \in R_{\delta+1})$.
From $\mathsf{k}^+(\alpha) \le \beta$ we get (3) $K\alpha[1] < \beta$.

1. $\alpha = 0$: trivial.

2. $\alpha = \alpha_0{+}1$:

From $\alpha_0 < \alpha$ & $K\alpha_0 = K\alpha[1] < \beta$ by (2) we obtain $\beta \in R_{\alpha_0+1} = R_\alpha$.

3. $\alpha \in Lim$ & $\tau_\alpha < \Omega$: By Lemma 2.1 and (1) we have $\tau_\alpha \leq \mathsf{k}(\alpha) \leq \beta$. From $0 < \xi < \tau_\alpha \leq \beta$ by Lemma 2.1 and (3) we conclude $\alpha[\xi] < \alpha$ & $K\alpha[\xi] \subseteq K\alpha[1] \cup K\xi < \beta$, and then by (2), $\beta \in R_{\alpha[\xi]+1}$. Hence $\beta \in \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} = R_\alpha$.

4. $\tau_\alpha = \Omega$: From $0 < \alpha$ & $K0 = \emptyset < \beta$ by (2) we get $\beta \in R_1$, thence $\beta \in Lim$. Similarly as above we obtain $\beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}$. Hence $\beta \in R_\alpha$.

The fixed-point-free functions $\overline{\phi}_\alpha$

Definition.

$\overline{\phi}_\alpha(\beta) := \phi_\alpha(\beta + \tilde{\imath}\alpha\beta)$ where

$\tilde{\imath}\alpha\beta := \begin{cases} 1 & \text{if } \beta = \beta_0 + n \text{ with } \phi_\alpha(\beta_0) \in K\alpha \cup \{\beta_0\} \\ 0 & \text{otherwise} \end{cases}$

$\overline{R}_\alpha := \mathrm{ran}(\overline{\phi}_\alpha)$

Notation. From now on we mostly write $\phi\alpha\beta$, $\overline{\phi}\alpha\beta$ for $\phi_\alpha(\beta)$, $\overline{\phi}_\alpha(\beta)$

Theorem 2.5.

(a) $\overline{\phi}_\alpha$ is order preserving.

(b) $\overline{R}_\alpha = \{\phi\alpha\beta : K\alpha \cup \{\beta\} < \phi\alpha\beta\} = \{\gamma \in R_\alpha \setminus R_{\alpha+1} : K\alpha < \gamma\}$.

(c) $\overline{\phi}\alpha\beta = \min\{\gamma \in R_\alpha : \forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma)$ & $K\alpha \cup \{\beta\} < \gamma\}$

Proof :

(a) If $\beta_1 < \beta_2$ then $\beta_1 + \tilde{\imath}\alpha\beta_1 < \beta_2$ or $\beta_1 + \tilde{\imath}\alpha\beta_1 = \beta_2$.

In the latter case $\tilde{\imath}\alpha\beta_2 = \tilde{\imath}\alpha\beta_1 = 1$.

(b) The first equation follows immediately from the definition, since $\mathsf{k}(\alpha) \leq \phi\alpha 0$ and $\eta{+}1 < \phi\alpha(\eta{+}1)$ for all $\eta < \Omega$. The second equation follows from the first, since $\phi\alpha\beta \in R_{\alpha+1} \Leftrightarrow \beta = \phi\alpha\beta$.

(c) Let $X := \{\gamma \in R_\alpha : \forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma)$ & $K\alpha \cup \{\beta\} < \gamma\}$. By (a) and (b) we have $\overline{\phi}\alpha\beta \in X$. It remains to prove $\forall\gamma \in X(\overline{\phi}\alpha\beta \leq \gamma)$. So let $\gamma \in X$, i.e. $\gamma = \phi\alpha\delta$ with $\forall\eta < \beta(\phi\alpha(\eta + \tilde{\imath}\alpha\eta) < \phi\alpha\delta)$ & $K\alpha \cup \{\beta\} < \phi\alpha\delta$ $(*)$.

To prove: $\overline{\phi}\alpha\beta \leq \phi\alpha\delta$, i.e. $\beta + \tilde{\imath}\alpha\beta \leq \delta$.

From $\forall\eta < \beta(\phi\alpha(\eta + \tilde{\imath}\alpha\eta) < \phi\alpha\delta)$ we get $\beta \leq \delta$. Therefore if $\beta < \delta$ or $\tilde{\imath}\alpha\beta = 0$, we are done.

Assume now $\beta = \delta$ & $\tilde{\imath}\alpha\beta = 1$. Then $\delta = \beta = \beta_0 + n$ with $\phi\alpha\beta_0 \in K\alpha \cup \{\beta_0\}$.

1. $0 < n$: Then $\eta := \beta_0 + (n{-}1) < \beta = \eta{+}1$ and therefore $\beta = \eta + \tilde{\imath}\alpha\eta \stackrel{(*)}{<} \delta = \beta$. Contradiction.

2. $n = 0$: Then $\phi\alpha\beta \in K\alpha \cup \{\beta\} \stackrel{(*)}{<} \phi\alpha\delta = \phi\alpha\beta$. Contradiction.

Corollary 2.6.
(a) $\xi < \alpha \;\&\; K\xi \cup \{\eta\} < \overline{\phi}\alpha\beta \;\Rightarrow\; \overline{\phi}\xi\eta < \overline{\phi}\alpha\beta$.
(b) $K\alpha \cup \{\beta\} < \overline{\phi}\alpha\beta$.

Proof :
(a) $\xi < \alpha \;\&\; K\xi \cup \{\eta\} < \overline{\phi}\alpha\beta \overset{2.5b}{\in} R_\alpha \;\Rightarrow\; \overline{\phi}\xi\eta \overset{2.5a}{<} \phi\xi\overline{\phi}\alpha\beta \overset{2.4}{=} \overline{\phi}\alpha\beta$.
(b) follows immediately from Theorem 2.5c.

Lemma 2.7. Let $\gamma_i = \overline{\phi}\alpha_i\beta_i$ $(i = 1, 2)$.
(a) $\gamma_1 < \gamma_2$ if, and only if, one of the following holds:
(i) $\alpha_1 < \alpha_2 \;\&\; K\alpha_1 \cup \{\beta_1\} < \gamma_2$;
(ii) $\alpha_1 = \alpha_2 \;\&\; \beta_1 < \beta_2$;
(iii) $\alpha_2 < \alpha_1 \;\&\; \gamma_1 \leq K\alpha_2 \cup \{\beta_2\}$.
(b) $\gamma_1 = \gamma_2 \;\Rightarrow\; \alpha_1 = \alpha_2 \;\&\; \beta_1 = \beta_2$.

Proof :
(a) Let $Q(\alpha_1, \beta_1, \alpha_2, \beta_2) :\equiv (i) \vee (ii) \vee (iii)$.
To prove: $\gamma_1 < \gamma_2 \;\Leftrightarrow\; Q(\alpha_1, \beta_1, \alpha_2, \beta_2)$.
From Theorem 2.5a and Corollary 2.6 we get the implications
(1) $Q(\alpha_1, \beta_1, \alpha_2, \beta_2) \;\Rightarrow\; \gamma_1 < \gamma_2$ and (2) $Q(\alpha_2, \beta_2, \alpha_1, \beta_1) \;\Rightarrow\; \gamma_2 < \gamma_1$.
Obviously, (3) $\neg Q(\alpha_1, \beta_1, \alpha_2, \beta_2) \;\Rightarrow\; Q(\alpha_2, \beta_2, \alpha_1, \beta_2) \vee (\alpha_1 = \alpha_2 \;\&\; \beta_1 = \beta_2)$.
From (2) and (3) we get: $\neg Q(\alpha_1, \beta_1, \alpha_2, \beta_2) \;\Rightarrow\; \neg(\gamma_1 < \gamma_2)$.
(b) Proof by contradiction. Assume $\gamma_1 = \gamma_2 \;\&\; \alpha_1 < \alpha_2$. Then by Lemma 2.6b we have $\alpha_1 < \alpha_2 \;\&\; K\alpha_1 \cup \{\beta_1\} < \gamma_1 = \gamma_2$. Hence $\gamma_1 < \gamma_2$ by Lemma 2.6a.

Lemma 2.8. For each $\gamma \in R_0 \cap \phi_\Lambda(0)$ there exists $\alpha < \Lambda$ such that $\gamma \in \overline{R}_\alpha$.

Proof :
Assume $\omega < \gamma$. Then $K\Lambda < \gamma \notin R_\Lambda$. Let α_1 be the least ordinal such that $K\alpha_1 < \gamma \notin R_{\alpha_1}$. Then by Theorem 2.4 there exists $\alpha < \alpha_1$ such that $K\alpha < \gamma \notin R_{\alpha+1}$. By minimality of α_1 we get $\gamma \in R_\alpha$. Hence $\gamma \in \overline{R}_\alpha$ by 2.5b.

The following will prove useful in §4.

Theorem 2.9. Let $\overline{\phi}\langle\Omega\alpha+\beta\rangle := \overline{\phi}\alpha\beta$ $(\alpha \leq \Lambda,\ \beta < \Omega)$. Then for all $\alpha < \Lambda+\Omega$, $\overline{\phi}\langle\alpha\rangle = \min\{\gamma \in R_0 : \forall\xi < \alpha(K\xi < \gamma \Rightarrow \overline{\phi}\langle\xi\rangle < \gamma) \;\&\; K\alpha < \gamma\}$

Proof :

$$\overline{\phi}\langle\Omega\alpha + \beta\rangle = \overline{\phi}\alpha\beta \overset{2.5c}{=} \min\{\gamma \in R_\alpha : \forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma) \;\&\; K\alpha \cup \{\beta\} < \gamma\} \overset{2.4}{=}$$
$$\min\{\gamma \in R_0 : \forall\xi < \alpha\forall\eta(K\xi \cup \{\eta\} < \gamma \Rightarrow \overline{\phi}\xi\eta < \gamma) \;\&$$
$$\forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma) \;\&\; K\alpha \cup \{\beta\} < \gamma\} \overset{(*)}{=}$$
$$\min\{\gamma \in R_0 : \forall\xi < \alpha\forall\eta(K\xi \cup K\eta < \gamma \Rightarrow \overline{\phi}\langle\Omega\xi + \eta\rangle < \gamma) \;\&$$
$$\forall\eta < \beta(K\alpha \cup K\eta < \gamma \Rightarrow \overline{\phi}\langle\Omega\alpha + \eta\rangle < \gamma) \;\&\; K\alpha \cup K\beta < \gamma\} =$$
$$\min\{\gamma \in R_0 : \forall\zeta < \Omega\alpha{+}\beta(K\zeta < \gamma \Rightarrow \overline{\phi}\langle\zeta\rangle < \gamma) \;\&\; K(\Omega\alpha + \beta) < \gamma\}.$$

$(*)$ For $\alpha = \beta = 0$ the equation is trivial. Otherwise it follows from the fact that for $1 < \gamma \in R_0$ we have $\forall\eta < \Omega(K\eta < \gamma \Leftrightarrow \eta < \gamma)$.

§3 Comparison of $\phi_\alpha, \overline{\phi}_\alpha$ with $\theta_\alpha, \overline{\theta}_\alpha$

In this section we will compare the Bachmann functions ϕ_α with Feferman's functions θ_α. We will prove that $\phi\alpha\beta = \theta\alpha(\widehat{\alpha} + \beta)$ for all $\alpha \le \Lambda$, $\beta < \Omega$, where $\widehat{\alpha} := \min\{\eta : \mathsf{k}^+(\alpha) \le \theta\alpha\eta\}$. This result is already stated in [Acz??, Theorem 3] [1] and, for $\alpha < \varepsilon_{\Omega+1}$, proved in [Wey76].

Before we can turn to the proper subject of this section we have to do some elementary ordinal arithmetic.

Definition $E_\Omega(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \in \{0, \Omega\} \\ \{\alpha\} & \text{if } \alpha \in \mathbb{E} \setminus \{\Omega\} \\ \bigcup_{i \le n} E_\Omega(\alpha_i) & \text{if } \gamma = \omega^{\alpha_0} \# \ldots \# \omega^{\alpha_n} \notin \mathbb{E} \end{cases}$

Definition. A set $C \subseteq On$ is *nice* iff
$0 \in C \;\&\; \forall n \forall \alpha_0, \ldots, \alpha_n (\omega^{\alpha_0} \# \ldots \# \omega^{\alpha_n} \in C \Leftrightarrow \{\alpha_0, \ldots, \alpha_n\} \subseteq C)$.

Lemma 3.1.
(a) $E_\Omega(\Omega + \alpha) = E_\Omega(\Omega \cdot \alpha) = E_\Omega(\Omega^\alpha) = E_\Omega(\alpha)$.
(b) $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta \eta \;\Rightarrow\; E_\Omega(\alpha) = E_\Omega(\gamma) \cup E_\Omega(\beta) \cup E_\Omega(\eta)$.
(c) If C is nice and $\Omega \in C$ then $\forall \alpha (\alpha \in C \Leftrightarrow E_\Omega(\alpha) \subseteq C)$.
(d) $\alpha < \varepsilon_{\Omega+1} \;\&\; \delta \in \mathbb{E} \;\Rightarrow\; (E_\Omega(\alpha) < \delta \Leftrightarrow K\alpha < \delta)$.

Proof :
(a) Let $\alpha = \omega^{\alpha_0} + \ldots + \omega^{\alpha_n}$ with $\alpha_1 \ge \ldots \ge \alpha_n$.
1. $E_\Omega(\Omega + \alpha) = \begin{cases} E_\Omega(\alpha) & \text{if } \Omega < \alpha_0 \\ E_\Omega(\Omega) \cup E_\Omega(\alpha) & \text{if } \Omega \ge \alpha_0 \end{cases}$

2. $E_\Omega(\Omega \cdot \alpha) = E_\Omega(\omega^{\Omega+\alpha_0} + \ldots + \omega^{\Omega+\alpha_n}) = \bigcup_{i \le n} E_\Omega(\Omega + \alpha_i) \overset{1.}{=} \bigcup_{i \le n} E_\Omega(\alpha_i) = E_\Omega(\alpha)$.

3. $E_\Omega(\Omega^\alpha) = E_\Omega(\omega^{\Omega \cdot \alpha}) = E_\Omega(\Omega \cdot \alpha) \overset{2.}{=} E_\Omega(\alpha)$.

(b) Let $\eta = \omega^{\eta_0} + \ldots + \omega^{\eta_m}$ with $\eta_0 \ge \ldots \ge \eta_m$.
Then $\Omega^\beta \eta = \omega^{\Omega \cdot \beta} \cdot (\omega^{\eta_0} + \ldots + \omega^{\eta_m}) = \omega^{\Omega\beta + \eta_0} + \ldots + \omega^{\Omega\beta + \eta_m}$.

Hence $E_\Omega(\Omega^\beta \eta) = \bigcup_{i \le m} E_\Omega(\Omega\beta + \eta_0) = \bigcup_{i \le m} (E_\Omega(\beta) \cup E_\Omega(\eta_i)) =$
$= E_\Omega(\beta) \cup \bigcup_{i \le m} E_\Omega(\eta_i) = E_\Omega(\beta) \cup E_\Omega(\eta)$.

(c) 1. $\alpha \in \{0, \Omega\}$: $E_\Omega(\alpha) = \emptyset \subseteq C$ and $\alpha \in C$.

2. $\alpha \in \mathbb{E}$: $E_\Omega(\alpha) = \{\alpha\}$.

3. $\alpha = \omega^{\alpha_0} \# \ldots \# \omega^{\alpha_n} \notin \mathbb{E}$: $E_\Omega(\alpha) = E_\Omega(\alpha_0) \cup \ldots \cup E_\Omega(\alpha_n)$ and therefore:
$E_\Omega(\alpha) \subseteq C \;\Leftrightarrow\; \forall i \le n (E_\Omega(\alpha_i) \subseteq C) \overset{\mathrm{IH}}{\Leftrightarrow} \forall i \le n (\alpha_i \in C) \overset{\mathrm{C\,nice}}{\Leftrightarrow} \alpha \in C$.

(d) 1. $\alpha \in \{0, \Omega\}$: $E_\Omega(\alpha) = \emptyset = K\alpha$.
2. $\alpha < \Omega$: $E_\Omega(\alpha) < \delta \Leftrightarrow \alpha < \delta \Leftrightarrow K\alpha < \delta$.
3. $\Omega < \alpha =_{\mathrm{NF}} \gamma + \Omega^\beta \eta$: $E_\Omega(\alpha) < \delta \overset{(b)}{\Leftrightarrow} E_\Omega(\gamma) \cup E_\Omega(\beta) \cup E_\Omega(\eta) < \delta \overset{\mathrm{IH}}{\Leftrightarrow} K\gamma \cup K\beta \cup K\eta < \delta \Leftrightarrow K\alpha < \delta$.

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[1] Actually Aczel's Theorem 3 looks somewhat different, but it implies the above formulated result. A proof of Theorem 3 can be extracted from the proof of Theorem 3.5 in [Bri75].

Basic properties of the functions θ_α

The functions $\theta_\alpha : On \to On$ and sets $C(\alpha,\beta) \subseteq On$ are defined simultaneously by recursion on α (cf. [Bri75, p.174], [Bu75, p.6], [Sch77, p.225]). Instead of giving this definition we present a list of basic properties which are sufficient for proving Theorems 3.6, 3.7 below. – *Notation:* $\theta\alpha\beta := \theta_\alpha(\beta)$.

(θ1) $\theta_\alpha : On \to On$ is a normal function and $\mathrm{In}_\alpha := \mathrm{ran}(\theta_\alpha)$

(θ2) (i) $\mathrm{In}_0 = \mathbb{H}$,

(ii) $\mathrm{In}_{\alpha+1} = \{\beta \in \mathrm{In}_\alpha : \alpha \in C(\alpha,\beta) \Rightarrow \beta = \theta\alpha\beta\}$,

(iii) $\mathrm{In}_\alpha = \bigcap_{\xi<\alpha} \mathrm{In}_\xi$ if $\alpha \in Lim$.

(θ3) $\theta\alpha\Omega = \Omega$.

(θ4) $\mathrm{In}_\alpha \cap \Omega = \{\beta \in \Omega : C(\alpha,\beta) \cap \Omega \subseteq \beta\}$

(θ5) $\{0\} \cup \beta \subseteq C(\alpha,\beta)$, and if $\alpha > 0$ then $C(\alpha,\beta)$ is nice and $\Omega \in C(\alpha,\beta)$.

(θ6) $\xi < \alpha \le \Lambda \;\&\; \Omega < \eta < \theta\xi\eta \;\Rightarrow\; (\xi,\eta \in C(\alpha,\beta) \Leftrightarrow \theta\xi\eta \in C(\alpha,\beta))$.

Remark. (θ4)-(θ6) are only needed for the proof of Lemma 3.3c (via 3.2 and 3.3a,b). Having established 3.3c we will make use only of (θ1)-(θ3) with ($\theta 2ii$) replaced by 3.3c.

Lemma 3.2.
(a) $\alpha < \theta\alpha(\Omega{+}1) \;\&\; \Omega \le \beta \;\Rightarrow\; (\beta \in \mathrm{In}_{\alpha+1} \Leftrightarrow \beta = \theta\alpha\beta)$.
(b) $0 < \alpha \le \Lambda \;\Rightarrow\; F_\alpha(\beta) = \theta\alpha(\Omega + 1 + \beta)$.

Proof :
(a) "$\Leftarrow$": immediate consequence of ($\theta 2ii$) (and (θ1)).
"$\Rightarrow$": Assume $\beta \in \mathrm{In}_\alpha$ and $(\alpha \in C(\alpha,\beta) \Rightarrow \beta = \theta\alpha\beta)$. For $\beta = \Omega$ the claim follows directly from (θ3). Otherwise:
$\theta\alpha\Omega \overset{(\theta 3)}{=} \Omega < \beta \in \mathrm{In}_\alpha \Rightarrow \theta\alpha(\Omega{+}1) \le \beta \Rightarrow \alpha < \beta \overset{(\theta 5)}{\Rightarrow} \alpha \in C(\alpha,\beta) \Rightarrow \beta = \theta\alpha\beta$.

(b) Let $J := \{\beta : \Omega < \beta\}$. We prove $\mathrm{ran}(F_\alpha) = \mathrm{In}_\alpha \cap J$ which is equivalent to the claim $\forall\beta(F_\alpha(\beta) = \theta\alpha(\Omega + 1 + \beta))$.

The proof proceeds by induction on α.

1. $\alpha = 1$: $\mathrm{ran}(F_1) = \{\beta : \beta = \Omega^\beta\} = \{\beta : \Omega < \beta = \omega^\beta\} \overset{(\theta 2)}{=} \mathrm{In}_1 \cap J$.

2. $\alpha = \alpha_0{+}1$ with $1 \le \alpha_0$: $\mathrm{ran}(F_\alpha) = \{\beta : \beta = F_{\alpha_0}(\beta)\} \overset{\mathrm{IH}}{=}$
$= \{\beta : \beta = \theta\alpha_0(\Omega{+}1{+}\beta)\} = \{\beta : \Omega < \beta = \theta\alpha_0\beta\} \overset{(*)}{=} \mathrm{In}_\alpha \cap J$.

$(*)$ $\alpha_0 < \Lambda \Rightarrow \alpha_0 < F_{\alpha_0}(0) \overset{\mathrm{IH}}{=} \theta_{\alpha_0}(\Omega{+}1) \overset{(a)}{\Rightarrow} \forall\beta > \Omega(\beta = \theta\alpha_0\beta \Leftrightarrow \beta \in \mathrm{In}_\alpha)$.

3. $\alpha \in Lim$: $\mathrm{ran}(F_\alpha) = \bigcap_{\xi<\alpha} \mathrm{ran}(F_\xi) \overset{\mathrm{IH}}{=} \bigcap_{\xi<\alpha} \mathrm{In}_\xi \cap J \overset{(\theta 2\mathrm{iii})}{=} \mathrm{In}_\alpha \cap J$.

Lemma 3.3. For $\alpha < \Lambda$ we have:
(a) $\xi < \alpha \;\&\; \eta < F_\xi(\eta) < \Lambda \;\Rightarrow\; (\xi,\eta \in C(\alpha,\beta) \;\Leftrightarrow\; F_\xi(\eta) \in C(\alpha,\beta))$.

(b) $\forall\delta \le \alpha(\delta \in C(\alpha,\beta) \;\Leftrightarrow\; K\delta \subseteq C(\alpha,\beta))$.

(c) $\mathrm{In}_{\alpha+1} = \{\beta \in \mathrm{In}_\alpha : K\alpha < \beta \Rightarrow \beta = \theta\alpha\beta\}$

Proof :

(a) For $\xi = 0$ the claim follows from Lemma 3.1a, c and (θ5).

Assume now $\xi > 0$ and let $\gamma := F_\xi(\eta)$.

Then $\xi, \eta_1 < \gamma = \theta\xi\eta_1$ with $\eta_1 := \Omega{+}1{+}\eta$.

By (θ5) and Lemma 3.1a,c we have $(\eta \in C(\alpha,\beta) \Leftrightarrow \eta_1 \in C(\alpha,\beta))$.

Hence: $\xi, \eta \in C(\alpha,\beta) \Leftrightarrow \xi, \eta_1 \in C(\alpha,\beta) \overset{(\theta 6)}{\Leftrightarrow} \gamma \in C(\alpha,\beta)$.

(b) Induction on δ: Assume $\delta \le \alpha$, and let $C := C(\alpha,\beta)$.

1. $\delta \in \{0,\Omega\}$: $\delta \in C$ & $K\delta = \emptyset$.
2. $\delta = \delta_0{+}1$: $\delta \in C \Leftrightarrow \delta_0 \in C$, and $K\delta = K\delta_0$.
3. $\delta \in Lim \cap \Omega$: $K\delta = \{\delta\}$.
4. $\delta =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$: $\delta \in C \overset{3.1c}{\Leftrightarrow} E_\Omega(\delta) \subseteq C \overset{3.1b}{\Leftrightarrow}$
$\Leftrightarrow E_\Omega(\gamma) \cup E_\Omega(\beta) \cup E_\Omega(\eta) \subseteq C \overset{3.1c}{\Leftrightarrow} \gamma, \beta, \eta \in C \overset{\mathrm{IH}}{\Leftrightarrow} K\gamma \cup K\beta \cup K\eta \subseteq C \Leftrightarrow K\delta \subseteq C$.
5. $\delta =_{\mathrm{NF}} F\xi\eta$: $\delta \in C \overset{(a)}{\Leftrightarrow} \xi, \eta \in C \overset{\mathrm{IH}}{\Leftrightarrow} K\xi \cup K\eta \subseteq C \overset{(*)}{\Leftrightarrow} K\delta \subseteq C$.

($*$) $\omega = \theta 0 1 \in C$.

(c) follows from ($\theta 2ii$), (θ4), (b) and the fact that $K\alpha \subseteq \Omega$.

Theorem 3.4. $\alpha \le \Lambda \;\Rightarrow\; \mathrm{In}_\alpha = \{\beta \in \mathbb{H} : \forall\xi < \alpha(K\xi < \beta \;\Rightarrow\; \theta\xi\beta = \beta)\}$.

Proof by induction on α:

1. $\alpha = 0$: By ($\theta 2i$) we have $\mathrm{In}_0 = \mathbb{H}$.
2. $\alpha = \alpha_0{+}1$: $\mathrm{In}_\alpha \overset{3.3c}{=} \{\beta \in \mathrm{In}_{\alpha_0} : K\alpha_0 < \beta \Rightarrow \beta = \theta\alpha_0\beta\} \overset{\mathrm{IH}}{=}$
$= \{\beta \in \mathbb{H} : \forall\xi < \alpha_0(K\xi < \beta \Rightarrow \beta = \theta\xi\beta)$ & $(K\alpha_0 < \beta \Rightarrow \beta = \theta\alpha_0\beta)\}$.
3. $\alpha \in Lim$: Then, by ($\theta 2iii$), $\mathrm{In}_\alpha = \bigcap_{\xi<\alpha} \mathrm{In}_\xi$ and the assertion follows immediately from the IH.

Definition. $\widehat{\alpha} := \min\{\eta : \mathsf{k}^+(\alpha) \le \theta\alpha\eta\}$.

Lemma 3.5. $\alpha \le \Lambda$ & $K\alpha < \theta\alpha\beta \;\Rightarrow\; (\theta\alpha(\widehat{\alpha} + \beta) = \beta \Leftrightarrow \theta\alpha\beta = \beta)$.

Proof :

"$\Rightarrow$": This follows from $\beta \le \theta\alpha\beta \le \theta\alpha(\widehat{\alpha} + \beta)$.

"$\Leftarrow$": If $K\alpha < \beta = \theta\alpha\beta$ then $\widehat{\alpha} \le \mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1 < \beta \in \mathbb{H}$ and thus $\widehat{\alpha}{+}\beta = \beta$.

Theorem 3.6. If $\alpha \le \Lambda$, then $R_\alpha = \{\gamma \in \Omega : \mathsf{k}^+(\alpha) \le \gamma \in \mathrm{In}_\alpha\}$,
and thus $\forall\beta < \Omega(\phi\alpha\beta = \theta\alpha(\widehat{\alpha} + \beta))$.

Proof by induction on α:

For $\beta < \Omega$ we have:

$\beta \in R_\alpha \overset{2.4}{\Leftrightarrow} \mathsf{k}^+(\alpha) \le \beta \in \mathbb{H}$ & $\forall\xi < \alpha(K\xi < \beta \Rightarrow \phi\xi\beta = \beta) \overset{\mathrm{IH}+3.5}{\Leftrightarrow}$
$\mathsf{k}^+(\alpha) \le \beta \in \mathbb{H}$ & $\forall\xi < \alpha(K\xi < \beta \Rightarrow \theta\xi\beta = \beta) \overset{3.4}{\Leftrightarrow} \mathsf{k}^+(\alpha) \le \beta \in \mathrm{In}_\alpha$.

The functions $\overline{\theta}_\alpha$

In [Bu75] the fixed-point-free functions $\overline{\theta}_\alpha$ are introduced, which are more suitable for proof-theoretic applications than the θ_α's. By definition, $\overline{\theta}_\alpha$ is the $<$-isomorphism from $\{\eta \in On : S\mu(\alpha) \le \eta\}$ onto $\overline{\mathrm{In}}_\alpha$ where $\overline{\mathrm{In}}_\alpha := \mathrm{In}_\alpha \setminus \mathrm{In}_{\alpha+1}$, $\mu(\alpha) := \min\{\eta : \theta\alpha\eta \in \overline{\mathrm{In}}_\alpha\}$, $S\mu(\alpha) := \min\{\xi : \mu(\alpha) < \Omega_{\xi+1}\}$.

As we will show in a moment, $S\mu(\alpha) = 0$ for all $\alpha < \Lambda$, and therefore, if $\alpha < \Lambda$ then $\overline{\theta}_\alpha$ is the ordering function of $\overline{\mathrm{In}}_\alpha$. On the other side, by Theorem 2.5, $\overline{\phi}_\alpha$ is the ordering function of $\overline{R}_\alpha = \{\gamma \in R_\alpha \setminus R_{\alpha+1} : K\alpha < \gamma\}$. Using Theorem 3.6 one easily sees that $\overline{R}_\alpha = \overline{\mathrm{In}}_\alpha \cap \Omega$. So we arrive at the following theorem.

Theorem 3.7. $\overline{\phi}\alpha\beta = \overline{\theta}\alpha\beta$ for all $\alpha < \Lambda$, $\beta < \Omega$.

Proof :

I. From $\alpha < \Lambda$ by Lemma 3.3c and $(\theta 3)$ we obtain $\forall \beta \in \Omega(\mathsf{k}(\alpha) \le \beta \Rightarrow \theta\alpha(\beta{+}1) \in \overline{\mathrm{In}}_\alpha \cap \Omega)$. Hence $S\mu(\alpha) = 0$, and $\overline{\mathrm{In}}_\alpha \cap \Omega$ is unbounded in Ω. This implies that $\overline{\theta}_\alpha{\restriction}\Omega$ is the ordering function of $\overline{\mathrm{In}}_\alpha \cap \Omega$.

II. As mentioned above, $\overline{\phi}_\alpha$ is the ordering function of $\overline{R}_\alpha$. So it remains to prove that $\overline{R}_\alpha = \overline{\mathrm{In}}_\alpha \cap \Omega$. First note that

(1) $\mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1 = \mathsf{k}^+(\alpha{+}1)$ and (2) $\forall \gamma \in \overline{\mathrm{In}}_\alpha(\mathsf{k}(\alpha) < \gamma)$ (by 3.3c).

Then for $\gamma < \Omega$ we get: $\gamma \in \overline{R}_\alpha \;\Leftrightarrow\; \mathsf{k}(\alpha) < \gamma \in R_\alpha \;\&\; \gamma \notin R_{\alpha+1} \overset{3.6,(1)}{\Leftrightarrow}$
$\mathsf{k}(\alpha) < \gamma \in \mathrm{In}_\alpha \;\&\; (\mathsf{k}(\alpha) < \gamma \Rightarrow \gamma \notin \mathrm{In}_{\alpha+1}) \overset{(2)}{\Leftrightarrow} \gamma \in \overline{\mathrm{In}}_\alpha$.

§4 The unary functions $\vartheta^{\mathbb{X}}$ and $\psi^{\mathbb{X}}$

As we have seen above, $\overline{\theta}_\alpha$ is the ordering function of $\overline{\mathrm{In}}_\alpha = \mathrm{In}_\alpha \setminus \mathrm{In}_{\alpha+1}$ (if $\alpha < \Lambda$). From this together with $(\theta 2ii)$ and $(\theta 4)$ one easily derives the following equation

(1) $\overline{\theta}\alpha 0 = \min\{\beta : C(\alpha,\beta) \cap \Omega \subseteq \beta \;\&\; \alpha \in C(\alpha,\beta)\}$

which motivates the definition of $\vartheta\alpha$ in [RW93]:

(2) $\vartheta\alpha := \min\{\beta : \tilde{C}(\alpha,\beta) \cap \Omega \subseteq \beta \;\&\; \alpha \in \tilde{C}(\alpha,\beta)\}$ $(\alpha < \varepsilon_{\Omega+1})$

where $\tilde{C}(\alpha,\beta)$ is the closure of $\{0,\Omega\} \cup \beta$ under $+$, $\lambda\xi.\omega^\xi$ and $\vartheta{\restriction}\alpha$.

On the other side, by Theorems 3.7, 2.9 we have:

(3) $\overline{\theta}\alpha 0 = \overline{\phi}\langle\Omega\alpha\rangle = \min\{\beta \in \mathbb{H} : \forall \xi < \Omega\alpha(K\xi < \beta \Rightarrow \overline{\phi}\langle\xi\rangle < \beta) \;\&\; K\alpha < \beta\}$.

In the light of (1)-(3) the following theorem suggests itself.

Theorem 4.1.

$\alpha < \varepsilon_{\Omega+1} \;\Rightarrow\; \vartheta\alpha = \min\{\beta \in \mathbb{E} : \forall \xi < \alpha(K\xi < \beta \Rightarrow \vartheta\xi < \beta) \,\&\, K\alpha < \beta\}$.

Proof :

I. From [RW93,1.1 and 1.2(1)-(4)] we obtain
$\vartheta\alpha \in \mathbb{E} \,\&\, \forall\xi < \alpha(E_\Omega(\xi) < \vartheta\alpha \Rightarrow \vartheta\xi < \vartheta\alpha) \,\&\, E_\Omega(\alpha) < \vartheta\alpha.$

II. Assume $\beta \in \mathbb{E}$ & $\forall\xi < \alpha(E_\Omega(\xi) < \beta \Rightarrow \vartheta\xi < \beta)$ & $E_\Omega(\alpha) < \beta$.
We will prove that $\vartheta\alpha \le \beta$.

For this let $Q := \{\gamma : E_\Omega(\gamma) \subseteq \beta\}$. Since $\beta \in \mathbb{E}$, we have $Q \subseteq \beta$. Moreover, as one easily sees, $\{0, \Omega\} \subseteq Q$ and Q is closed under $+$, $\lambda\xi.\omega^\xi$ and $\vartheta{\restriction}\alpha$. Hence $\tilde{C}(\alpha,\beta) \subseteq Q$ and thus $\tilde{C}(\alpha,\beta) \cap \Omega \subseteq Q \cap \Omega \subseteq \beta$. It remains to show that $\alpha \in \tilde{C}(\alpha,\beta)$. But this follows immediately from $E_\Omega(\alpha) \subseteq \beta \subseteq \tilde{C}(\alpha,\beta)$ and [RW93, 1.2(4)].

From I. and II. we get
$\vartheta\alpha = \min\{\beta \in \mathbb{E} : \forall\xi < \alpha(E_\Omega(\xi) < \beta \Rightarrow \vartheta\xi < \beta) \,\&\, E_\Omega(\alpha) < \beta\}$,
which together with Lemma 3.1d yields the claim.

Relativization

Comparing the recursion equations for $\vartheta\alpha$ and $\overline{\phi}\langle\alpha\rangle$ in Theorems 4.1, 2.9 one notices that these equations are almost identical. The only difference is that in the equation for $\vartheta\alpha$ there appears $\mathbb{E}$ where in the equation for $\overline{\phi}\langle\alpha\rangle$ we have R_0 (i.e. $\mathbb{H}$). In order to establish the exact relationship between ϑ and $\overline{\phi}$ we go back to the definition of the Bachmann hierarchy in §1 and replace the initial clause "$R_0 := \mathbb{H} \cap \Omega$" of this definition by "$R_0 := \mathbb{X} \cap \Omega$" where here and in the sequel $\mathbb{X}$ *always denotes a subclass of* $\{1\} \cup Lim$ *such that* $\mathbb{X} \cap \Omega$ *is* Ω*-club.* Then the whole of §§1,2 remains valid as it stands. To make the dependency on $\mathbb{X}$ visible we write $R^{\mathbb{X}}_\alpha, \overline{R}^{\mathbb{X}}_\alpha, \phi^{\mathbb{X}}_\alpha, \overline{\phi}^{\mathbb{X}}_\alpha, \phi^{\mathbb{X}}\langle\alpha\rangle, \overline{\phi}^{\mathbb{X}}\langle\alpha\rangle$ instead of $R_\alpha, \overline{R}_\alpha, \ldots$.

Remark.

Theorems 4.1, 2.9 yield $\vartheta\alpha = \overline{\phi}^{\mathbb{E}}\langle\alpha\rangle$ and $\vartheta(\Omega\alpha+\beta) = \overline{\phi}^{\mathbb{E}}_\alpha(\beta)$ $\;(\alpha < \varepsilon_{\Omega+1},\, \beta < \Omega)$

The previous explanations motivate the following definition.

Definition.

$\vartheta^{\mathbb{X}}\alpha := \min\{\beta \in \mathbb{X} : \forall\xi < \alpha(K\xi < \beta \Rightarrow \vartheta^{\mathbb{X}}\xi < \beta) \,\&\, K\alpha < \beta\}$ $\;(\alpha \le \Lambda)$.

Theorem 4.1 now reads: $\vartheta\alpha = \vartheta^{\mathbb{E}}\alpha$ for $\alpha < \varepsilon_{\Omega+1}$.
Further, by Theorem 2.9 we have
($\vartheta 0$) $\;\vartheta^{\mathbb{X}}(\Omega\alpha + \beta) = \overline{\phi}^{\mathbb{X}}_\alpha(\beta)$, if $\beta < \Omega$.

Therefore, properties of $\vartheta^{\mathbb{X}}$ can be proved by deriving them from corresponding properties of $\overline{\phi}$. But for various reasons it is also advisable to work directly from the above definition.

Let us first mention that for $\beta < \Omega$ the set $\{\xi < \alpha : K\xi < \beta\}$ is countable too, and therefore $\vartheta^{\mathbb{X}}\alpha < \Omega$. Moreover, directly from the definition of $\vartheta^{\mathbb{X}}$ we obtain:

(ϑ1) $K\alpha < \vartheta^{\mathbb{X}}\alpha \in \mathbb{X}$,

(ϑ2) $\alpha_0 < \alpha \;\&\; K\alpha_0 < \vartheta^{\mathbb{X}}\alpha \;\Rightarrow\; \vartheta^{\mathbb{X}}\alpha_0 < \vartheta^{\mathbb{X}}\alpha$,

(ϑ3) $\beta \in \mathbb{X} \;\&\; K\alpha < \beta < \vartheta^{\mathbb{X}}\alpha \;\Rightarrow\; \exists\xi < \alpha(K\xi < \beta \leq \vartheta^{\mathbb{X}}\xi)$,

and then

(ϑ4) $\vartheta^{\mathbb{X}}\alpha_0 = \vartheta^{\mathbb{X}}\alpha_1 \Rightarrow \alpha_0 = \alpha_1$ [from (ϑ1),(ϑ2)],

(ϑ5) $\beta \in \mathbb{X} \;\&\; \beta < \vartheta^{\mathbb{X}}\Lambda \;\Rightarrow\; \exists\xi < \Lambda(\beta = \vartheta^{\mathbb{X}}\xi)$.

Proof of (ϑ5): If $\beta \leq \omega$ then $\beta \in \{\vartheta 0, \vartheta 1\}$. Otherwise we have $K\Lambda < \beta < \vartheta^{\mathbb{X}}\Lambda$, and the assertion follows by transfinite induction from (ϑ3).

Note on Klammersymbols. As we mentioned above, §§1,2 remain valid if ϕ is replaced by $\phi^{\mathbb{X}}$. So by Theorem 1.8, for $A = \begin{pmatrix}\xi_0 & \dots & \xi_n \\ \alpha_0 & \dots & \alpha_n\end{pmatrix}$ and $\alpha = \Omega^{\alpha_n}\xi_n + \ldots + \Omega^{\alpha_0}\xi_0$ we have $\phi^{\mathbb{X}}_0 A = \phi^{\mathbb{X}}\langle\alpha\rangle$ from which one easily derives $\overline{\phi^{\mathbb{X}}_0}A = \overline{\phi}^{\mathbb{X}}\langle\alpha\rangle$ [2], whence (by Theorem 2.9) $\overline{\phi^{\mathbb{X}}_0}A = \vartheta^{\mathbb{X}}\alpha$. Via Theorem 4.1 this fits together with Schütte's result $\overline{\phi^{\mathbb{E}}_0}A = \vartheta\alpha$ in [Sch92].

The function $\psi^{\mathbb{X}}$

In [Bu86] (actually already in [Bu81]) the author introduced the functions $\psi_\sigma : On \to \Omega_{\sigma+1}$ and proved, via an ordinal analysis of ID_ν, that $\psi_0\varepsilon_{\Omega_\nu+1} = \theta_{\varepsilon_{\Omega_\nu+1}}(0)$. In [BS88] ordinal analyses of several impredicative subsystems of 2nd order arithmetic are carried out by means of the ψ_σ's. The definition of ψ_σ in [BS88] differs in some minor respects from that in [Bu86]; for example, $\lambda\xi.\omega^\xi$ is a basic function in [BS88] but not in [Bu86]. In [RW93] Rathjen and Weiermann compare their ϑ with $\psi_0{\restriction}\varepsilon_{\Omega+1}$ from [BS88] which they abbreviate by ψ. In §5 we will present a refinement of this comparison which is based on Schütte's definition of the Veblen function φ (below Γ_0) in terms of ψ, given in §7 of [BS88].

Similarly as Theorem 4.1 one can prove

$\psi\alpha = \min\{\beta \in \mathbb{E} : \forall\xi < \alpha(K\xi < \beta \Rightarrow \psi\xi < \beta)\}$, for $\alpha < \varepsilon_{\Omega+1}$.

This motivates the following

Definition of $\psi^{\mathbb{X}}\alpha$ for $\alpha \leq \Lambda{+}1$

$\psi^{\mathbb{X}}\alpha := \min\{\beta \in \mathbb{X} : \forall\xi < \alpha(K\xi < \beta \Rightarrow \psi^{\mathbb{X}}\xi < \beta)\}$.

For the rest of this section we assume $\mathbb{X}$ to be fixed, and write ϑ, ψ for $\vartheta^{\mathbb{X}}, \psi^{\mathbb{X}}$.

Remark. Immediately from the definitions it follows that $\psi\alpha \leq \vartheta\alpha$.

Before turning to the announced exact comparison of ϑ and ψ, we prove a somewhat weaker (but still very useful) result which can be obtained with much less effort. This corresponds to [RW93, p.64] which in turn stems from [BS83] and [BS76].

---

2 $\overline{\varphi}A$ is the 'fixed-point-free version' of φA defined in [Sch54,§3].

Lemma 4.2. For $\alpha \leq \Lambda$.

(a) $\alpha_0 \leq \alpha \;\Rightarrow\; \psi\alpha_0 \leq \psi\alpha$.

(b) $\alpha_0 < \alpha$ & $K\alpha_0 < \psi\alpha \;\Rightarrow\; \psi\alpha_0 < \psi\alpha$.

(c) $\psi\alpha < \psi(\alpha{+}1) \;\Leftrightarrow\; K\alpha < \psi\alpha$.

(d) $\alpha \in Lim \;\Rightarrow\; \psi\alpha = \sup_{\xi<\alpha}\psi\xi$.

(e) $\psi\alpha = \min\{\gamma \in \mathbb{X} : \forall\xi < \alpha(K\xi < \psi\xi \Rightarrow \psi\xi < \gamma)\}$.

Proof :

(a), (b) follow directly from the definition.

(c) "$\Rightarrow$": Assume $\neg(K\alpha < \psi\alpha)$. Then from $\psi\alpha \in \mathbb{X}$ & $\forall\xi < \alpha(K\xi < \psi\alpha \Rightarrow \psi\xi < \psi\alpha)$ we conclude $\psi\alpha \in \mathbb{X}$ & $\forall\xi < \alpha{+}1(K\xi < \psi\alpha \Rightarrow \psi\xi < \psi\alpha)$, and thus $\psi(\alpha{+}1) \leq \psi\alpha$.

"$\Leftarrow$": From $\alpha < \alpha{+}1$ & $K\alpha < \psi\alpha \leq \psi(\alpha{+}1)$ we conclude $\psi\alpha < \psi(\alpha{+}1)$ by (b).

(d) By (a) we have $\gamma := \sup_{\xi<\alpha}\psi\xi \leq \psi\alpha$. Assume $\gamma < \psi\alpha$. Then $\gamma \in \mathbb{X} \cap \psi\alpha$, and therefore by definition of $\psi\alpha$ there exists $\xi < \alpha$ with $K\xi < \gamma \leq \psi\xi$. Hence by (c), $\exists\xi < \alpha(\gamma < \psi(\xi{+}1))$. Contradiction

(e) 1. We have $\psi\alpha \in \mathbb{X}$ and, by (a),(b), $\forall\xi < \alpha(K\xi < \psi\xi \Rightarrow \psi\xi < \psi\alpha)$.
2. Notice that $(K\xi < \psi\xi \Rightarrow \psi\xi < \gamma)$ implies $(K\xi < \gamma \Rightarrow \psi\xi < \gamma)$. Therefore, if $\gamma \in \mathbb{X}$ & $\forall\xi < \alpha(K\xi < \psi\xi \Rightarrow \psi\xi < \gamma)$ then $\gamma \in \mathbb{X}$ & $\forall\xi < \alpha(K\xi < \gamma \Rightarrow \psi\xi < \gamma)$ which yields $\psi\alpha \leq \gamma$.

Definition.

Let $\alpha \leq \Lambda$ with $K\alpha < \psi\Lambda$.

Then by Lemma 4.2d there exists $\xi < \Lambda$ such that $K\alpha < \psi\xi$, and we can define

$\tilde{\mathsf{g}}(\alpha) := \min\{\xi < \Lambda : K\alpha < \psi\xi\}$,

$\mathsf{g}(\alpha) := \tilde{\mathsf{g}}(\alpha) \dot{-} 1$, where $\beta \dot{-} 1 := \begin{cases} \beta_0 & \text{if } \beta = \beta_0{+}1 \\ \beta & \text{otherwise} \end{cases}$,

$\mathsf{h}(\alpha) := \mathsf{g}(\alpha) + \Omega^\alpha$. (Note that $\mathsf{h}(\alpha) \leq \Lambda$.)

Lemma 4.3. Assume $\alpha \leq \Lambda$ & $K\alpha < \psi\Lambda$.

(a) $\psi 0 \leq K\alpha \;\Rightarrow\; \psi\mathsf{g}(\alpha) \leq K\alpha < \psi(\mathsf{g}(\alpha){+}1)$.

(b) $K\mathsf{g}(\alpha) < \psi\mathsf{g}(\alpha)$.

(c) $K\mathsf{h}(\alpha) < \psi\mathsf{h}(\alpha)$.

(d) $\alpha_0 < \alpha$ & $K\alpha_0 < \psi\mathsf{h}(\alpha) \;\Rightarrow\; \psi\mathsf{h}(\alpha_0) < \psi\mathsf{h}(\alpha)$.

Proof :

(a) From $\psi 0 \leq K\alpha$ and Lemma 4.2d it follows that $0 < \tilde{\mathsf{g}}(\alpha) \notin Lim$.

Therefore $\tilde{\mathsf{g}}(\alpha) = \mathsf{g}(\alpha){+}1$, which yields the assertion.

(b) Follows from (a) and Lemma 4.2c.

(c) $K(\mathsf{g}(\alpha) + \Omega^\alpha) \subseteq K\mathsf{g}(\alpha) \cup K\alpha \overset{(b),(a)}{<} \psi(\mathsf{g}(\alpha) + 1) \leq \psi(\mathsf{g}(\alpha) + \Omega^\alpha)$.

(d) From $\alpha_0 < \alpha$ & $K\alpha_0 < \psi\mathsf{h}(\alpha)$ by (a) we obtain $\alpha_0 < \alpha$ & $\mathsf{g}(\alpha_0) < \mathsf{h}(\alpha) = \mathsf{g}(\alpha) + \Omega^\alpha$ and then $\mathsf{h}(\alpha_0) = \mathsf{g}(\alpha_0) + \Omega^{\alpha_0} < \mathsf{h}(\alpha)$. This together with $K\mathsf{h}(\alpha_0) < \psi\mathsf{h}(\alpha_0)$ (cf. (c)) yields $\psi\mathsf{h}(\alpha_0) < \psi\mathsf{h}(\alpha)$ by Lemma 4.2a,b.

Theorem 4.4. $\alpha \le \Lambda \;\&\; K\alpha < \psi\Lambda \;\Rightarrow\; \vartheta\alpha \le \psi\mathsf{h}(\alpha)$.

Proof by induction on α:
By Lemma 4.3a,d, $K\alpha < \psi\mathsf{h}(\alpha) \in \mathbb{X} \;\&\; \forall\xi < \alpha(K\xi < \psi\mathsf{h}(\alpha) \Rightarrow \psi\mathsf{h}(\xi) < \psi\mathsf{h}(\alpha))$.
Hence by IH, $K\alpha < \psi\mathsf{h}(\alpha) \in \mathbb{X} \;\&\; \forall\xi < \alpha(K\xi < \psi\mathsf{h}(\alpha) \Rightarrow \vartheta\xi < \psi\mathsf{h}(\alpha))$ which yields $\vartheta\alpha \le \psi\mathsf{h}(\alpha)$.

Corollary 4.5.
(a) $\alpha = \Omega^\alpha \le \Lambda \;\&\; K\alpha < \psi\alpha \;\Rightarrow\; \vartheta\alpha = \psi\alpha$.
(b) $\vartheta\varepsilon_{\Omega+1} = \psi\varepsilon_{\Omega+1} \;\&\; \vartheta\Lambda = \psi\Lambda$.

Proof :
(a) $K\alpha < \psi\alpha \;\&\; \alpha = \Omega^\alpha \;\Rightarrow\; \mathsf{g}(\alpha) < \alpha = \Omega^\alpha \;\Rightarrow\; \mathsf{h}(\alpha) = \mathsf{g}(\alpha) + \Omega^\alpha = \alpha \overset{4.4}{\Rightarrow} \vartheta\alpha \le \psi\alpha \le \vartheta\alpha$.
(b) are instances of (a).

Note. In the appendix of [Btz13] it is shown that $\psi^{\mathrm{SC}}\Lambda$ equals Bachmann's $\varphi_{F_{\omega_2+1}(1)}(1)$. In the present context this equation can be derived as follows
$\psi^{\mathrm{SC}}\Lambda \overset{\text{Cor.4.5}}{=} \vartheta^{\mathrm{SC}}\Lambda \overset{(\vartheta 0)}{=} \overline{\phi}^{\mathrm{SC}}_\Lambda(0) = \phi^{\mathrm{SC}}_\Lambda(0) \overset{\text{L.4.6}}{=} \phi^{\mathbb{H}}_\Lambda(0) =$ ‘$\varphi_{F_{\omega_2+1}(1)}(1)$’.

Lemma 4.6.
(a) $K\gamma = \emptyset \;\&\; \mathbb{Y} \cap \Omega = R^{\mathbb{X}}_\gamma \;\Rightarrow\; \phi^{\mathbb{Y}}_\alpha = \phi^{\mathbb{X}}_{\gamma+\alpha}$.
(b) $\mathrm{SC} \cap \Omega = R^{\mathbb{H}}_\Omega$.

Proof :
(a) Induction on α using Theorem 2.4 and the fact that $K\gamma = \emptyset$ implies $K(\gamma + \alpha) = K\alpha$ and $\mathsf{k}^+(\gamma + \alpha) = \mathsf{k}^+(\alpha)$ for all α.
(b) By definition we have $\forall\alpha < \Omega(\phi^{\mathbb{H}}_\alpha = \varphi_\alpha)$, which together with Lemma 1.5d yields $\mathrm{SC} \cap \Omega = \{\alpha \in \Omega : \phi^{\mathbb{H}}_\alpha(0) = \alpha\} = R^{\mathbb{H}}_\Omega$.

Corollary 4.7.
(i) $K\gamma = \emptyset \;\&\; \mathbb{Y} \cap \Omega = R^{\mathbb{X}}_\gamma \;\Rightarrow\; \overline{\phi}^{\mathbb{Y}}_\alpha = \overline{\phi}^{\mathbb{X}}_{\gamma+\alpha}$ and $\vartheta^{\mathbb{Y}}\alpha = \vartheta^{\mathbb{X}}(\Omega\gamma + \alpha)$.
(ii) $\phi^{\mathbb{E}}_\alpha = \phi^{\mathbb{H}}_{1+\alpha}$, $\phi^{\mathrm{SC}}_\alpha = \phi^{\mathbb{H}}_{\Omega+\alpha}$, $\vartheta^{\mathbb{E}}\alpha = \vartheta^{\mathbb{H}}(\Omega + \alpha)$, and
$\vartheta^{\mathrm{SC}}\alpha = \vartheta^{\mathbb{H}}(\Omega^2 + \alpha) = \vartheta^{\mathbb{E}}(\Omega^2 + \alpha)$.

Proof :
(i) follows from Lemma 4.6a by Theorem 2.5b and $(\vartheta 0)$.
(ii) follows from Lemma 4.6, (i), and $\mathbb{E} \cap \Omega = R^{\mathbb{H}}_1$.

§5 Exact comparison of ϑ and ψ

Let $\mathbb{X} \subseteq \mathbb{H}$ be fixed such that $\mathbb{X} \cap \Omega$ is Ω-club.
As before we write ϑ, ψ for $\vartheta^{\mathbb{X}}, \psi^{\mathbb{X}}$.

In this section we always assume $\alpha < \Lambda$ and $K\alpha \cup \{\beta\} < \psi\Lambda$.

Lemma 5.1.
(a) $\alpha_0 < \alpha \;\&\; \forall\xi(\alpha_0 \le \xi < \alpha \to \psi\xi = \psi(\xi{+}1)) \Rightarrow \psi\alpha_0 = \psi\alpha$.
(b) $\psi\alpha_0 < \psi\alpha \;\Rightarrow\; \exists\alpha_1(\alpha_0 \le \alpha_1 < \alpha \;\&\; K\alpha_1 < \psi\alpha_1 = \psi\alpha_0)$.

Proof :

(a) follows from Lemma 4.2a,d by induction on α.

(b) From $\psi\alpha_0 < \psi\alpha$ by Lemmata 4.2a, 5.1a we obtain
$\exists\xi(\alpha_0 \le \xi < \alpha \;\&\; \psi\xi < \psi(\xi{+}1))$. Let $\alpha_1 := \min\{\xi \ge \alpha_0 : \psi\xi < \psi(\xi{+}1)\}$.
Then $\alpha_0 \le \alpha_1 < \alpha$ and, by (a) and Lemma 4.2c, $K\alpha_1 < \psi\alpha_1 = \psi\alpha_0$.

Lemma 5.2.
(a) $\psi\alpha < \gamma \in \mathbb{X} \;\Rightarrow\; \psi(\alpha{+}1) \le \gamma$.
(b) $\gamma \in \mathbb{X} \cap \psi\Lambda \;\Rightarrow\; \exists\alpha(K\alpha < \psi\alpha = \gamma)$.
(c) $\Omega^\alpha|\gamma \;\&\; \delta < \Omega^\alpha \;\&\; K(\gamma+\delta) < \psi(\gamma+\delta) \;\Rightarrow\; K\gamma < \psi\gamma$.
(d) $\Omega^\alpha|\gamma \;\&\; \psi\gamma < \psi(\gamma+\Omega^\alpha) \;\Rightarrow\; K\gamma < \psi\gamma$.

Proof :

(a) $\mathbb{X} \ni \gamma < \psi(\alpha+1) \;\Rightarrow\; \exists\xi < \alpha{+}1(K\xi < \gamma \le \psi\xi) \;\Rightarrow\; \gamma \le \psi\alpha$.

(b) By Lemma 4.2a,d it follows that $\psi\alpha \le \gamma < \psi(\alpha{+}1)$ for some $\alpha < \Lambda$.
By (a) it follows that $\psi\alpha = \gamma$.

(c) Induction on δ: Since $\Omega^\alpha|\gamma \;\&\; \delta < \Omega^\alpha$ we have $K\gamma \subseteq K(\gamma+\delta)$. Therefore, if $\psi\gamma = \psi(\gamma+\delta)$ then $K\gamma < \psi\gamma$. If $\psi\gamma < \psi(\gamma+\delta)$, then by Lemma 5.1b there exists $\delta_0 < \delta$ such that $K(\gamma+\delta_0) < \psi(\gamma+\delta_0)$; thence, by IH, $K\gamma < \psi\gamma$.

(d) By Lemma 5.1b there exists $\delta < \Omega^\alpha$ such that $K(\gamma+\delta) < \psi(\gamma+\delta)$.
Hence $K\gamma < \psi\gamma$ by (c).

Lemma 5.3. $\delta =_{\mathrm{NF}} \gamma + \Omega^\alpha\xi \;\;\&\;\; K(\Omega^\alpha\xi) < \psi(\gamma+\Omega^{\alpha+1}) \;\;\Rightarrow\;\; K(\Omega^\alpha\xi) < \psi\delta$.

Proof :

For $\psi\delta = \psi(\gamma+\Omega^{\alpha+1})$ the claim is trivial. Otherwise, by Lemma 5.1b there exists δ_1 with $\delta \le \delta_1 < \gamma+\Omega^{\alpha+1}$ and $K\delta_1 < \psi\delta_1 = \psi\delta$. Then $\delta_1 = \gamma+\Omega^\alpha\beta+\delta_2$ with $\xi \le \beta < \Omega$ and $\delta_2 < \Omega^\alpha$. Hence $K(\Omega^\alpha\beta) \subseteq K\delta_1 < \psi\delta$. Now assume $\beta > 0$. Then $K\alpha \cup K\beta = K(\Omega^\alpha\beta) < \psi\delta$ which together with $\xi \le \beta < \Omega$ yields $K(\Omega^\alpha\xi) \subseteq K\alpha \cup K\xi < \psi\delta$.

Definitions.
1. $\dot{\psi}\alpha := \begin{cases} 0 & \text{if } \alpha = 0 \\ \psi\alpha & \text{if } \alpha > 0 \end{cases}$

2. If $\alpha \le \beta$ then $-\alpha+\beta$ denotes the unique γ such that $\alpha+\gamma = \beta$.

The following definition is an extension and modification of the corresponding definition on p.26 of [BS88].

Definition of $[\alpha,\beta] < \Lambda$

By Lemma 4.2a,d there exists $\eta < \Lambda$ such that
$\dot\psi(\Omega^{\alpha+1}\eta) \le K\alpha \cup \{\beta\} < \psi(\Omega^{\alpha+1}(\eta{+}1))$.

Let $\gamma := \Omega^{\alpha+1}\eta$. Then $\dot\psi\gamma \le K\alpha \cup \{\beta\} < \psi(\gamma + \Omega^{\alpha+1})$.

If $\Omega\alpha + \beta < \omega$ then $[\alpha,\beta] := \beta$, else
$[\alpha,\beta] := \gamma + \Omega^\alpha(1{+}\xi)$ with $\xi := \begin{cases} -\dot\psi\gamma + \beta & \text{if } K\alpha < \dot\psi\gamma \\ \beta & \text{otherwise} \end{cases}$

Remark. $\omega \le \Omega\alpha + \beta \;\Rightarrow\; \omega \le [\alpha,\beta]$.

Lemma 5.4. (a) $K[\alpha,\beta] < \psi[\alpha,\beta]$; (b) $K(\Omega\alpha+\beta) < \psi[\alpha,\beta]$.

Proof :
Assume $\omega \le \Omega\alpha + \beta$ (otherwise $K[\alpha,\beta] = \emptyset$ and $K(\Omega\alpha+\beta) = \emptyset$). Then $[\alpha,\beta] =_{\mathrm{NF}} \gamma{+}\Omega^\alpha(1{+}\xi)$ with $\psi\gamma \le K\alpha{\cup}\{\beta\} < \psi(\gamma{+}\Omega^{\alpha+1})$ and $\xi \le \beta \le \dot\psi\gamma{+}\xi$.

(a) By Lemmata 5.2d and 4.2a,b we obtain $K\gamma < \psi\gamma < \psi[\alpha,\beta]$.
$K(\Omega^\alpha(1{+}\xi)) = K\alpha \cup K\xi \;\&\; \xi \le \beta < \Omega \;\&\; K\alpha \cup \{\beta\} < \psi(\gamma + \Omega^{\alpha+1}) \;\Rightarrow$ $K(\Omega^\alpha(1{+}\xi)) < \psi(\gamma + \Omega^{\alpha+1})$.
$[\alpha,\beta] =_{\mathrm{NF}} \gamma + \Omega^\alpha(1{+}\xi) \;\&\; K(\Omega^\alpha(1{+}\xi)) < \psi(\gamma + \Omega^{\alpha+1}) \overset{5.3}{\Rightarrow}$
$K(\Omega^\alpha(1{+}\xi)) < \psi[\alpha,\beta]$.

(b) By (proof of) (a) we have $K\alpha \cup \{\xi\} \subseteq K[\alpha,\beta] < \psi[\alpha,\beta]$ and $\psi\gamma < \psi[\alpha,\beta]$. From this together with $\beta \le \dot\psi\gamma{+}\xi$ and $\psi[\alpha,\beta] \in \mathbb{X} \subseteq \mathbb{H}$, we obtain $K(\Omega\alpha{+}\beta) = K\alpha \cup \{\beta\} < \psi[\alpha,\beta]$.

Lemma 5.5.
$\Omega\alpha_0 + \beta_0 < \Omega\alpha_1 + \beta_1 \;\&\; K(\Omega\alpha_0 + \beta_0) < \psi[\alpha_1,\beta_1] \;\Rightarrow\; [\alpha_0,\beta_0] < [\alpha_1,\beta_1]$.

Proof :
1. $\Omega\alpha_1 + \beta_1 < \omega$: Then $[\alpha_0,\beta_0] = \beta_0 < \beta_1 = [\alpha_1,\beta_1]$.
2. $\Omega\alpha_0 + \beta_0 < \omega \le \Omega\alpha_1 + \beta_1$: Then $[\alpha_0,\beta_0] = \beta_0 < \omega \le [\alpha_1,\beta_1]$.
3. $\omega \le \Omega\alpha_0 + \beta_0$: Then $[\alpha_i,\beta_i] =_{\mathrm{NF}} \gamma_i + \Omega^{\alpha_i}(1+\xi_i)$ $(i = 0,1)$, and
$\dot\psi\gamma_0 \le K\alpha_0 \cup \{\beta_0\} < \psi[\alpha_1,\beta_1]$.

3.1. $\alpha := \alpha_0 = \alpha_1 \;\&\; \beta_0 < \beta_1$:
3.1.1. $\gamma_0 < \gamma_1$: Then $[\alpha_0,\beta_0] = \gamma_0 + \Omega^\alpha(1{+}\xi_0) < \gamma_0 + \Omega^{\alpha+1} \le \gamma_1 \le [\alpha_1,\beta_1]$.
3.1.2. $\gamma := \gamma_0 = \gamma_1$: To prove $\xi_0 < \xi_1$. We have $\xi_i = \begin{cases} -\dot\psi\gamma + \beta_i & \text{if } K\alpha < \dot\psi\gamma \\ \beta_i & \text{otherwise} \end{cases}$.
Hence $\xi_0 < \xi_1$ follows from $\beta_0 < \beta_1$.

3.2. $\alpha_0 < \alpha_1$: From $\dot\psi\gamma_0 < \psi[\alpha_1,\beta_1]$ and $0 < \alpha_1$ we get $\gamma_0 < [\alpha_1,\beta_1] = \gamma_1 + \Omega^{\alpha_1}(1{+}\xi_1)$, and then $\gamma_0 + \Omega^{\alpha_1} \le [\alpha_1,\beta_1]$. Further we have $[\alpha_0,\beta_0] = \gamma_0 + \Omega^{\alpha_0}(1{+}\xi_0) < \gamma_0 + \Omega^{\alpha_0+1} \le \gamma_0 + \Omega^{\alpha_1}$.

Lemma 5.6. $\vartheta(\Omega\alpha+\beta) \le \psi([\alpha,\beta]) < \psi\Lambda$.

Proof by induction on $\Omega\alpha+\beta$:

Let $\gamma_0 := \psi[\alpha,\beta]$.

To prove: $\gamma_0 \in \mathbb{X}$ & $K(\Omega\alpha+\beta) < \gamma_0$ & $\forall\zeta < \Omega\alpha+\beta(K\zeta < \gamma_0 \Rightarrow \vartheta\zeta < \gamma_0)$.

1. By definition of ψ and Lemma 5.4b we have $\gamma_0 \in \mathbb{X}$ & $K(\Omega\alpha+\beta) < \gamma_0$.

2. Assume $\Omega\xi+\eta < \Omega\alpha+\beta$ & $K(\Omega\xi+\eta) < \gamma_0$. Then, by L.5.5, $[\xi,\eta] < [\alpha,\beta]$. From this by Lemmata 5.4a, 4.2a,b and the IH we obtain $\vartheta(\Omega\xi+\eta) \le \psi[\xi,\eta] < \psi[\alpha,\beta] = \gamma_0$.

Definition of $\overline{\delta} < \Lambda$ for $\delta < \Lambda$ [3]

1. If $\delta < \omega$ then $\overline{\delta} := \delta$.

2. If $\omega \le \delta =_{\mathrm{NF}} \gamma+\Omega^\alpha(1+\xi)$ then $\overline{\delta} := \Omega\alpha+\beta$ with $\beta := \begin{cases} \dot{\psi}\gamma+\xi & \text{if } K\alpha < \dot{\psi}\gamma \\ \xi & \text{otherwise} \end{cases}$

Remark. $\omega \le \delta \;\Rightarrow\; \omega \le \overline{\delta}$.

Lemma 5.7. $\overline{[\alpha,\beta]} = \Omega\alpha+\beta$.

Proof :

$[\alpha,\beta] =_{\mathrm{NF}} \gamma+\Omega^\alpha(1+\xi)$ with $\xi = \begin{cases} -\dot{\psi}\gamma+\beta & \text{if } K\alpha < \dot{\psi}\gamma \\ \beta & \text{otherwise} \end{cases}$

Hence $\overline{[\alpha,\beta]} = \Omega\alpha+\tilde{\beta}$ with $\tilde{\beta} := \begin{cases} \dot{\psi}\gamma+\xi & \text{if } K\alpha < \dot{\psi}\gamma \\ \xi & \text{otherwise} \end{cases}$. Obviously $\tilde{\beta} = \beta$.

Lemma 5.8. Let $\delta,\delta' < \Lambda$.

(a) $K\delta < \psi\delta$ & $\overline{\delta} = \Omega\alpha+\beta \;\Rightarrow\; \delta = [\alpha,\beta]$.

(b) $K\delta < \psi\delta \;\Rightarrow\; \vartheta\overline{\delta} \le \psi\delta$.

(c) $K\delta < \psi\delta$ & $K\delta' < \psi\delta'$ & $\overline{\delta} = \overline{\delta'} \;\Rightarrow\; \delta = \delta'$.

Proof :

(a) 1. $\delta < \omega$: Then $\Omega\alpha+\beta = \overline{\delta} = \delta < \omega$ and thus $[\alpha,\beta] = \beta = \delta$.

2. Otherwise: Then $\omega \le \delta =_{\mathrm{NF}} \gamma+\Omega^\alpha(1+\xi)$ with $\beta = \begin{cases} \dot{\psi}\gamma+\xi & \text{if } K\alpha < \dot{\psi}\gamma \\ \xi & \text{otherwise} \end{cases}$.

The latter yields $\dot{\psi}\gamma \le K\alpha \cup \{\beta\}$. From $K\delta < \psi\delta$ by Lemma 5.2c we get $K\gamma < \psi\gamma$ and then $\psi\gamma < \psi\delta$. Now we have $K\alpha \cup K\xi \subseteq K\delta < \psi\delta \in \mathbb{H}$ & $\psi\gamma < \psi\delta$ which implies $K\alpha \cup \{\beta\} < \psi\delta \le \psi(\gamma+\Omega^{\alpha+1})$.

It follows that $[\alpha,\beta] = \gamma+\Omega^\alpha(1+\tilde{\xi})$ where $\tilde{\xi} := \begin{cases} -\dot{\psi}\gamma+\beta & \text{if } K\alpha < \dot{\psi}\gamma \\ \beta & \text{otherwise} \end{cases}$.

Obviously $\tilde{\xi} = \xi$ and therefore $[\alpha,\beta] = \delta$.

(b) Take α,β such that $\overline{\delta} = \Omega\alpha+\beta$. Then by Lemma 5.6 and (a) we obtain $\vartheta\overline{\delta} = \vartheta(\Omega\alpha+\beta) \le \psi[\alpha,\beta] = \psi\delta$.

[3] This definition is closely related to clause 5 in Definition 3.6 of [RW93]. But be aware that $\overline{\delta}$ there has a different meaning than here.

(c) By (a) there are $\alpha, \beta, \alpha', \beta'$ such that
$\overline{\delta} = \Omega\alpha + \beta$ & $\delta = [\alpha, \beta]$ and $\overline{\delta}' = \Omega\alpha' + \beta'$ & $\delta' = [\alpha', \beta']$.
Therefore from $\overline{\delta} = \overline{\delta'}$ one concludes $\alpha = \alpha'$ & $\beta = \beta'$ and then $\delta = \delta'$.

Theorem 5.9. $\delta < \Lambda$ & $K\delta < \psi\delta \;\Rightarrow\; \vartheta\overline{\delta} = \psi\delta$.

Proof by induction on δ:
By Lemma 5.8b we have $\vartheta\overline{\delta} \le \psi\delta$. Assumption: $\vartheta\overline{\delta} < \psi\delta$. Then by Lemma 5.2b there exists γ s.t. $K\gamma < \psi\gamma = \vartheta\overline{\delta} < \psi\delta$. Hence $\gamma < \delta$ and therefore, by IH, $\psi\gamma = \vartheta\overline{\gamma}$. From $\vartheta\overline{\delta} = \psi\gamma = \vartheta\overline{\gamma}$ & $K\delta < \psi\delta$ & $K\gamma < \psi\gamma$ by (ϑ4) and Lemma 5.8c we obtain $\delta = \gamma$. Contradiction.

Corollary 5.10.
(a) $\vartheta(\Omega\alpha + \beta) = \psi[\alpha, \beta]$.
(b) $K\alpha < \psi\Omega^{\alpha+1} \Rightarrow \vartheta(\Omega\alpha) = \psi\Omega^{\alpha}$.

Proof :
(a) Let $\delta := [\alpha, \beta]$. Then by L.5.4a $K\delta < \psi\delta$, and therefore $\vartheta(\Omega\alpha + \beta) \overset{\text{L.5.7}}{=} \vartheta\overline{\delta} = \psi\delta = \psi[\alpha, \beta]$.
(b) $\alpha < \Lambda$ & $K\alpha < \psi\Omega^{\alpha+1} \;\Rightarrow\; \vartheta(\Omega\alpha) = \psi[\alpha, 0] = \psi(\Omega^{\alpha}(1 + 0)) = \psi\Omega^{\alpha}$.

§6 Defining the Bachmann hierarchy by functionals of higher type

This section is based on [Wey76, (3.2.9)-(3.2.11),(3.2.15)].

Convention. n ranges over natural numbers ≥ 1.

Definition. Let M be an arbitrary nonempty set.
1. $\mathsf{M}^1 := \mathsf{M}$. 2. $\mathsf{M}^{n+1} :=$ set of all functions $F : \mathsf{M}^n \to \mathsf{M}^n$.

Notation. If $1 \le m < n$ and $F_i \in \mathsf{M}^i$ for $m \le i \le n$,
then $F_n F_{n-1} \dots F_m := F_n(F_{n-1}) \dots (F_m)$.

Abbreviation: $\mathrm{Id}^{n+1} := \mathrm{Id}_{\mathsf{M}^n} \in \mathsf{M}^{n+1}$.

Assumption.
∇ is an operation such that for every family $(X_\xi)_{\xi<\alpha}$ with $0 < \alpha \le \Omega$ the following holds: $\forall \xi < \alpha (X_\xi \in \mathsf{M}^1) \;\Rightarrow\; \nabla_{\xi<\alpha} X_\xi \in \mathsf{M}^1$.

Definition. If $n > 1$ and $\forall \xi < \alpha (F_\xi \in \mathsf{M}^{n+1})$ then
$\nabla_{\xi<\alpha} F_\xi \in \mathsf{M}^{n+1}$ is defined by $(\nabla_{\xi<\alpha} F_\xi) G := \nabla_{\xi<\alpha}(F_\xi G)$.

Lemma 6.1. If $0 < \alpha \le \Omega$ & $\forall \xi < \alpha (F_\xi \in \mathsf{M}^{n+1})$ & $H \in \mathsf{M}^{n+1}$, then
$(\nabla_{\xi<\alpha} F_\xi) \circ H = \nabla_{\xi<\alpha}(F_\xi \circ H)$.

Proof :
For each $G \in \mathsf{M}^n$ we have

$((\nabla_{\xi<\alpha}F_\xi)\circ H)G = (\nabla_{\xi<\alpha}F_\xi)(HG) = \nabla_{\xi<\alpha}(F_\xi(HG)) = \nabla_{\xi<\alpha}((F_\xi\circ H)G) = (\nabla_{\xi<\alpha}(F_\xi\circ H))G$.

Definition. For $F \in \mathsf{M}^{n+1}$ and $\alpha \leq \Omega$ we define $F^{(\alpha)} \in \mathsf{M}^{n+1}$ by
(i) $F^{(0)} := \mathrm{Id}^{n+1}$; (ii) $F^{(\alpha+1)} := F\circ F^{(\alpha)}$; (iii) $F^{(\alpha)} := \nabla_{\xi<\alpha}F^{(1+\xi)}$ if $\alpha \in Lim$.

Definition.
(i) Let $\mathrm{I}_2 \in \mathsf{M}^2$ be given;
(ii) For $m \geq 2$ we define $\mathrm{I}_{m+1} \in \mathsf{M}^{m+1}$ by $\mathrm{I}_{m+1}F := F^{(\Omega)}$.

Definition of $[\![\alpha]\!]_m$
For $m \geq 2$ and $\alpha < \varepsilon_{\Omega+1}$ we define $[\![\alpha]\!]_m \in \mathsf{M}^m$ by recursion on α:
(i) $[\![0]\!]_m := \mathrm{Id}^m$; (ii) If $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta\eta$, then $[\![\alpha]\!]_m := ([\![\beta]\!]_{m+1}\mathrm{I}_m)^{(\eta)} \circ [\![\gamma]\!]_m$.

Lemma 6.2. For $m \geq 2$ and $\alpha < \varepsilon_{\Omega+1}$:
(a) $[\![\alpha{+}1]\!]_m = \mathrm{I}_m \circ [\![\alpha]\!]_m$;
(b) $\alpha \in Lim \;\Rightarrow\; [\![\alpha]\!]_m = \nabla_{\xi<\tau(\alpha)}[\![\alpha[\xi]]\!]_m$.

Proof :
(a) $[\![\gamma{+}\Omega^0(\eta{+}1)]\!]_m = ([\![0]\!]_{m+1}\mathrm{I}_m)^{(\eta+1)}\circ[\![\gamma]\!]_m = \mathrm{I}_m^{(\eta+1)}\circ[\![\gamma]\!]_m = \mathrm{I}_m\circ(\mathrm{I}_m^{(\eta)}\circ[\![\gamma]\!]_m) = \mathrm{I}_m \circ [\![\gamma + \Omega^0{\cdot}\eta]\!]_m$.

(b) Induction on α:
1. $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta\eta$ with $\eta \in Lim$: Then $\tau(\alpha) = \eta$ and $\alpha[\xi] = \gamma + \Omega^\beta(1{+}\xi)$.
$[\![\alpha]\!]_m = ([\![\beta]\!]_{m+1}\mathrm{I}_m)^{(\eta)} \circ [\![\gamma]\!]_m =$
$(\nabla_{\xi<\eta}([\![\beta]\!]_{m+1}\mathrm{I}_m)^{(1+\xi)}) \circ [\![\gamma]\!]_m =$
$\nabla_{\xi<\eta}(([\![\beta]\!]_{m+1}\mathrm{I}_m)^{(1+\xi)} \circ [\![\gamma]\!]_m) = \nabla_{\xi<\eta}[\![\alpha[\xi]]\!]_m$.
2. $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta(\eta{+}1)$ with $\beta = \beta_0{+}1$:
Then $\tau(\alpha) = \Omega$ and $\alpha[\xi] = \gamma + \Omega^\beta\eta + \Omega^{\beta_0}(1{+}\xi)$.
$[\![\alpha]\!]_m = ([\![\beta]\!]_{m+1}\mathrm{I}_m)^{(\eta+1)} \circ [\![\gamma]\!]_m =$
$([\![\beta_0{+}1]\!]_{m+1}\mathrm{I}_m) \circ ([\![\beta]\!]_{m+1}\mathrm{I}_m)^{(\eta)} \circ [\![\gamma]\!]_m \overset{\text{(a)}}{=}$
$(\mathrm{I}_{m+1}([\![\beta_0]\!]_{m+1}\mathrm{I}_m)) \circ [\![\gamma + \Omega^\beta\eta]\!]_m =$
$(\nabla_{\xi<\Omega}([\![\beta_0]\!]_{m+1}\mathrm{I}_m)^{(1+\xi)}) \circ [\![\gamma + \Omega^\beta\eta]\!]_m =$
$\nabla_{\xi<\Omega}(([\![\beta_0]\!]_{m+1}\mathrm{I}_m)^{(1+\xi)} \circ [\![\gamma + \Omega^\beta\eta]\!]_m) = \nabla_{\xi<\Omega}[\![\alpha[\xi]]\!]_m$.
3. $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta(\eta{+}1)$ with $\beta \in Lim$:
Then $\tau(\alpha) = \tau(\beta)$ and $\alpha[\xi] = \gamma + \Omega^\beta\eta + \Omega^{\beta[\xi]}$.
$[\![\alpha]\!]_m = ([\![\beta]\!]_{m+1}\mathrm{I}_m)^{(\eta+1)} \circ [\![\gamma]\!]_m =$
$([\![\beta]\!]_{m+1}\mathrm{I}_m) \circ ([\![\beta]\!]_{m+1}\mathrm{I}_m)^{(\eta)} \circ [\![\gamma]\!]_m =$
$([\![\beta]\!]_{m+1}\mathrm{I}_m) \circ [\![\gamma + \Omega^\beta\eta]\!]_m \overset{\text{IH}}{=}$
$(\nabla_{\xi<\tau(\beta)}([\![\beta[\xi]]\!]_{m+1}\mathrm{I}_m)) \circ [\![\gamma + \Omega^\beta\eta]\!]_m =$
$\nabla_{\xi<\tau(\beta)}(([\![\beta[\xi]]\!]_{m+1}\mathrm{I}_m) \circ [\![\gamma + \Omega^\beta\eta]\!]_m) = \nabla_{\xi<\tau(\alpha)}[\![\alpha[\xi]]\!]_m$.

Corollary 6.3. For $X \in \mathsf{M}^1$ and $\alpha < \varepsilon_{\Omega+1}$ the following holds:

(i) $[\![0]\!]_2 X = X$;

(ii) $[\![\alpha{+}1]\!]_2 X = \mathrm{I}_2([\![\alpha]\!]_2 X)$;

(iii) $[\![\alpha]\!]_2 X = \nabla_{\xi<\tau(\alpha)}([\![\alpha[\xi]]\!]_2 X)$ if $\alpha \in Lim$.

Now we fix M, I_2 and ∇ as follows:

1. $\mathsf{M} :=$ set of all Ω-club subsets of Ω.

2. $\mathrm{I}_2 : \mathsf{M} \to \mathsf{M}$, $I_2(X) := \{\beta \in \Omega : \mathrm{en}_X(\beta) = \beta\}$,
   where en_X is the ordering function of X.

3. If $\forall \xi < \alpha (X_\xi \in \mathsf{M})$ then
   $$\nabla_{\xi<\alpha} X_\xi := \begin{cases} \bigcap_{\xi<\alpha} X_\xi & \text{if } \alpha < \Omega \\ \{\beta \in \Omega \cap Lim : \beta \in \bigcap_{\xi<\beta} X_\xi\} & \text{if } \alpha = \Omega \end{cases}$$

Then by transfinite induction on α from the above Corollary and the definition of R^X_α we conclude

Theorem 6.4. $R^X_\alpha = [\![\alpha]\!]_2 X$, for all $\alpha < \varepsilon_{\Omega+1}$ and $X \in \mathsf{M}$.

Appendix

This appendix is devoted to the proof of Lemmata 1.1, 1.2d.

Lemma A1.

(a) $\lambda \in Lim \;\Rightarrow\; 0 < \lambda[0]$.

(b) $\gamma + \Omega^\beta < \Omega^\alpha \;\&\; \eta < \Omega \;\Rightarrow\; \gamma + \Omega^\beta \eta < \Omega^\alpha$.

(c) $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta \eta \notin \mathrm{ran}(F_0) \;\&\; \Omega^\alpha < \lambda \;\Rightarrow\; \Omega^\alpha \le \lambda[0]$.

Proof of (c):

From $\Omega^\alpha < \lambda = \gamma + \Omega^\beta \eta$ by (b) we get $\Omega^\alpha \le \gamma + \Omega^\beta$. If $\eta \in Lim$ then $\lambda[0] = \gamma + \Omega^\beta$. If $1 < \eta = \eta_0{+}1$ then $\gamma + \Omega^\beta \le \gamma + \Omega^\beta \eta_0 \le \lambda[0]$. If $\eta = 1$ then $0 < \gamma$ (since $\lambda \notin \mathrm{ran}(F_0)$) and therefore $\Omega^{\beta+1} \le \gamma$ which together with $\Omega^\alpha < \lambda = \gamma + \Omega^\beta$ yields $\Omega^\alpha \le \gamma \le \lambda[0]$.

Lemma A2. $\lambda =_{\mathrm{NF}} F_\alpha(\beta) \;\&\; 0 < \beta \;\Rightarrow\; F_\alpha(\beta[n]) \le \lambda[n]$.

Proof :

1. $\beta \in Lim$: $F_\alpha(\beta[n]) = \lambda[n]$.

2. $\beta = \beta_0{+}1$:

2.1. $\alpha = 0$: $F_\alpha(\beta[n]) = \Omega^{\beta_0} \le \Omega^{\beta_0} \cdot (1{+}n) = \lambda[n]$.

2.2. $\alpha > 0$: $F_\alpha(\beta[n]) = F_\alpha(\beta_0) < \lambda^- \le \lambda[n]$.

Lemma A3. $F_\zeta(\mu) < \lambda \le F_\zeta(\mu{+}1) \;\Rightarrow\; F_\zeta(\mu) \le \lambda[0]$.

Proof :

0. $\lambda = F_\zeta(\mu+1)$:

0.1. $\zeta = 0$: $\lambda = \Omega^{\mu+1}$, $\lambda[\xi] = \Omega^\mu(1{+}\xi)$, $\lambda[0] = F_0(\mu)$.

0.2. $\zeta > 0$: $F_\zeta(\mu) < \lambda^- < F_{\zeta[0]}(\lambda^-) = \lambda[0]$.

1. $\lambda < F_\zeta(\mu+1)$:

1.1. $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$:

$F_\zeta(\mu) \in \mathrm{ran}(F_0)$ & $F_\zeta(\mu) < \lambda \overset{\text{L.A1c}}{\Rightarrow} F_\zeta(\mu) \le \lambda[0]$.

1.2. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$: Then $\alpha < \zeta$ and thus $F_\alpha(F_\zeta(\mu)) = F_\zeta(\mu) < F_\alpha(\beta)$. Hence $F_\zeta(\mu) < \beta$ and therefore $F_\zeta(\mu) \overset{\text{IH}}{\le} \beta[0] \le F_\alpha(\beta[0]) \overset{\text{A2}}{\le} \lambda[0]$.

Definition. $r(\gamma) := \begin{cases} -1 & \text{if } \gamma \notin \mathrm{ran}(F_0) \\ \alpha & \text{if } \gamma =_{\mathrm{NF}} F_\alpha(\beta) \\ \gamma & \text{if } \gamma = \Lambda \end{cases}$

Lemma A4.

(a) $r(F_\alpha(\beta)) = \max\{\alpha, r(\beta)\}$.

(b) $\lambda[0] < \delta < \lambda \;\Rightarrow\; r(\delta) \le r(\lambda)$.

(c) $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$ & $\beta \notin Lim$ & $\lambda^- < \eta < \lambda \;\Rightarrow\; \lambda^- \le \eta[1]$.

Proof :

(a) 1. $\beta < F_\alpha(\beta)$:

Then $r(F_\alpha(\beta)) = \alpha$ and ($r(\beta) = -1$ or $\beta =_{\mathrm{NF}} F_{\beta_0}(\beta_1)$ with $\beta_0 \le \alpha$).

2. $\beta = F_\alpha(\beta)$: Then $\beta =_{\mathrm{NF}} F_{\beta_0}(\beta_1)$ with $\alpha < \beta_0 = r(\beta) = r(F_\alpha(\beta))$.

(b) 1. $\lambda =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$:

1.1. $\eta \in Lim$: $\gamma + \Omega^\beta = \lambda[0] < \delta < \gamma + \Omega^\beta\eta \overset{\text{A1b}}{\Rightarrow} \delta \notin \mathrm{ran}(F_0)$.

1.2. $\eta = \eta_0{+}1$:

$\gamma + \Omega^\beta\eta_0 < \lambda[0] < \delta < \gamma + \Omega^\beta(\eta_0{+}1) \notin \mathrm{ran}(F_0)$ & $\Omega^{\beta+1}|\gamma \;\Rightarrow\; \delta \notin \mathrm{ran}(F_0)$.

2. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$: If $\lambda < F_{\alpha+1}(0)$ then also $\delta < F_{\alpha+1}(0)$ and thus $r(\delta) \le \alpha = r(\lambda)$. Otherwise there exists μ such that $F_{\alpha+1}(\mu) < \lambda < F_{\alpha+1}(\mu{+}1)$. Then by L.A3 we get $F_{\alpha+1}(\mu) \le \lambda[0] < \delta < F_{\alpha+1}(\mu{+}1)$ and thus $\delta \notin \mathrm{ran}(F_{\alpha+1})$, i.e. $r(\delta) \le \alpha = r(\lambda)$.

3. $\lambda = \Lambda$: $r(\delta) < \Lambda = r(\Lambda)$.

(c) For $\beta = 0 \vee \eta = \eta_0{+}1$ the claim is trivial. Assume now $\beta = \beta_0{+}1$ & $\eta \in Lim$. $F_\alpha(\beta_0) < \eta < F_\alpha(\beta_0{+}1) \overset{\text{L.A3}}{\Rightarrow} \lambda^- = F_\alpha(\beta_0){+}1 \le \eta[0] + 1 \le \eta[1]$.

Lemma 1.1. $\lambda =_{\mathrm{NF}} F_\alpha(\beta)$ & $\beta \in Lim$ & $1 \le \xi < \tau_\beta \;\Rightarrow\; \lambda[\xi] =_{\mathrm{NF}} F_\alpha(\beta[\xi])$.

Proof :

We have $\lambda[\xi] = F_\alpha(\beta[\xi])$ & $\beta[0] < \beta[\xi] < \beta$. By L.A4b this yields $\lambda[\xi] = F_\alpha(\beta[\xi])$ & $r(\beta[\xi]) \le r(\beta) \le \alpha$, whence $\lambda[\xi] =_{\mathrm{NF}} F_\alpha(\beta[\xi])$.

Lemma 1.2d. $\xi{+}1 < \tau_\lambda \;\&\; \lambda[\xi] < \delta \le \lambda[\xi{+}1] \;\Rightarrow\; \lambda[\xi] \le \delta[1]$.

Proof by induction on $\delta\#\lambda$:

If $r(\delta) < r(\lambda[\xi])$ then, by L.A4b, $\lambda[\xi] \le \delta[0]$. (Proof: $\delta[0] < \lambda[\xi] < \delta \overset{\text{L.A4b}}{\Rightarrow} r(\lambda[\xi]) \le r(\delta)$.)

Assume now that $r(\lambda[\xi]) \le r(\delta)$ **(**$\dagger$**)**.

1. $\lambda =_{\text{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$.

1.1. $\eta \in \mathit{Lim}$:

$\gamma + \Omega^\beta(1{+}\xi) = \lambda[\xi] < \delta < \lambda[\xi{+}1] = \gamma + \Omega^\beta(1{+}\xi) + \Omega^\beta \;\Rightarrow\; \lambda[\xi] \le \delta[0]$.

1.2. $\eta = \eta_0{+}1$: $\gamma + \Omega^\beta\eta_0 + \Omega^\beta[\xi] = \lambda[\xi] < \delta \le \lambda[\xi{+}1] = \gamma + \Omega^\beta\eta_0 + \Omega^\beta[\xi{+}1] \;\Rightarrow$
$\delta = (\gamma + \Omega^\beta\eta_0) + \delta_0$ with $\Omega^\beta[\xi] < \delta_0 \le \Omega^\beta[\xi{+}1] \;\Rightarrow$
$\delta[0] = \gamma + \Omega^\beta\eta_0 + \delta_0[0]$ with $\Omega^\beta[\xi] \overset{\text{IH}}{\le} \delta_0[0] \;\Rightarrow\; \lambda[\xi] \le \delta[0]$.

2. $\lambda =_{\text{NF}} F_\alpha(\beta) \;\&\; \beta \in \mathit{Lim}$: Then (1) $\lambda[\xi] = F_\alpha(\beta[\xi])$, and (2) $\lambda[\xi] < \delta < \lambda$.
From $\alpha \overset{(1)}{\le} r(\lambda[\xi]) \overset{(\dagger)}{\le} r(\delta) \overset{(2),\text{L.A4b}}{\le} r(\lambda) = \alpha$ we get $r(\delta) = \alpha$, i.e. $\delta =_{\text{NF}} F_\alpha(\eta)$ for some η. Now from $\lambda[\xi] < \delta \le \lambda[\xi{+}1]$ we conclude $\beta[\xi] < \eta \le \beta[\xi{+}1]$ and then, by IH, $\beta[\xi] \le \eta[0]$. Hence $\lambda[\xi] \le F_\alpha(\eta[0]) \overset{\text{L.A2}}{\le} \delta[0]$.

3. $\lambda =_{\text{NF}} F_\alpha(\beta) \;\&\; \beta \notin \mathit{Lim}$:

3.1. $\alpha = 0$: Then $\beta = \beta_0{+}1$, and $\lambda[\xi] = \Omega^{\beta_0}(1{+}\xi) < \delta \le \Omega^{\beta_0}(1{+}\xi) + \Omega^{\beta_0}$ implies $\lambda[\xi] \le \delta[0]$.

3.2. $\alpha = \alpha_0{+}1$: Then $\lambda[\xi] = F^{\xi+1}_{\alpha_0}(\lambda^-)$.
Hence, by ($\dagger$), $\delta =_{\text{NF}} F_\zeta(\eta)$ with $\alpha_0 \le \zeta$.

3.2.1. $\alpha_0 < \zeta$: $\lambda^- < F_\zeta(\eta) \;\Rightarrow\; \lambda[\xi{+}1] = F^{\xi+2}_{\alpha_0}(\lambda^-) < F_\zeta(\eta)$. Contradiction.

3.2.2. $\zeta = \alpha_0$: Then from $F^{\xi+1}_{\alpha_0}(\lambda^-) = \lambda[\xi] < \delta = F_{\alpha_0}(\eta) \le \lambda[\xi{+}1]$ we conclude $F^{\xi}_{\alpha_0}(\lambda^-) < \eta \le \lambda[\xi]$. As we will show, this implies $F^{\xi}_{\alpha_0}(\lambda^-) \le \eta[1]$, thence $F^{\xi+1}_{\alpha_0}(\lambda^-) \le F_\zeta(\eta[1]) \overset{\text{L.A2}}{\le} \delta[1]$.

Proof of $F^{\xi}_{\alpha_0}(\lambda^-) \le \eta[1]$:
(i) $\xi = n{+}1$: Then the claim follows by IH from $\lambda[n] = F^{\xi}_{\alpha_0}(\lambda^-) < \eta \le \lambda[n{+}1]$.
(ii) $\xi = 0$: $\lambda^- < \eta < \lambda \overset{\text{L.A4c}}{\Rightarrow} \lambda^- \le \eta[1]$.

3.3. $\alpha \in \mathit{Lim}$: $\lambda[\xi] = F_{\alpha[\xi]}(\lambda^-)$, and by ($\dagger$) we have $\delta =_{\text{NF}} F_\zeta(\eta)$ with $\alpha[\xi] \le \zeta$.

3.3.1. $\alpha[\xi{+}1] < \zeta$: $\lambda^- < F_\zeta(\eta) \;\Rightarrow\; F_{\alpha[\xi+1]}(\lambda^-) < F_\zeta(\eta) = \delta$. Contradiction.

3.3.2. $\alpha[\xi] < \zeta \le \alpha[\xi{+}1]$:

(i) $\eta \in \mathit{Lim}$: Then $\lambda^- < \delta[1] = F_\zeta(\eta[1])$ (for $\beta = 0$, $\lambda^- = 0$. If $\beta = \beta_0{+}1$, then $F_\alpha(\beta_0) < \delta < F_\alpha(\beta_0{+}1)$ and thus, by L.A3, $F_\alpha(\beta_0) \le \delta[0]$).

$\alpha[\xi] < \zeta \;\&\; \lambda^- < \delta[1] \;\Rightarrow\; \lambda[\xi] = F_{\alpha[\xi]}(\lambda^-) < \delta[1]$.

(ii) $\eta \notin \mathit{Lim}$: By IH $\alpha[\xi] \le \zeta[1]$. Further $\lambda^- \le \delta^-$.

Proof of $\lambda^- \leq \delta^-$: Assume $\beta = \beta_0{+}1$.

$F_\alpha(\beta_0) < \delta = F_\zeta(\eta)$ & $\zeta < \alpha \Rightarrow 0 < \eta \Rightarrow \eta = \eta_0{+}1$.

$F_\alpha(\beta_0) < F_\zeta(\eta_0{+}1)$ & $\zeta < \alpha \;\Rightarrow\; F_\alpha(\beta_0) \leq F_\zeta(\eta_0)$.

From $\alpha[\xi] \leq \zeta[1]$ and $\lambda^- \leq \delta^-$ we conclude $\lambda[\xi] = F_{\alpha[\xi]}(\lambda^-) \leq F_{\zeta[1]}(\delta^-) \leq \delta[1]$.

3.3.3. $\zeta = \alpha[\xi]$: This case is similar to 3.2.2(ii):

$\lambda[\xi] = F_\zeta(\lambda^-) < F_\zeta(\eta) < F_\alpha(\beta) \;\Rightarrow\; \lambda^- < \eta < F_\alpha(\beta) \;\Rightarrow$

$\lambda[\xi] = F_\zeta(\lambda^-) \overset{\text{L.A4c}}{\leq} F_\zeta(\eta[1]) \overset{\text{L.A2}}{\leq} \delta[1]$.

4. $\lambda = \Lambda$: This case is very similar to 3.3, but considerably simpler.

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