Explaining the Gentzen-Takeuti reduction steps

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April 27, 1999

Introduction

In [Bu97] a precise explanation of Gentzen's reduction steps for derivations in 1st order arithmetic Z (cf. [Ge38]) in terms of (cut-elimination for) infinitary derivations in ω-arithmetic was given. Even more, Gentzen's reduction steps and ordinal assignment were *derived* from infinitary proof theory. In the present paper we will extend this work to an impredicative subsystem BI_1^- of 2nd order arithmetic thereby explaining Takeuti's consistency proof for Π_1^1-CA in terms of the infinitary approach (with $\Omega_{\mu+1}$-rules) from [BS88]. Our goal is to explain Takeuti's reduction steps occurring in the consistency proofs for **ISN** on pp. 320-341 of [Tak87] in the same way as we have explained Gentzen's reduction steps in [Bu97]. Actually BI_1^- is a rather weak subsystem of Π_1^1-CA (it has the strength of ID_1 only), but we claim that the results of the present paper can be extended to full Π_1^1-CA.

Contents: In §1 after some preliminaries about syntax we introduce a special kind of representing derivations in (Tait style) sequent calculi, which is particularly suited for the purposes of this paper. In §2 we introduce infinitary systems BI_0^∞ and BI_1^∞. The system BI_0^∞ is just cutfree ω-arithmetic augmented by the "Repetition Rule" (Rep). The main feature of the system BI_1^∞ (which extends BI_0^∞) is the Ω_1-rule, which is an infinitary rule of "higher order", since it has uncountably infinite "arity" (the premises of an Ω_1-inference form a family $(\Gamma_\iota)_{\iota\in I}$ where I is (essentially) the set of all BI_0^∞-derivations). We then define certain operators $\mathcal{E}, \mathcal{D}_0, \mathcal{S}_X^{\mathcal{F}}$ on BI_1^∞-derivations. By means of $\mathcal{E}$ and $\mathcal{D}_0$ every BI_1^∞-derivation with an arithmetical endsequent Γ can be transformed into a BI_0^∞-derivation of Γ. The operator $\mathcal{S}_X^{\mathcal{F}}$ is needed for the embedding $h \mapsto h^\infty$ of BI_1^- into BI_1^∞. In §3 we introduce the finitary proof system BI_1^* which is an extension of BI_1^- by certain inference rules $(\mathsf{E}), (\mathsf{D}_0), (\mathsf{S}_X^{\mathcal{F}})$ corresponding to the operators $\mathcal{E}, \mathcal{D}_0, \mathcal{S}_X^{\mathcal{F}}$. The embedding $h \mapsto h^\infty$ is extended to BI_1^* by interpreting $(\mathsf{E}), (\mathsf{D}_0), (\mathsf{S}_X^{\mathcal{F}})$ by means of $\mathcal{E}, \mathcal{D}_0, \mathcal{S}_X^{\mathcal{F}}$. In BI_1^* it is possible to define an operation $(h, i) \mapsto h[i]$ which assigns to every BI_1^*-derivation h a family of BI_1^*-derivations $h[i]$ such that $h[i]^\infty$ is the i^{th} immediate subderivation of h^∞. In §4 we show that if h is a (hypothetical) BI_1^*-derivation of the empty sequent (contradiction) then h^∞ ends with a Rep, and $h[0]$ results from h by application of a Gentzen-Takeuti reduction step as described in [Tak87, pp. 320-341]. Since $h[0]^\infty$ is a proper subtree of (the wellfounded tree) h^∞, we thus have obtained an "ordinal free" termination proof for Takeuti's reduction procedure for BI_1^*. An ordinal assignment $\mathrm{BI}_1^* \ni h \mapsto \mathrm{o}(h)$ such that $\mathrm{o}(h[i]) < \mathrm{o}(h)$ will be introduced in §5. Since we do not use ordinal diagrams but a different system of notations, this assignment can not fully coincide with Takeuti's, but it will be very similar to that. The important point is, that our ordinal assignment is directly derived from the infinitary approach, and thus (to some degree) also yields a certain explanation for Takeuti's assignment.

Acknowledgement: [Bu97] and the present paper have profited a lot by other authors' work on the Gentzen-Takeuti method and on the relationship between finitary and infinitary derivations, e.g. [Ar88], [Ar96a], [Mi75], [Mi75a], [Mi79], [Mi93]. The describtion of Gentzen's ordinal assignment in [Bu97] is similar to that in [Mi79, Definition 1]. Our $\mathrm{o}(\psi_n(d))$ corresponds to $O_n(d)$ in [Mi79], and Lemma 5 in [Bu97] corresponds to Lemma 5 in [Mi79]. The E-rule already occurs in [Ar96a] under the name "height rule", but there no interpretation of E as cut-elimination operator is given.

§1 Preliminaries

Syntax

Let $\mathcal{L}_0$ be the 1st-order language of arithmetic which has predicate symbols for primitive recursive relations, but no function symbols except the constant 0 and the unary function symbol S (successor). The language $\mathcal{L}_1$ (which is the formal language of all proof systems considered below) is obtained from $\mathcal{L}_0$ by adding infinitely many 1-ary predicate variables $X, Y, Z, ...$ and a restricted kind of 2nd-order quantification. *Atomic formulas* are of the form $pt_1...t_n$ or Xt where p is an n-ary predicate symbol and $t, t_1, ..., t_n$ are $\mathcal{L}_0$-terms. *Literals* are expressions of the shape A or $\neg A$ where A is an atomic formula. *Formulas* are built up from literals by means of $\wedge, \vee, \forall x, \exists x, \forall X, \exists X$, where the use of $\forall X$, $\exists X$ is *restricted* as follows: If A is a formula then $\forall XA$ and $\exists XA$ are formulas *only if* A contains no 2nd order quantifier and no predicate variable except X. The *negation* $\neg C$ of a formula C is defined via de Morgan's laws. The *rank* $\mathrm{rk}(C)$ of a formula C is defined as follows: $\mathrm{rk}(C) := 0$ if C is a literal or a formula $\forall XA$ or $\exists XA$, $\mathrm{rk}(A_0 \wedge A_1) := \mathrm{rk}(A_0 \vee A_1) := \max\{\mathrm{rk}(A_0), \mathrm{rk}(A_1)\}+1$, $\mathrm{rk}(\forall xA) := \mathrm{rk}(\exists xA) := \mathrm{rk}(A)+1$. By $\mathrm{FV}(\theta)$ [$\mathrm{FV}_0(\theta)$, resp.] we denote the set of all free variables [free number variables, resp.] of the formula or term θ. A formula or term θ is called *closed* iff $\mathrm{FV}_0(\theta) = \emptyset$. $\theta(x/t)$ denotes the result of replacing every free occurrence of x in θ by t (renaming bound variables of θ if necessary). The only closed terms are the *numerals* $0, S0, SS0, ...$ We identify numerals and natural numbers. Finite sets of formulas are called *sequents*. We use the following *syntactic variables*: s, t for terms, A, B, C, D, F for formulas, Γ, Δ for sequents, α, β, γ for ordinals, i, j, k, l, m, n for natural numbers (and numerals). As far as sequents are concerned we usually write $A_1, ..., A_n$ for $\{A_1, ..., A_n\}$, and A, Γ, Δ for $\{A\} \cup \Gamma \cup \Delta$, etc. P is used as syntactic variable for formulas $\forall XA$. (Note that due to our general restriction on the formation of formulas, a formula $P = \forall XA$ contains no 2nd order variable except X !) $\mathcal{F}$ is a syntactic variable for expressions λxF where F is a formula. If $P = \forall XA$ then $P[\mathcal{F}]$ denotes the formula $A(X/\mathcal{F})$, i.e. the result of replacing in A every subformula Xt by $F(x/t)$. We write $P[Y]$ for $P[\lambda x.Yx]$.

TRUE_0 := set of all true closed $\mathcal{L}_0$-literals.
Var_1 := set of all predicate variables.
$\mathcal{A}$:= set of all formulas which contain no 2nd order quantifiers

Proof systems

A proof system $\mathfrak{S}$ is given by

- a set of formal expressions called *inference symbols* (syntactic variable $\mathcal{I}$)
- for each inference symbol $\mathcal{I}$ a set $|\mathcal{I}|$ (the *arity* of $\mathcal{I}$), a sequent $\Delta(\mathcal{I})$ and a family of sequents $(\Delta_\iota(\mathcal{I}))_{\iota\in|\mathcal{I}|}$. The elements of $\Delta(\mathcal{I})$ [$\bigcup_{\iota\in|\mathcal{I}|} \Delta_\iota(\mathcal{I})$] are called the *principal formulas* [*minor formulas*] of $\mathcal{I}$.
- for each inference symbol $\mathcal{I}$ a set $\mathrm{Eig}(\mathcal{I})$ which is either empty or a singleton $\{u\}$ with u a variable not in $\mathrm{FV}(\Delta(\mathcal{I}))$; in the latter case u is called the *eigenvariable* of $\mathcal{I}$.

NOTATION

By writing

$$(\mathcal{I}) \quad \frac{\ldots\Delta_\iota\ldots(\iota{\in}I)}{\Delta} \ [\ !u!\ ]$$

we express that $\mathcal{I}$ is an inference symbol with $|\mathcal{I}| = I$, $\Delta(\mathcal{I}) = \Delta$, $\Delta_\iota(\mathcal{I}) = \Delta_\iota$, and $\mathrm{Eig}(\mathcal{I}) = \emptyset$ [$\mathrm{Eig}(\mathcal{I}) = \{u\}$, resp.].

If $I = \{0, ..., n{-}1\}$ we write $\dfrac{\Delta_0 \ \ \Delta_1 \ \ldots \ \Delta_{n-1}}{\Delta}$, instead of $\dfrac{\ldots\Delta_\iota\ldots(\iota{\in}I)}{\Delta}$.

Inference symbols $\mathcal{I}$ with $|\mathcal{I}| = \emptyset$ will be called *axioms*.

Example:

By (Cut_C) $\dfrac{C \qquad \neg C}{\emptyset}$ we express that for each formula C, the expression $\mathcal{I} := \mathsf{Cut}_C$ is an inference symbol with $|\mathcal{I}| = \{0, 1\}$, $\Delta(\mathcal{I}) = \emptyset$, $\Delta_0(\mathcal{I}) = \{C\}$, $\Delta_1(\mathcal{I}) = \{\neg C\}$.

Inductive definition of $\mathfrak{S}$-derivations
If $\mathcal{I}$ is an inference symbol of $\mathfrak{S}$, and $(d_\iota)_{\iota\in|\mathcal{I}|}$ is a family of $\mathfrak{S}$-*derivations* such that $\mathrm{Eig}(\mathcal{I}) \cap \mathrm{FV}(\Gamma) = \emptyset$ where $\Gamma := \Delta(\mathcal{I}) \cup \bigcup_{\iota\in|\mathcal{I}|}(\Gamma(d_\iota) \setminus \Delta_\iota(\mathcal{I}))$, then $d := \mathcal{I}(d_\iota)_{\iota\in|\mathcal{I}|}$ is an $\mathfrak{S}$-*derivation* with $\Gamma(d) := \Gamma$ *(endsequent of d)* and $\mathrm{last}(d) := \mathcal{I}$ *(last inference of d).*

Instead of $\mathcal{I}(d_\iota)_{\iota\in|\mathcal{I}|}$ we also write $\dfrac{\ldots d_\iota \ldots (\iota\in|\mathcal{I}|)}{\mathcal{I}}$ or $\mathcal{I}d_0...d_{n-1}$ or $\dfrac{d_0 \ldots d_{n-1}}{\mathcal{I}}$ if $|\mathcal{I}| = \{0, ..., n-1\}$.

Abbreviation: $\mathfrak{S} \ni d \vdash \Gamma \;:\Longleftrightarrow\;$ d is an $\mathfrak{S}$-derivation with $\Gamma(d) \subseteq \Gamma$.
So $\mathfrak{S}$ also denotes the set of all $\mathfrak{S}$-derivations.

Remark
Our notion of derivation differs from the usual one in so far as our derivations have inferences (inference symbols) and not sequents assigned to their nodes. The sequent "belonging" to a node τ of a derivation d is not explicitely displayed, but can be computed by tree recursion from d (similarly as the free assumptions in a natural deduction style derivation).

The Tait-style inference rules in their traditional form $\dfrac{\ldots \Gamma, \Delta_\iota(\mathcal{I}) \ldots}{\Gamma, \Delta(\mathcal{I})}$ are reobtained here as follows:
If $\mathcal{I} \in \mathfrak{S}$ and $\mathrm{Eig}(\mathcal{I}) \cap \mathrm{FV}(\Gamma) = \emptyset$, then
from $\ldots \mathfrak{S} \ni d_\iota \vdash \Gamma, \Delta_\iota(\mathcal{I}) \ldots (\iota \in |\mathcal{I}|)$ we conclude $\mathfrak{S} \ni \mathcal{I}(d_\iota)_{\iota\in|\mathcal{I}|} \vdash \Gamma, \Delta(\mathcal{I})$.

§2 The infinitary systems BI_0^∞ and BI_1^∞

In BI_0^∞ and BI_1^∞ only closed formulas (i.e., formulas A with $\mathrm{FV}_0(A) = \emptyset$) are allowed.

The inference symbols of BI_0^∞ are

(Ax_Δ) $\dfrac{}{\Delta}$ if $\Delta = \{A\} \subseteq \mathsf{TRUE}_0$ or $\Delta = \{\neg C, C\}$

$(\bigwedge_{A_0\wedge A_1})$ $\dfrac{A_0 \qquad A_1}{A_0 \wedge A_1}$ $\qquad$ $(\bigvee^k_{A_0\vee A_1})$ $\dfrac{A_k}{A_0 \vee A_1}$ $(k \in \{0,1\})$

$(\bigwedge_{\forall xA})$ $\dfrac{\ldots A(x/i) \ldots (i\in\mathbb{N})}{\forall xA}$ $\qquad$ $(\bigvee^k_{\exists xA})$ $\dfrac{A(x/k)}{\exists xA}$ $(k \in \mathbb{N})$ $\qquad$ (Rep) $\dfrac{\emptyset}{\emptyset}$

The inference symbols of BI_1^∞ are those of BI_0^∞ together with

$(\bigwedge^Y_P)$ $\dfrac{P[Y]}{P}$ $!Y!$ $\qquad$ $(\Omega_{\neg P})$ $\dfrac{\ldots \Delta^P_q \ldots (q \in |P|)}{\neg P}$

(Cut_C) $\dfrac{C \qquad \neg C}{\emptyset}$

$(\widetilde{\Omega}^Y_{\neg P})$ $\dfrac{P[Y] \qquad \ldots \Delta^P_q \ldots (q \in |P|)}{\emptyset}$ $!Y!$

with $|P| := \{(\mathtt{d}, X) \in \mathrm{BI}_0^\infty \times \mathrm{Var}_1 : \Gamma(\mathtt{d}) \subseteq \mathcal{A} \;\&\; X \notin \mathrm{FV}(\Delta^P_{(\mathtt{d},X)})\}$ and $\Delta^P_{(\mathtt{d},X)} := \Gamma(\mathtt{d}) \setminus \{P[X]\}$.

Remark
1. The pair $(\mathtt{d}, X)$ plays the role of a derivation $\bigwedge^X_P \mathtt{d}$ with $\Gamma(\bigwedge^X_P \mathtt{d}) = \{P\} \cup \Delta^P_{(\mathtt{d},X)}$.
2. Obviously an inference $\widetilde{\Omega}^Y_{\neg P}$ is just a combination of $\bigwedge^Y_P$, $\Omega_{\neg P}$, Cut_P. It corresponds to the Ω_1-rule in [BS88] (pg. 39). For a better understanding of $\Omega_{\neg P}$ it may be helpful to look at the formation of BI_1^∞-derivations by means of $\Omega_{\neg P}$:
By the general definition of $\mathfrak{S}$-derivations we have
$\forall q \in |P|\big(\mathrm{BI}_1^\infty \ni \mathtt{e}_q \vdash \Gamma, \Delta^P_q\big) \Longrightarrow \mathrm{BI}_1^\infty \ni (\mathtt{e}_q)_{q\in|P|} \vdash \Gamma, \neg P$,

i.e.,
$\forall(\mathtt{d},X)\forall\Delta\subseteq\mathcal{A}\big(X\notin \mathrm{FV}(\Delta)\ \&\ \mathrm{BI}_0^\infty\ni\mathtt{d}\vdash\Delta,P[X]\ \Rightarrow\ \mathrm{BI}_1^\infty\ni\mathtt{e}_{(\mathtt{d},X)}\vdash\Delta,\Gamma\big)\ \Longrightarrow\ \mathrm{BI}_1^\infty\ni(\mathtt{e}_q)_{q\in|P|}\vdash\neg P,\Gamma.$

Definition
The *degree* (or *cutrank*) $\deg(\mathtt{d})$ of a BI_1^∞-derivation $\mathtt{d}$ is defined by:
$\deg(\mathcal{I}(\mathtt{d}_\iota)_{\iota\in I}):=\sup(\{\deg(\mathcal{I})\}\cup\{\deg(\mathtt{d}_\iota):\iota\in I\})$ with $\deg(\mathcal{I}):=\begin{cases}\mathrm{rk}(C)+1 & \text{if } \mathcal{I}=\mathsf{Cut}_C\\ 0 & \text{otherwise}\end{cases}$

Abbreviation
$\mathtt{d}\vdash_m\Gamma\ :\Leftrightarrow\ \mathrm{BI}_1^\infty\ni\mathtt{d}\vdash\Gamma\ \&\ \deg(\mathtt{d})\le m$

Theorem 1 and Definition (Cutelimination)
As in [Bu97] we define operators $\mathcal{R}_C$ and $\mathcal{E}$ on BI_1^∞-derivations such that
a) $\mathtt{d}_0\vdash_m\Gamma,C\ \&\ \mathtt{d}_1\vdash_m\Gamma,\neg C\ \&\ \mathrm{rk}(C)\le m\ \Longrightarrow\ \mathcal{R}_C(\mathtt{d}_0,\mathtt{d}_1)\vdash_m\Gamma,$
b) $\mathtt{d}\vdash_{m+1}\Gamma\ \Longrightarrow\ \mathcal{E}(\mathtt{d})\vdash_m\Gamma.$

Proof:
a) As in [Bu97], except for the following additional cases:
1. $\mathtt{d}_0=\mathsf{Ax}_{\{\neg C,C\}}$: Then $\neg C,C\subseteq\Gamma(\mathtt{d}_0)\subseteq\Gamma,C$ and thus $\neg C\in\Gamma$.
Hence the claim holds for $\mathcal{R}_C(\mathtt{d}_0,\mathtt{d}_1):=\mathsf{Rep}\,\mathtt{d}_1$.
2. $C=P$ and $\mathrm{last}(\mathtt{d}_0)=\bigwedge_P^Y$, $\mathrm{last}(\mathtt{d}_1)=\Omega_{\neg P}$:
Then $\mathtt{d}_{00}\vdash_m\Gamma,P,P[Y]$, and $\mathtt{d}_{1q}\vdash_m\Gamma,\neg P,\Delta_q^P$ for all $q\in|P|$.
If $Y\in\mathrm{FV}(\Gamma(\mathtt{d}_1))$ let $\mathtt{d}_{00}'$ be the BI_1^*-derivation resulting from $\mathtt{d}_{00}$ by interchanging Y with some $Z\notin\mathrm{FV}(\Gamma)$; otherwise let $\mathtt{d}_{00}':=\mathtt{d}_{00}$ and $Z:=Y$.
By IH we get $\mathcal{R}_C(\mathtt{d}_{00}',\mathtt{d}_1)\vdash_m\Gamma,P[Z]$, and $\mathcal{R}_C(\mathtt{d}_0,\mathtt{d}_{1q})\vdash_m\Gamma,\Delta_q^P$ for all $q\in|P|$.
$$\mathcal{R}_C(\mathtt{d}_0,\mathtt{d}_1):=\frac{\mathcal{R}_C(\mathtt{d}_{00}',\mathtt{d}_1)\qquad\ldots\mathcal{R}_C(\mathtt{d}_0,\mathtt{d}_{1q})\ldots(q\in|P|)}{\widetilde{\Omega}_{\neg P}^Z}$$
b) As in [Bu97]: For $\mathtt{d}=\mathcal{I}(\mathtt{d}_\iota)_{\iota\in I}$ let $\mathcal{E}(\mathtt{d}):=\begin{cases}\mathsf{Rep}\,\mathcal{R}_C(\mathcal{E}(\mathtt{d}_0),\mathcal{E}(\mathtt{d}_1)) & \text{if } \mathcal{I}=\mathsf{Cut}_C\\ \mathcal{I}\big(\mathcal{E}(\mathtt{d}_i)\big)_{\iota\in I} & \text{otherwise}\end{cases}.$

Theorem 2 and Definition (Collapsing)
We define an operation $\mathcal{D}_0$ such that: $\mathtt{d}\vdash_0\Gamma\ \&\ \Gamma\subseteq\mathcal{A}\ \Rightarrow\ \mathrm{BI}_0^\infty\ni\mathcal{D}_0(\mathtt{d})\vdash\Gamma.$
Proof by induction on $\mathtt{d}$:
Main case: Let $\mathtt{d}=\widetilde{\Omega}_{\neg P}^Y\big(\mathtt{d}_\iota\big)_{\iota\in\{0\}\cup|P|}$ and (w.l.o.g.) $\Gamma(\mathtt{d})=\Gamma$.
Then $Y\notin\mathrm{FV}(\Gamma)$, $\mathtt{d}_0\vdash_0\Gamma,P[Y]$, and $\mathtt{d}_q\vdash_0\Gamma,\Delta_q^P$ for all $q\in|P|$ (†).
By IH $\mathrm{BI}_0^\infty\ni\mathcal{D}_0(\mathtt{d}_0)\vdash\Gamma,P[Y]$, and thus $Y\notin\mathrm{FV}(\Gamma(\mathcal{D}_0(\mathtt{d}_0))\setminus\{P[Y]\})$.
Hence $q_0:=(\mathcal{D}_0(\mathtt{d}_0),Y)\in|P|$ and $\Delta_{q_0}^P\subseteq\Gamma$.
Now (†) yields $\mathtt{d}_{q_0}\vdash_0\Gamma$, and by IH we get $\mathrm{BI}_0^\infty\ni\mathcal{D}_0(\mathtt{d}_{q_0})\vdash\Gamma$.
So $\mathcal{D}_0(\mathtt{d}):=\mathsf{Rep}\,\mathcal{D}_0(\mathtt{d}_{q_0})$ is a derivation as required.

Other cases: $\mathcal{D}_0\big(\mathcal{I}(\mathtt{d}_\iota)_{\iota\in I}\big):=\mathcal{I}\big(\mathcal{D}_0(\mathtt{d}_\iota)\big)_{\iota\in I}$.

Theorem 3 and Definition (Substitution)
$\mathrm{BI}_0^\infty\ni\mathtt{d}\vdash\Gamma\ \Rightarrow\ \mathrm{BI}_0^\infty\ni\mathcal{S}_X^{\mathcal{F}}(\mathtt{d})\vdash\Gamma(X/\mathcal{F}).$
Proof:
$\mathcal{S}_X^{\mathcal{F}}\big(\mathcal{I}(\mathtt{d}_\iota)_{\iota\in|\mathcal{I}|}\big):=\mathcal{I}^\circ\big(\mathcal{S}_X^{\mathcal{F}}(\mathtt{d}_\iota)\big)_{\iota\in|\mathcal{I}^\circ|}$ with $\mathcal{I}^\circ:=\mathcal{I}(X/\mathcal{F})$,
i.e., $\mathsf{Rep}^\circ:=\mathsf{Rep}$, $(\mathsf{Ax}_\Delta)^\circ:=\mathsf{Ax}_{\Delta^\circ}$, $(\bigwedge_C)^\circ:=\bigwedge_{C^\circ}$, $(\bigvee_C^k)^\circ:=\bigvee_{C^\circ}^k$, where $C^\circ:=C(X/\mathcal{F})$.

§3 The finitary proof systems BI_1^- and BI_1^*

Let $\mathrm{Ax}(Z)$ be "the" set of arithmetic axioms (except induction) in sequent form.
So $\mathrm{Ax}(Z)$ is a (prim. rec.) set of sequents in the language $\mathcal{L}_1$ such that
(i) $\Delta \in \mathrm{Ax}(Z)$ & $A \in \Delta \;\Rightarrow\; A$ is a literal,
(ii) $\Delta \in \mathrm{Ax}(Z) \;\Rightarrow\; \Delta(\vec{x}/\vec{t}) \in \mathrm{Ax}(Z)$,
(iii) $\Delta \in \mathrm{Ax}(Z)$ & $\mathrm{FV}_0(\Delta) = \emptyset \;\Rightarrow\; \Delta \cap \mathsf{TRUE}_0 \neq \emptyset$.

The inference symbols of BI_1^- are

$(\mathsf{Ax}^*_\Delta)\ \dfrac{}{\Delta}$ if $\Delta \in \mathrm{Ax}(Z)$ or $\Delta = \{\neg C, C\}$

$(\bigwedge_{A_0\wedge A_1})\ \dfrac{A_0 \qquad A_1}{A_0\wedge A_1}$ $\qquad$ $(\bigvee^k_{A_0\vee A_1})\ \dfrac{A_k}{A_0\vee A_1}$ $(k \in \{0,1\})$

$(\bigwedge^y_{\forall x A})\ \dfrac{A(x/y)}{\forall x A}$ $!y!$ $\qquad$ $(\bigvee^t_{\exists x A})\ \dfrac{A(x/t)}{\exists x A}$

$(\bigwedge^Y_P)\ \dfrac{P[Y]}{P}$ $!Y!$ $\qquad$ $(\bigvee^{\mathcal{F}}_{\neg P})\ \dfrac{\neg P[\mathcal{F}]}{\neg P}$

$(\mathsf{Ind}^{y,t}_F)\ \dfrac{\neg F, F(y/Sy)}{\neg F(y/0), F(y/t)}$ $!y!$

$(\mathsf{R}_C)\ \dfrac{C \qquad \neg C}{\emptyset}$.

The inference symbols of BI_1^* are those of BI_1^- together with
$(\mathsf{E})\ \dfrac{\emptyset}{\emptyset}$, $\qquad (\mathsf{D}_0)\ \dfrac{\emptyset}{\emptyset}$, $\qquad (\mathsf{S}^{\mathcal{F}}_X)\ \left[\dfrac{\Gamma}{\Gamma(X/\mathcal{F})}\right]$

For $\mathsf{S}^{\mathcal{F}}_X$ no sequents $\Delta(\mathsf{S}^{\mathcal{F}}_X)$ and $\Delta_\iota(\mathsf{S}^{\mathcal{F}}_X)$ are defined.
Instead we define directly $\Gamma(\mathsf{S}^{\mathcal{F}}_X h_0) := \Gamma(h_0)(X/\mathcal{F})$ and $|\mathsf{S}^{\mathcal{F}}_X| := \{0\}$.

Remark The combination $\mathsf{S}^{\mathcal{F}}_X \mathsf{D}_0$ correponds to Takeuti's substitution rule.

Definition

For each BI_1^*-derivation $h = \mathcal{I}h_0...h_{n-1}$ we set $\deg(h) := \begin{cases} \max\{\mathrm{rk}(C), \deg(h_0), \deg(h_1)\} & \text{if } \mathcal{I} = \mathsf{R}_C \\ \max\{\mathrm{rk}(F), \deg(h_0)\} & \text{if } \mathcal{I} = \mathsf{Ind}^{y,t}_F \\ \max\{\mathrm{rk}(P[\mathcal{F}]), \deg(h_0)\} & \text{if } \mathcal{I} = \bigvee^{\mathcal{F}}_{\neg P} \\ \deg(h_0) \dot{-} 1 & \text{if } \mathcal{I} = \mathsf{E} \\ \sup_{i<n} \deg(h_i) & \text{otherwise} \end{cases}$

Note that this definition is different from the definition of $\deg(\mathtt{d})$ for BI_1^∞-derivations $\mathtt{d}$!

Definition of $h(x/k)$

For $h = \mathcal{I}h_0...h_{n-1}$ let $h(x/k) := \begin{cases} h & \text{if } \mathrm{Eig}(\mathcal{I}) = \{x\} \\ \mathcal{I}(x/k)h_0(x/k)...h_{n-1}(x/k) & \text{otherwise} \end{cases}$
where $\mathcal{I}(x/k)$ is defined as expected (cf. [Bu97]).

Inductive Definition of proper BI_1^*-derivations
If $\mathcal{I}$ is an n-ary BI_1^*-inference symbol and $h_0, ..., h_{n-1}$ are proper BI_1^*-derivations then $h := \mathcal{I}h_0...h_{n-1}$ is a proper BI_1^*-derivation if the following conditions (in addition to the eigenvariable conditions) are satisfied
– $\mathcal{I} = \mathsf{D}_0 \;\Rightarrow\; \deg(h_0) = 0$ & $\Gamma(h_0) \subseteq \mathcal{A}$,
– $\mathcal{I} = \mathsf{S}^{\mathcal{F}}_X \;\Rightarrow\; h_0$ is of the form $\mathsf{D}_0 h_{00}$

Definition

A proper BI_1^*-derivation h is called *good* if it satisfies the following conditions:

(1) every free number variable x occurring in h is the eigenvariable of an inference below that occurrence,

(2) all eigenvariables in h are distinct and none of them occurs below the inference in which it is used as an eigenvariable.

Definition

$\mathsf{BI}_1^* :=$ set of all proper BI_1^*-derivations

$\mathsf{BI}_1^{*\prime} :=$ set of all good $h \in \mathsf{BI}_1^*$

Proposition 1

a) Every $h \in \mathsf{BI}_1^*$ with $\mathrm{FV}_0(\Gamma(h)) = \emptyset$ can be transformed into an $h' \in \mathsf{BI}_1^{*\prime}$ such that $\Gamma(h') \subseteq \Gamma(h)$ and $\deg(h') = \deg(h)$.

b) For each $h = \mathcal{I}h_0...h_{n-1} \in \mathsf{BI}_1^{*\prime}$ the following holds:

(i) $\mathrm{FV}_0(\Gamma(h)) = \emptyset$.

(ii) If $\mathrm{Eig}(\mathcal{I}) = \emptyset$ or $\mathcal{I} = \bigwedge_P^Y$ then $h_0, ..., h_{n-1} \in \mathsf{BI}_1^{*\prime}$.

(iii) If $\mathrm{Eig}(\mathcal{I}) = \{y\}$ then $h_0(y/k) \in \mathsf{BI}_1^{*\prime}$.

(iv) If $\mathcal{I} = \mathsf{Ind}_F^{y,t}$ then $\mathrm{FV}_0(F(y/t)) = \emptyset$ and therefore w.l.o.g. t is a numeral.

(v) If $h = \mathsf{S}_X^{\mathcal{F}} h_0$ and $Y \in \{X\} \cup \mathrm{FV}(\mathcal{F})$ then Y does not occur as eigenvariable in h_0.

Interpretation of $\mathsf{BI}_1^{*\prime}$ in BI_1^∞

For each $h = \mathcal{I}h_0...h_{n-1} \in \mathsf{BI}_1^{*\prime}$ we define its interpretation $h^\infty \in \mathrm{BI}_1^\infty$ as follows:

0. $(\mathsf{Ax}_\Delta^*)^\infty := \mathsf{Ax}_{\Delta'}$ with suitable $\Delta' \subseteq \Delta$,
1. $(\bigwedge_{\forall x A}^{y} h_0)^\infty := \bigwedge_{\forall x A}\big(h_0(y/i)^\infty\big)_{i \in \mathbb{N}}$,
2. $(\bigvee_{\neg P}^{\mathcal{F}} h_0)^\infty := \Omega_{\neg P}\big(\mathcal{R}_{P[\mathcal{F}]}(\mathcal{S}_X^{\mathcal{F}}(\mathtt{d}), h_0^\infty)\big)_{(\mathtt{d},X) \in |P|}$
3. $(\mathsf{Ind}_F^{y,n} h_0)^\infty := \mathsf{Rep}\,\mathsf{e}_n$ with
   $\mathsf{e}_0 := \mathsf{Ax}_{\{\neg F(y/0), F(y/0)\}}$, $\mathsf{e}_1 := h_0(y/0)^\infty$, $\mathsf{e}_{i+1} := \mathcal{R}_{F(y/i)}(\mathsf{e}_i, h_0(y/i)^\infty)$ for $i > 0$.
4. $(\mathsf{R}_C h_0 h_1)^\infty := \mathcal{R}_C(h_0^\infty, h_1^\infty)$, $(\mathsf{E}h_0)^\infty := \mathcal{E}(h_0^\infty)$, $(\mathsf{D}_0 h_0)^\infty := \mathcal{D}_0(h_0^\infty)$, $(\mathsf{S}_X^{\mathcal{F}} h_0)^\infty := \mathcal{S}_X^{\mathcal{F}}(h_0^\infty)$
5. Otherwise: $(\mathcal{I}h_0...h_{n-1})^\infty := \mathcal{I}h_0^\infty \dots h_{n-1}^\infty$

Theorem 4

$h \in \mathsf{BI}_1^{*\prime} \;\Rightarrow\; h^\infty \vdash_{\deg(h)} \Gamma(h)$.

Proof for $h = \bigvee_{\neg P}^{\mathcal{F}} h_0$:

Let $\Gamma := \Gamma(h)$, $m := \deg(h) = \max\{\mathrm{rk}(P[\mathcal{F}]), \deg(h_0)\}$.

Then $\neg P \in \Gamma$, $\Gamma(h_0) \subseteq \Gamma, \neg P[\mathcal{F}]$, and, by IH, $h_0^\infty \vdash_m \Gamma, \neg P[\mathcal{F}]$ $(*)$.

Now the following implications hold:

$(\mathtt{d}, X) \in |P| \;\Rightarrow\; \mathrm{BI}_0^\infty \ni \mathtt{d} \vdash \Delta_{(\mathtt{d},X)}^P, P[X]$ with $X \notin \mathrm{FV}(\Delta_{(\mathtt{d},X)}^P) \Rightarrow$

$\Rightarrow \mathcal{S}_X^{\mathcal{F}}(\mathtt{d}) \vdash_0 \Delta_{(\mathtt{d},X)}^P, P[\mathcal{F}] \underset{(*)}{\Rightarrow} \mathcal{R}_{P[\mathcal{F}]}(\mathcal{S}_X^{\mathcal{F}}(\mathtt{d}), h_0^\infty) \vdash_m \Delta_{(\mathtt{d},X)}^P, \Gamma$.

Hence $h^\infty \vdash_m \Gamma$, since $\neg P \in \Gamma$.

Definition
$|P|^* := \{(\mathsf{D}_0 h, X) : h \in \mathsf{BI}_1^* \ \&\ \Gamma(h) \subseteq \mathcal{A} \ \&\ \deg(h) = 0 \ \&\ X \notin \mathrm{FV}(\Delta^P_{(\mathsf{D}_0 h, X)})\}$
$|\Omega_{\neg P}|^* := |P|^*$, $|\widetilde{\Omega}^Y_{\neg P}|^* := \{0\} \cup |P|^*$; in all other cases $|\mathcal{I}|^* := |\mathcal{I}|$.

Definition of $\mathrm{tp}(h)$ and $h[\iota]$ for $h \in \mathsf{BI}_1^{*\prime}$ and $\iota \in |\mathrm{tp}(h)|^*$
By primitive recursion on the build-up of $h \in \mathsf{BI}_1^{*\prime}$ we define an inference symbol $\mathrm{tp}(h) \in \mathrm{BI}_1^\infty$, and proper BI_1^*-derivation(s) $h[\iota]$ in such a way that $\mathrm{tp}(h) = \mathrm{last}(h^\infty)$ and $(h[z])^\infty = h^\infty(z^\infty)$ for all $z \in |\mathrm{tp}(h)|^*$. Here $i^\infty := i$ for $i \in \mathbb{N}$, $(d, X)^\infty := (d^\infty, X)$, and $h^\infty(\iota) :=$ the "ι^{th}" immediate subderivation of h^∞. So, for $\mathrm{tp}(h) \neq \Omega_{\neg P}, \widetilde{\Omega}^Y_{\neg P}$ we have $h^\infty = \mathrm{tp}(h)\big(h[i]^\infty\big)_{i \in |\mathrm{tp}(h)|}$.

The definition clauses for $h = \mathsf{R}_C h_0 h_1$, $\mathsf{E} h_0$, $\mathsf{D}_0 h_0$, $\mathsf{S}^{\mathcal{F}}_X h_0$ can be read off from the corresponding clauses in the definitions of $\mathcal{R}_C$, $\mathcal{E}$, $\mathcal{D}_0$, $\mathcal{S}^{\mathcal{F}}_X$.

1.1. $h = \mathsf{Ax}^*_\Delta$: $\mathrm{tp}(h) := \mathsf{Ax}_{\Delta'}$ with suitable $\Delta' \subseteq \Delta$.
1.2. $h = \bigwedge_C h_0 h_1$: $\mathrm{tp}(h) := \bigwedge_C$, $h[i] := h_i$.
1.3. $h = \bigwedge^Y_P h_0$: $\mathrm{tp}(h) := \bigwedge^Y_P$, $h[0] := h_0$.
1.4. $h = \bigvee^k_C h_0$: $\mathrm{tp}(h) := \bigvee^k_C$, $h[0] := h_0$.
1.5. $h = \bigwedge^y_C h_0$: $\mathrm{tp}(h) := \bigwedge_C$, $h[i] := h_0(y/i)$.
1.6. $h = \bigvee^{\mathcal{F}}_{\neg P} h_0$: $\mathrm{tp}(h) := \Omega_{\neg P}$, $h[(d, X)] := \mathsf{R}_{P[\mathcal{F}]}(\mathsf{S}^{\mathcal{F}}_X d) h_0$.
2. $h = \mathsf{Ind}^{y,n}_F h_0$: $\mathrm{tp}(h) := \mathsf{Rep}$, $h[0] := e_n$ with
$e_0 := \mathsf{Ax}^*_{\{\neg F(0), F(0)\}}$, $e_1 := h_0(y/0)$, $e_{i+1} := \mathsf{R}_{F(i)} e_i h_0(y/i)$ for $i > 0$.
3. $h = \mathsf{R}_C h_0 h_1$:
3.1. $C \notin \Delta(\mathrm{tp}(h_0))$: $\mathrm{tp}(h) := \mathrm{tp}(h_0)$, $h[\iota] := \mathsf{R}_C h_0[\iota] h_1$.
3.2. $\neg C \notin \Delta(\mathrm{tp}(h_1))$: $\mathrm{tp}(h) := \mathrm{tp}(h_1)$, $h[\iota] := \mathsf{R}_C h_0 h_1[\iota]$.
3.3. $C \in \Delta(\mathrm{tp}(h_0))$ and $\neg C \in \Delta(\mathrm{tp}(h_1))$:
3.3.1. $\mathrm{tp}(h_0) = \mathsf{Ax}_{\{\neg C, C\}}$: $\mathrm{tp}(h) = \mathsf{Rep}$ and $h[0] = h_1$.
3.3.2. $\mathrm{tp}(h_1) = \mathsf{Ax}_{\{\neg C, C\}}$: $\mathrm{tp}(h) = \mathsf{Rep}$ and $h[0] = h_0$.
3.3.3. $C = \forall x A$: Then $\mathrm{tp}(h_1) = \bigvee^k_{\neg C}$ for some $k \in \mathbb{N}$.
$\mathrm{tp}(h) := \mathsf{Cut}_{A(x/k)}$, $h[0] := \mathsf{R}_C h_0[k] h_1$, $h[1] := \mathsf{R}_C h_0 h_1[0]$.
3.3.4. $C = \exists x A$ or $A_0 \wedge A_1$ or $A_0 \vee A_1$: analogous to 3.3.3.
3.3.5. $C = P$: Then $\mathrm{tp}(h_0) = \bigwedge^Y_P$ and $\mathrm{tp}(h_1) = \Omega_{\neg P}$.
$\mathrm{tp}(h) := \widetilde{\Omega}^Y_{\neg P}$, $h[0] := \mathsf{R}_P h_0[0] h_1$, $h[q] := \mathsf{R}_P h_0 h_1[q]$ for $q \in |P|^*$.
3.3.6. $C = \neg P$: analogous to 3.3.5.
4. $h = \mathsf{E} h_0$:
4.1. $\mathrm{tp}(h_0) = \mathsf{Cut}_C$: $\mathrm{tp}(h) := \mathsf{Rep}$, $h[0] := \mathsf{R}_C \mathsf{E} h_0[0] \mathsf{E} h_0[1]$,
4.2. otherwise: $\mathrm{tp}(h) := \mathrm{tp}(h_0)$, $h[\iota] := \mathsf{E} h_0[\iota]$.
5. $h = \mathsf{D}_0 h_0$:
5.1. $\mathrm{tp}(h_0) = \widetilde{\Omega}^Y_{\neg P}$: $\mathrm{tp}(h) := \mathsf{Rep}$, $h[0] = \mathsf{D}_0 h_0[(\mathsf{D}_0 h_0[0], Y)]$.
5.2. otherwise: $\mathrm{tp}(h) := \mathrm{tp}(h_0)$, $h[\iota] := \mathsf{D}_0 h_0[\iota]$.
6. $h = \mathsf{S}^{\mathcal{F}}_X d$: $\mathrm{tp}(h) := \mathrm{tp}(d)(X/\mathcal{F})$, $h[\iota] := \mathsf{S}^{\mathcal{F}}_X d[\iota]$.

Remark Clause 5.1 "$h[0] := \mathsf{D}_0 h_0[(\mathsf{D}_0 h_0[0], Y)]$" seems not to fall under the scheme of primitive recursion, since it is of the form "$f(n+1, 0) := g(f(n, \tilde{g}(f(n, 0), n)))$" with previously defined $g, \tilde{g}$. But the situation is resolved as follows. The above definition is divided into three stages: First $\mathrm{tp}(h)$ is defined for all h. Then $h[\iota]$ is defined for all h with $\mathrm{tp}(h) = \Omega_{\neg P}$ or $\widetilde{\Omega}^Y_{\neg P}$ (clauses 1.6, 3.1, 3.2, 3.3.5, 4.2, 5.2). Finally $h[i]$ is defined for the remaining h's.

Proposition 2
If $\mathrm{tp}(h) = \bigwedge^Y_P$ or $\widetilde{\Omega}^Y_{\neg P}$, then Y occurs as eigenvariable in h.

Abbreviation $h \vdash^*_m \Gamma \;:\Longleftrightarrow\; h \in \mathsf{BI}^*_1 \;\&\; \Gamma(h) \subseteq \Gamma \;\&\; \deg(h) \le m$.

Theorem 5

If $\mathsf{BI}^{*\prime}_1 \ni h \vdash^*_m \Gamma$ and $\mathcal{I} = \mathrm{tp}(h)$ then:

a) $\Delta(\mathcal{I}) \subseteq \Gamma$

b) $h[\iota] \vdash^*_m \Gamma, \Delta_\iota(\mathcal{I})$ for all $\iota \in |\mathcal{I}|$

c) $\mathcal{I} = \mathsf{Cut}_C \;\Rightarrow\; \mathrm{rk}(C) < \deg(h)$

d) $\mathcal{I} \in \{\bigwedge^Y_P, \widetilde{\Omega}^Y_{\neg P}\} \;\Rightarrow\; Y \notin \mathrm{FV}(\Gamma(h))$

Proof by induction on the build up of h (only the interesting cases):

1. $h = \mathsf{R}_P h_0 h_1$, $\mathrm{tp}(h_0) = \bigwedge^Y_P$, $\mathrm{tp}(h_1) = \Omega_{\neg P}$, $\mathcal{I} = \widetilde{\Omega}^Y_{\neg P}$:

a),c) are trivial.

b) We have $\mathsf{BI}^{*\prime}_1 \ni h_0 \vdash_m \Gamma, P$ and $\mathsf{BI}^{*\prime}_1 \ni h_1 \vdash_m \Gamma, \neg P$ and therefore (by IH) $h_0[0] \vdash^*_m \Gamma, P, P[Y]$, and $h_1[q] \vdash^*_m \Gamma, \neg P, \Delta^P_q$ for all $q \in |P|^*$. Hence $h[0] = \mathsf{R}_P h_0[0] h_1 \vdash^*_m \Gamma, P[Y]$ and $h[q] = \mathsf{R}_P h_0 h_1[q] \vdash^*_m \Gamma, \Delta^P_q$.

d) By IH $Y \notin \mathrm{FV}(\Gamma(h_0))$. By Proposition 2 Y occurs as eigenvariable in h_0. Since h is good , this implies $Y \notin \mathrm{FV}(\Gamma(h_0))$ and thus $Y \notin \mathrm{FV}(\Gamma(h))$.

2. $h = \mathsf{D}_0 h_0$, $\mathrm{tp}(h_0) = \widetilde{\Omega}^Y_{\neg P}$, $\Gamma \subseteq \mathcal{A}$, $\mathcal{I} = \mathsf{Rep}$: w.l.o.g. we may assume that $\Gamma = \Gamma(h)$.

a),c),d) are trivial.

b) We have $\mathsf{BI}^{*\prime}_1 \ni h_0 \vdash_0 \Gamma$ with $\Gamma \subseteq \mathcal{A}$, and by IH (1) $h_0[0] \vdash^*_0 \Gamma, P[Y]$, (2) $h_0[q] \vdash^*_0 \Gamma, \Delta^P_q$ for all $q \in |P|^*$, (3) $Y \notin \mathrm{FV}(\Gamma)$. By definition $h[0] = \mathsf{D}_0 h_0[q_0]$ with $q_0 = (\mathsf{D}_0 h_0[0], Y)$. From (1) we get $\mathsf{D}_0 h_0[0] \vdash^*_0 \Gamma, P[Y]$ and $\Delta^P_{q_0} \subseteq \Gamma$. Hence $q_0 \in |P|^*$, since $Y \notin \mathrm{FV}(\Gamma)$. By (2) we obtain now $h_0[q_0] \vdash^*_0 \Gamma$ which yields $h[0] = \mathsf{D}_0 h_0[q_0] \vdash^*_0 \Gamma$. (Cf. Proof of Collapsing Theorem above)

3. $h = \bigvee^{\mathcal{F}}_{\neg P} h_0$, $\mathcal{I} = \Omega_{\neg P}$, $h[(d,X)] = \mathsf{R}_{P[\mathcal{F}]}(\mathsf{S}^{\mathcal{F}}_X d) h_0$:

a) $\Delta(\mathcal{I}) = \{\neg P\} \subseteq \Gamma(h) \subseteq \Gamma$.

b) Let $q = (d, X) \in |P|^*$. Then $d \vdash^*_0 \Delta^P_q, P[X]$ with $\Delta^P_q \subseteq \mathcal{A}$ and $X \notin \mathrm{FV}(\Delta^P_q)$. Hence $\mathsf{S}^{\mathcal{F}}_X d \vdash^*_0 \Delta^P_q, P[\mathcal{F}]$, and together with $h_0 \vdash^*_m \Gamma, \neg P[\mathcal{F}]$ we get $\mathsf{R}_{P[\mathcal{F}]}(\mathsf{S}^{\mathcal{F}}_X d) h_0 \vdash^*_m \Delta^P_q, \Gamma$. Note that $\mathrm{rk}(P[\mathcal{F}]) \le \deg(h) \le m$.

c),d) are trivial.

4. $h = \mathsf{S}^{\mathcal{F}}_X d$ with $d = \mathsf{D}_0 h_0$, $\deg(d) = \deg(h_0) = 0$, $\Gamma(d) = \Gamma(h_0) \subseteq \mathcal{A}$, $\mathcal{I} = \mathrm{tp}(d)(X/\mathcal{F})$: Let $\Gamma_d := \Gamma(d)$.

a) $\Delta(\mathrm{tp}(d)) \overset{\mathrm{IHa}}{\subseteq} \Gamma_d \;\Rightarrow\; \Delta(\mathcal{I}) = \Delta(\mathrm{tp}(d))(X/\mathcal{F}) \subseteq \Gamma_d(X/\mathcal{F}) = \Gamma(h) \subseteq \Gamma$.

b) By IHa it follows that $\mathrm{tp}(d) \ne \Omega_{\neg P}, \bigwedge^Y_P$, since $\Gamma(d) \subseteq \mathcal{A}$. By definition of $\mathrm{tp}(\mathsf{D}_0 h_0)$ we also have $\mathrm{tp}(d) \ne \widetilde{\Omega}^Y_{\neg P}$. Hence $\mathrm{tp}(d)$ is an BI^∞_0-inference, i.e. one of $\mathsf{Ax}_\Delta, \bigwedge_C, \bigvee^k_C, \mathsf{Rep}$. By IHb) $d[i] \vdash^*_0 \Gamma_d, \Delta_i(\mathrm{tp}(d))$. Hence $h[i] = \mathsf{S}^{\mathcal{F}}_X d[i] \vdash^*_0 \Gamma_d(X/\mathcal{F}), \Delta_i(\mathrm{tp}(d))(X/\mathcal{F})$, i.e. $h[i] \vdash^*_0 \Gamma, \Delta_i(\mathcal{I})$.

c) By IHb $\mathrm{tp}(d) \ne \mathsf{Cut}_C$ and thus $\mathcal{I} \ne \mathsf{Cut}_C$.

d) As shown under b), $\mathrm{tp}(d) \notin \{\bigwedge^Y_P, \widetilde{\Omega}^Y_{\neg P}\}$ and therefore also $\mathcal{I} \notin \{\bigwedge^Y_P, \widetilde{\Omega}^Y_{\neg P}\}$.

Definition

$\mathcal{A}_0$:= set of all closed literals

$\mathsf{BI}^*_0 := \{\mathsf{D}_0 h : h \in \mathsf{BI}^*_1 \;\&\; \deg(h) = 0 \;\&\; \Gamma(h) \subseteq \mathcal{A}_0\}$, $\mathsf{BI}^{*\prime}_0 := \{d \in \mathsf{BI}^*_0 : d \text{ good }\}$

Corollary

$h \in \mathsf{BI}^{*\prime}_0 \;\&\; \mathrm{tp}(h) \ne \mathsf{Ax}^* \;\Longrightarrow\; \mathrm{tp}(h) = \mathsf{Rep}$ and $h[0] \in \mathsf{BI}^*_0$.

Proof: Since $h \in \mathsf{BI}^{*\prime}_0$, we have $\mathrm{tp}(h) \ne \widetilde{\Omega}^Y_{\neg P}$. Together with $\deg(h) = 0$, $\Gamma(h) \subseteq \mathcal{A}_0$, and Theorem 5 this yields the claim.

Remark

Since $h[0]^\infty$ is a proper subtree of (the wellfounded tree) h^∞, we thus have obtained an "ordinal free" termination proof for the reduction procedure $h \mapsto h[0]'$ ($h \in \mathsf{BI}_0^{*\prime}$). ($h[0]' \in \mathsf{BI}_0^{*\prime}$ results from $h[0] \in \mathsf{BI}_0^*$ by renaming of eigenvariables, and substituting 0 for free number variables; therefore it has the same underlying tree structure as $h[0]$.)

§4 Takeuti's main reduction step

We now want to demonstrate that Takeuti's reduction steps occurring in the consistency proofs for **ISN** on pp. 320-341 of [Tak87] are essentially the same as the transition from h to $h[0]$ (for $h \in \mathsf{BI}_0^{*\prime}$) in our approach. We will only treat the most interesting case which is "(6) Case 1." on pp. 327,328 or "(7) Case 1." on pp. 339,340 (resp.) of [Tak87].

Definition (Nominal forms for derivations)

1. $\diamond$ is a nominal form, and $Q(\diamond) := \emptyset$.
2. If $\mathfrak{a}$ is a nominal form, and $h \in \mathsf{BI}_1^*$ then $\mathsf{R}_C\mathfrak{a}h$ and $\mathsf{R}_C h\mathfrak{a}$ are nominal forms and $Q(\mathsf{R}_C\mathfrak{a}h) := Q(\mathsf{R}_C h\mathfrak{a}) := \{\mathsf{R}_C\} \cup Q(\mathfrak{a})$.
3. If $\mathfrak{a}$ is a nominal form, then $\mathsf{E}\mathfrak{a}$, $\mathsf{D}_0\mathfrak{a}$, $\mathsf{S}_X^{\mathcal{F}}\mathfrak{a}$ are nominal forms, and $Q(\mathsf{X}\mathfrak{a}) := \{\mathsf{X}\} \cup Q(\mathfrak{a})$.

An $\mathsf{R}(,\mathsf{E})$-form is a nominal form $\mathfrak{a}$ with $Q(\mathfrak{a}) \subseteq \{\mathsf{R}_C : C \text{ formula }\}(\cup\{\mathsf{E}\})$.

We use $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ as syntactic variables for nominal forms.

$$\mathfrak{a}\{h\} := \begin{array}{c} h \\ \cdot\,\dot{\cdot}\,\cdot \\ \mathfrak{a} \end{array} := \text{the result of substituting } h \text{ for } \diamond \text{ in } \mathfrak{a} .$$

Now let $h \in \mathsf{BI}_0^{*\prime}$ with $\mathrm{tp}(h) = \mathsf{Rep}$. By looking through the definition of $\mathrm{tp}(h)$ and $h[\iota]$ one easily verifies that there is a nominal form $\mathfrak{a}$ such that one of the following cases holds:

(I) $h = \mathfrak{a}\{\mathsf{Ind}_F^{y;n} h_0\}$, $h[0] = \mathfrak{a}\{(\mathsf{Ind}_F^{y;n} h_0)[0]\}$,

(II) $h = \mathfrak{a}\{\mathsf{E}\mathfrak{b}\{\mathsf{R}_C h_0 h_1\}\}$, $h[0] = \mathfrak{a}\{\mathsf{R}_{C[k]}\mathsf{E}\mathfrak{b}\{\mathsf{R}_C h_0^- h_1\}\mathsf{E}\mathfrak{b}\{\mathsf{R}_C h_0 h_1^-\}\}$ for some R-form $\mathfrak{b}$,

(III) $h = \mathfrak{a}\{\mathsf{R}_C h_0 h_1\}$, $h[0] = \mathfrak{a}\{h_i\}$ with $i \in \{0,1\}$ and $\mathrm{tp}(h_{1-i}) = \mathsf{Ax}_{\{\neg C, C\}}$,

(IV) $h = \mathfrak{a}\{\mathsf{D}_0 h'\}$, $h[0] = \mathfrak{a}\{(\mathsf{D}_0 h')[0]\}$ with $\mathrm{tp}(h') = \Omega_{\neg P}^X$.

We only consider case (IV) (for (I) and (II) cf. [Bu97]):

Again by inspecting the definition of $\mathrm{tp}(h)$ and $h[\iota]$ one sees that there is an $\{\mathsf{R},\mathsf{E}\}$-form $\mathfrak{b}$ such that $h' = \mathfrak{b}\{\mathsf{R}_P h_0 h_1\}$ and (w.l.o.g.) $\mathrm{tp}(h_0) = \bigwedge_P^X$, $\mathrm{tp}(h_1) = \Omega_{\neg P}$ and

($\Diamond$1) $h'[0] = \mathfrak{b}\{\mathsf{R}_P h_0[0] h_1\}$, $h'[q] = \mathfrak{b}\{\mathsf{R}_P h_0 h_1[q]\}$.

Further there is a nominal form $\mathfrak{c}_0$ and an $\{\mathsf{R},\mathsf{E}\}$-form $\mathfrak{c}_1$ such that

($\Diamond$2) $h_0 = \mathfrak{c}_0\{\bigwedge_P^X h_{00}\}$, $h_1 = \mathfrak{c}_1\{\bigvee_{\neg P}^{\mathcal{F}} h_{10}\}$

($\Diamond$3) $h_0[0] = \mathfrak{c}_0\{h_{00}\}$, $h_1[(d,X)] = \mathfrak{c}_1\{\mathsf{R}_{P[\mathcal{F}]}(\mathsf{S}_X^{\mathcal{F}} d) h_{10}\}$.

Putting things together we get

$h[0] \overset{(\mathrm{IV})}{=} \mathfrak{a}\{(\mathsf{D}_0 h')[0]\} \overset{5.1.}{=}$

$\mathfrak{a}\{\mathsf{D}_0 h'[(\mathsf{D}_0 h'[0], X)]\} \overset{(\Diamond 1)}{=}$

$\mathfrak{a}\{\mathsf{D}_0\mathfrak{b}\{\mathsf{R}_P h_0 h_1[(\mathsf{D}_0\mathfrak{b}\{\mathsf{R}_P h_0[0] h_1\}, X)]\}\} \overset{(\Diamond 3)}{=}$

$\mathfrak{a}\{\mathsf{D}_0\mathfrak{b}\{\mathsf{R}_P h_0 \mathfrak{c}_1\{\mathsf{R}_{P[\mathcal{F}]}\mathsf{S}_X^{\mathcal{F}}\mathsf{D}_0\mathfrak{b}\{\mathsf{R}_P\mathfrak{c}_0\{h_{00}\}h_1\}h_{10}\}\}\}$.

Altogether we have

$h \quad = \mathfrak{a}\{\mathsf{D}_0\mathfrak{b}\{\mathsf{R}_P\mathfrak{c}_0\{\bigwedge_P^X h_{00}\}\mathfrak{c}_1\{\bigvee_{\neg P}^{\mathcal{F}} h_{10}\}\}\}$,

$h[0] = \mathfrak{a}\{\mathsf{D}_0\mathfrak{b}\{\mathsf{R}_P\mathfrak{c}_0\{\bigwedge_P^X h_{00}\}\mathfrak{c}_1\{\mathsf{R}_{P[\mathcal{F}]}\mathsf{S}_X^{\mathcal{F}}\mathsf{D}_0\mathfrak{b}\{\mathsf{R}_P\mathfrak{c}_0\{h_{00}\}\mathfrak{c}_1\{\bigvee_{\neg P}^{\mathcal{F}} h_{10}\}\}h_{10}\}\}\}$.

($h[0]$ results from h by replacing $\bigvee_{\neg P}^{\mathcal{F}}$ by $\mathsf{R}_{P[\mathcal{F}]}\mathsf{S}_X^{\mathcal{F}}\mathsf{D}_0\mathfrak{b}\{\mathsf{R}_P\mathfrak{c}_0\{h_{00}\}\mathfrak{c}_1\{\bigvee_{\neg P}^{\mathcal{F}} h_{10}\}\}$.)

In tree form this is:

$$h:\quad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{h_{00}}{\bigwedge^X_P}\\ \vdots\\ \mathfrak{c}_0\end{array}\qquad\begin{array}{c}\dfrac{h_{10}}{\bigvee^{\mathcal{F}}_{\neg P}}\\ \vdots\\ \mathfrak{c}_1\end{array}}{\mathsf{R}_P}\\ \vdots\\ \mathfrak{b}\end{array}}{\mathsf{D}_0}\\ \vdots\\ \mathfrak{a}\end{array}
\qquad\qquad
h[0]:\quad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{h_{00}}{\bigwedge^X_P}\\ \vdots\\ \mathfrak{c}_0\end{array}\qquad\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}h_{00}\\ \vdots\\ \mathfrak{c}_0\end{array}\quad\begin{array}{c}\dfrac{h_{10}}{\bigvee^{\mathcal{F}}_{\neg P}}\\ \vdots\\ \mathfrak{c}_1\end{array}}{\mathsf{R}_P}\\ \vdots\\ \mathfrak{b}\end{array}}{\mathsf{S}^{\mathcal{F}}_X\mathsf{D}_0}\qquad h_{10}}{\mathsf{R}_{P[\mathcal{F}]}}\\ \vdots\\ \mathfrak{c}_1\end{array}}{\mathsf{R}_P}\\ \vdots\\ \mathfrak{b}\end{array}}{\mathsf{D}_0}\\ \vdots\\ \mathfrak{a}\end{array}$$

or

$$h:\quad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}h_{00}\\ \dfrac{\Gamma_0,P[X]}{\Gamma_0,P}\\ \vdots\\ \mathfrak{c}_0\\ \Gamma,P\end{array}\qquad\begin{array}{c}h_{10}\\ \dfrac{\Gamma_1,\neg P[\mathcal{F}]}{\Gamma_1,\neg P}\\ \vdots\\ \mathfrak{c}_1\\ \Gamma,\neg P\end{array}}{\Gamma}\\ \vdots\\ \mathfrak{b}\\ \Lambda\end{array}}{\Lambda}\\ \vdots\\ \mathfrak{a}\end{array}
\qquad\qquad
h[0]:\quad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}h_{00}\\ \dfrac{\Gamma_0,P[X]}{\Gamma_0,P}\\ \vdots\\ \mathfrak{c}_0\\ \Gamma,P\end{array}\qquad\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}h_{00}\\ \Gamma_0,P[X]\\ \vdots\\ \mathfrak{c}_0\\ \Gamma,P,P[X]\end{array}\quad\begin{array}{c}h_{10}\\ \dfrac{\Gamma_1,\neg P[\mathcal{F}]}{\Gamma_1,\neg P}\\ \vdots\\ \mathfrak{c}_1\\ \Gamma,\neg P\end{array}}{\Gamma,P[X]}\\ \vdots\\ \mathfrak{b}\\ \Lambda,P[X]\end{array}}{\Lambda,P[\mathcal{F}]}\qquad\begin{array}{c}h_{10}\\ \Gamma_1,\neg P[\mathcal{F}]\end{array}}{\Lambda,\Gamma_1}\\ \vdots\\ \mathfrak{c}_1\\ \Lambda,\Gamma,\neg P\end{array}}{\Lambda,\Gamma}\\ \vdots\\ \mathfrak{b}\\ \Lambda\end{array}}{\Lambda}\\ \vdots\\ \mathfrak{a}\end{array}$$

Now compare this with the proof figures on pp. 327,328 and pp. 339,340 in [Tak87].

§5 Ordinal assignment

In this we section carry through an ordinal analysis of the infinitary system BI_1^∞, and then derive from that an ordinal assignment $h \mapsto \mathrm{o}(h)$ for BI_1^*-derivations such that $\forall\iota \in |\mathrm{tp}(h)|^*(\,\mathrm{o}(h[\iota]) < \mathrm{o}(h)\,)$ can be proved by finitary means.
$\alpha, \beta, \gamma, \xi$ range over ordinals $< \varepsilon_{\Omega+1}$. As usual we assume $\alpha = \{\xi : \xi < \alpha\}$.
We use the collapsing function ϑ from [RW93].

Definition
$\vartheta\alpha := \min\{\beta : C(\alpha,\beta) \cap \Omega \subseteq \beta \;\&\; \alpha \in C(\alpha,\beta)\}$
with $C(\alpha,\beta) :=$ closure of $\{0,\Omega\} \cup \beta$ under $+$, $\lambda\xi.\omega^\xi$ and $\vartheta{\restriction}\alpha$ $(:= (\xi \mapsto \vartheta\xi)_{\xi<\alpha})$.

Definition of a finite set $E_\Omega(\alpha) \subseteq \Omega$ for each α
1. $E_\Omega(0) := E_\Omega(\Omega) := \{0\}$,
2. $E_\Omega(\alpha) := \{\alpha\}$, if $\alpha = \omega^\alpha < \Omega$,
3. $E_\Omega(\alpha\#\beta) := E_\Omega(\alpha) \cup E_\Omega(\beta)$,
4. $E_\Omega(\omega^\alpha) := E_\Omega(\alpha)$.

Basic properties of $\vartheta\alpha$ (cf. [RW93])
$(\vartheta.1)$ $E_\Omega(\alpha) \subseteq \vartheta\alpha < \Omega \;\&\; \omega^{\vartheta\alpha} = \vartheta\alpha$,
$(\vartheta.2)$ $\alpha < \beta \;\&\; E_\Omega(\alpha) \subseteq \vartheta\beta \;\Rightarrow\; \vartheta\alpha < \vartheta\beta$,

Lemma 1
a) $\max E_\Omega(\alpha) = \begin{cases} 0 & \text{if } \alpha < \varepsilon_0 \\ \varepsilon_\xi & \text{if } \varepsilon_\xi \le \alpha < \varepsilon_{\xi+1} < \Omega \end{cases}$
b) $\vartheta\alpha < \vartheta(\alpha\#\beta)$ if $\beta \neq 0$, and $\vartheta\alpha < \vartheta\omega^\alpha$ if $\alpha < \omega^\alpha$.

Definition $\alpha \ll_\xi \beta \;:\Leftrightarrow\; \alpha < \beta \;\&\; E_\Omega(\alpha) \setminus (\xi{+}1) \subseteq \vartheta\beta$.

Lemma 2
$(\ll.0)$ $\alpha \ll_\xi \beta \;\&\; \xi \le \eta \implies \alpha \ll_\eta \beta$,
$(\ll.1)$ $\alpha < \beta < \Omega \implies \alpha \ll_0 \beta$,
$(\ll.2)$ $\alpha \ll_\xi \beta \;\&\; \xi < \vartheta\beta \;\Rightarrow\; \vartheta\alpha < \vartheta\beta$,
$(\ll.3)$ $\alpha_0 \ll_\xi \alpha \;\Rightarrow\; \alpha_0\#\beta \ll_\xi \alpha\#\beta \;\&\; \omega^{\alpha_0} \ll_\xi \omega^\alpha$,
$(\ll.4)$ $\alpha_0, \alpha_1 \ll_0 \alpha \;\Rightarrow\; \omega^{\alpha_0}\#\omega^{\alpha_1} \ll_0 \omega^\alpha$.
Proof:
1. $\alpha < \beta < \Omega \Rightarrow \max E_\Omega(\alpha) \le \max E_\Omega(\beta) < \vartheta\beta \Rightarrow \alpha \ll_0 \beta$.
2. $\alpha \ll_\xi \beta \;\&\; \xi \le \vartheta\beta \Rightarrow \alpha < \beta \;\&\; E_\Omega(\alpha) \subseteq \vartheta\beta \Rightarrow \vartheta\alpha < \vartheta\beta$.
3. $\alpha_0 \ll_\xi \alpha \;\Rightarrow\; E_\Omega(\alpha_0) \setminus (\xi{+}1) \subseteq \vartheta\alpha \subseteq \vartheta(\alpha\#\beta) \;\&\; E_\Omega(\beta) \subseteq \vartheta\beta \subseteq \vartheta(\alpha\#\beta) \;\Rightarrow$
$\Rightarrow\; E_\Omega(\alpha_0\#\beta) \setminus (\xi{+}1) \subseteq \vartheta(\alpha\#\beta) \;\Rightarrow\; \alpha_0\#\beta \ll_\xi \alpha\#\beta$, since $\alpha_0\#\beta < \alpha\#\beta$.
4. $\alpha_0, \alpha_1 \ll_0 \alpha \;\Rightarrow\; \omega^{\alpha_0}\#\omega^{\alpha_1} < \omega^\alpha \;\&\; E_\Omega(\omega^{\alpha_0}\#\omega^{\alpha_1}) = E_\Omega(\alpha_0) \cup E_\Omega(\alpha_1) \subseteq \vartheta(\alpha) \subseteq \vartheta(\omega^\alpha)$.

Definition of $\mathtt{d} \triangleleft \alpha$ for BI_1^∞-derivations $\mathtt{d}$
For $\mathtt{d} \in \mathsf{BI}_0^\infty$ let $\|\mathtt{d}\|$ be the length (or depth) of $\mathtt{d}$, i.e., $\|(\mathtt{d}_i)_{i\in I}\| := \sup\{\|\mathtt{d}_i\| + 1 : i \in I\}$.
Then we set $\|i\| := 0$ $(i \in \mathbb{N})$, $\|(\mathtt{d}, X)\| := \|\mathtt{d}\|$
and define $\mathtt{d} \triangleleft \alpha$ for $\mathtt{d} \in \mathsf{BI}_1^\infty$ by

$\mathcal{I}(\mathtt{d}_\iota)_{\iota\in I} \triangleleft \alpha \;:\Leftrightarrow\; \forall\iota \in I \exists\alpha_\iota (\mathtt{d}_\iota \triangleleft \alpha_\iota \ll_{\|\iota\|} \alpha)$

Proposition If $\mathtt{d} \in \mathsf{BI}_0^\infty$ and $\alpha < \Omega$ then $(\mathtt{d} \triangleleft \alpha \;\Leftrightarrow\; \|\mathtt{d}\| \le \alpha)$.

Theorem 6
a) $\mathtt{d}_0 \lhd \alpha$ & $\mathtt{d}_1 \lhd \beta \;\Rightarrow\; \mathcal{R}_C(\mathtt{d}_0, \mathtt{d}_1) \lhd \alpha \# \beta$
b) $\mathtt{d} \lhd \alpha \;\Rightarrow\; \mathcal{E}(\mathtt{d}) \lhd \omega^{\alpha}$ & $\mathcal{D}_0(\mathtt{d}) \lhd \vartheta\alpha$ & $\mathcal{S}^{\mathcal{F}}_X(\mathtt{d}) \lhd \alpha$ (if $\mathcal{D}_0(\mathtt{d})$, $\mathcal{S}^{\mathcal{F}}_X(\mathtt{d})$, resp. is defined)
c) $\forall i \leq n(\, h_0(y/i)^{\infty} \lhd \alpha\,) \;\Rightarrow\; (\mathsf{Ind}^{y,n}_F h_0)^{\infty} \lhd \omega^{\alpha+1}$
d) $h_0^{\infty} \lhd \alpha \;\Rightarrow\; (\bigvee^{\mathcal{F}}_{\neg P} h_0)^{\infty} \lhd \alpha \# \Omega$
Proof:
The proof is straightforward (using Lemma 2). We only consider three cases.
b) Induction on α:
1. Let $\mathtt{d} = \widetilde{\Omega}^{Y}_{\neg P}(\mathtt{d}_\iota)_{\iota \in \{0\} \cup |P|}$:
Then $\mathtt{d}_0 \lhd \alpha_0 \ll_0 \alpha$ and $\forall q \in |P| (\mathtt{d}_q \lhd \alpha_q \ll_{\|q\|} \alpha)$
$\mathtt{d}_0 \lhd \alpha_0 \overset{\text{IH}}{\Rightarrow} \mathcal{D}_0(\mathtt{d}_0) \lhd \xi := \vartheta\alpha_0 < \vartheta\alpha \;\Rightarrow\; \|q_0\| \leq \xi$ with $q_0 := (\mathcal{D}_0(\mathtt{d}_0), Y) \;\Rightarrow\; \mathtt{d}_{q_0} \lhd \alpha_{q_0} \ll_{\xi} \alpha \overset{\text{IH}+\xi<\vartheta\alpha}{\Rightarrow}$
$\mathcal{D}_0(\mathtt{d}_{q_0}) \lhd \vartheta(\alpha_{q_0}) < \vartheta\alpha \;\Rightarrow\; \mathcal{D}_0(\mathtt{d}) = \mathsf{Rep}\, \mathcal{D}_0(\mathtt{d}_{q_0}) \lhd \vartheta\alpha$.
2. $\mathtt{d} = \mathcal{I}\big(\mathcal{D}_0(\mathtt{d}_i)\big)_{i \in I}$ and $\forall i \in I \exists \alpha_i (\mathtt{d}_i \lhd \alpha_i \ll_0 \alpha)$:
By IH $\forall i \in I \exists \alpha_i (\mathcal{D}_0(\mathtt{d}_i) \lhd \vartheta\alpha_i \ll_0 \vartheta\alpha))$ and thus $\mathcal{D}_0(\mathtt{d}) = \mathcal{I}\big(\mathcal{D}_0(\mathtt{d}_i)\big)_{i \in I} \lhd \vartheta\alpha$.
d) $(\bigvee^{\mathcal{F}}_{\neg P} h_0)^{\infty} = \Omega_{\neg P}\big(\mathcal{R}_{P[\mathcal{F}]}(\mathcal{S}^{\mathcal{F}}_X(\mathtt{d}), h_0^{\infty})\big)_{(\mathtt{d},X) \in |P|}$:
$q = (\mathtt{d}, X) \in |P| \;\Rightarrow\; \|q\| = \|\mathtt{d}\| < \Omega \;\Rightarrow\; \mathtt{d} \lhd \|q\| \overset{\text{a})+h_0^{\infty} \lhd \alpha}{\Longrightarrow} \mathcal{R}_{P[\mathcal{F}]}(\mathcal{S}^{\mathcal{F}}_X(\mathtt{d}), h_0^{\infty}) \lhd \|q\| \# \alpha \ll_{\|q\|} \Omega \# \alpha$.

Now, with Theorem 6 in mind, we assign to each BI_1^*-derivation h an ordinal (notation) $\mathrm{o}(h)$ in such a way that $h^{\infty} \lhd \mathrm{o}(h)$ holds.

Definition of an ordinal $\mathrm{o}(h)$ for each BI_1^*-derivation h

$$
\mathrm{o}(h) := \begin{cases}
1 & \text{if } \mathcal{I} = \mathsf{Ax}^*_{\Delta} \\
\mathrm{o}(h_0)+1 & \text{if } \mathcal{I} = \bigwedge^{y}_{\forall x A} \text{ or } \bigwedge^{Y}_{P} \text{ or } \bigvee^{k}_{C} \\
\omega^{\mathrm{o}(h_0)+1} & \text{if } \mathcal{I} = \mathsf{Ind}^{y,t}_F \\
\mathrm{o}(h_0)\#\Omega & \text{if } \mathcal{I} = \bigvee^{\mathcal{F}}_{\neg P} \\
\mathrm{o}(h_0)\#\mathrm{o}(h_1) & \text{if } \mathcal{I} = \bigwedge_{A_0 \wedge A_1} \text{ or } \mathsf{R}_C \\
\omega^{\mathrm{o}(h_0)} & \text{if } \mathcal{I} = \mathsf{E} \\
\vartheta\mathrm{o}(h_0) & \text{if } \mathcal{I} = \mathsf{D}_0 \\
\mathrm{o}(h_0) & \text{if } \mathcal{I} = \mathsf{S}^{\mathcal{F}}_X
\end{cases}
$$

For technical reasons we also set $\mathrm{o}((d,X)) := \mathrm{o}(d)$ and $\mathrm{o}(i) := 0$ for $i \in \mathbb{N}$.

Theorem 7
$h \in \mathsf{BI}_1^{*\prime}$ & $\iota \in |\mathrm{tp}(h)|^* \;\Rightarrow\; \mathrm{o}(h[\iota]) \ll_{\mathrm{o}(\iota)} \mathrm{o}(h)$
Proof by induction on the definition of $h[\iota]$:
1. $h = \mathsf{D}_0 h_0$: Let $\alpha := \mathrm{o}(h_0)$ and $\alpha_\iota := \mathrm{o}(h_0[\iota])$.
1.1. $\mathrm{tp}(h_0) = \widetilde{\Omega}^{Y}_{\neg P}$: Then $\mathrm{tp}(h) = \mathsf{Rep}$ and $h[0] = \mathsf{D}_0 h_0[q_0]$ with $q_0 = (\mathsf{D}_0 h_0[0], Y)$, and therefore $\mathrm{o}(h[0]) = \vartheta\alpha_{q_0}$ and $\mathrm{o}(q_0) = \vartheta\alpha_0 =: \xi$. By IH $\alpha_\iota \ll_{\mathrm{o}(\iota)} \alpha$ for all $\iota \in |\mathrm{tp}(h_0)|^*$.
Hence $\alpha_0 \ll_0 \alpha$ & $\alpha_{q_0} \ll_{\xi} \alpha$ which yields $\xi = \vartheta\alpha_0 < \vartheta\alpha$ & $\alpha_{q_0} \ll_{\xi} \alpha$, and then $\mathrm{o}(h[0]) = \vartheta\alpha_{q_0} \ll_0 \vartheta\alpha = \mathrm{o}(h)$.
1.2. Otherwise: Then $|\mathrm{tp}(h)|^* = |\mathrm{tp}(h_0)|^* \subseteq \mathbb{N}$ and $h[i] = \mathsf{D}_0 h_0[i]$.
By IH $\alpha_i \ll_0 \alpha$ and thus $\mathrm{o}(h[i]) = \vartheta\alpha_i \ll_0 \vartheta\alpha = \mathrm{o}(h)$.
2. $h = \bigvee^{\mathcal{F}}_{\neg P} h_0$: Then $\mathrm{tp}(h) = \Omega_{\neg P}$ and $h[(d,X)] = \mathsf{R}_{P[\mathcal{F}]} \mathsf{S}^{\mathcal{F}}_X d\, h_0$.
Hence $\mathrm{o}(h[(d,X)]) = \mathrm{o}(d)\#\mathrm{o}(h_0) \ll_{\mathrm{o}(d)} \Omega\#\mathrm{o}(h_0) = \mathrm{o}(h)$ for all $(d,X) \in |P|^*$.

References

[Ar88] Arai, T.: On Takeuti's fundamental conjecture. English translation of the revised and reduced version of the original text in Japanese [Sugaku 40, No.4, pp. 322-337 (1988)]

[Ar96a] Arai, T.: Consistency proof via pointwise induction. Arch. Math. Logic 37, pp. 149-165 (1998)

[Bu91] Buchholz, W.: Notation systems for infinitary derivations. Arch. Math. Logic 30, pp. 277-296 (1991)

[Bu97] Buchholz, W.: Explaining Gentzen's Consistency Proof within infinitary Proof Theory. In: Gottlob, G., Leitsch, A., Mundici, D. (eds.) Computational Logic and Proof Theory. 5th Kurt Gödel Colloquium, KGC'97, Lecture Notes in Computer Science, Vol.1298. Springer 1997

[BS88] Buchholz, W. and Schütte, K.: Proof Theory of Impredicative Subsystems of Analysis. Studies in Proof Theory, Monographs 2. Napoli: Bibliopolis 1988

[Ge38] Gentzen, G.: Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie. Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften. Neue Folge 4, pp. 19-44 (1938)

[Mi75] Mints, G.: Finite Investigations of Transfinite Derivations. In: Mints,G., Selected Papers in Proof Theory. Studies in Proof Theory, Monographs 3. Napoli: Bibliopolis 1992. Russian original: Zapiski Nauchnyh seminarov LOMI 49 (1975) pp. 67-122

[Mi75a] Mints, G.: Normalization of Finite Terms and Derivations via Infinite Ones. In: Mints,G., Selected Papers in Proof Theory. Studies in Proof Theory, Monographs 3. Napoli: Bibliopolis 1992. Original: Abstracst of the Congress on Logic, Methodology, and Philosophy of Science, Bukarest 1975.

[Mi79] Mints, G.: A New Reduction Sequence for Arithmetic. In: Mints,G., Selected Papers in Proof Theory. Studies in Proof Theory, Monographs 3. Napoli: Bibliopolis 1992. Russian original: Zapiski Nauchnyh seminarov LOMI 88 (1979) pp. 130-?

[Mi93] Mints, G.: A normal form for logical derivations implying one for arithmetic derivations. APAL 62 (1993) pp. 65-79

[RW93] Rathjen, M. and Weiermann, A.: Proof-theoretic investigations on Kruskal's theorem. APAL 60 (1993) pp. 49-88

[Tak87] Takeuti, Proof Theory. 2nd edition. Amsterdam: North-Holland 1987