An intuitionistic fixed point theory

Wilfried Buchholz

Mathematisches Institut, Universität München, Theresienstr. 39, D-80333 München, Germany
email: buchholz@rz.mathematik.uni-muenchen.de

Introduction

We will prove that a certain intuitionistic fixed-point theory $\widehat{\mathrm{ID}}_1^i$ is conservative over HA for almost negative formulas. At first sight this result is a little bit surprising, since $\widehat{\mathrm{ID}}_1^i$ with classical logic (i.e. the theory $\widehat{\mathrm{ID}}_1$) is proof-theoretically equivalent to Σ^1_1-AC, a theory much stronger than Peano-arithmetic PA (cf. [4]), while on the other hand for theories of (iterated) inductive definitions the intuitionistic and classical versions have the same proof-theoretic strength. A closer inspection reveals, that the difference in strength between $\widehat{\mathrm{ID}}_1^i$ and $\widehat{\mathrm{ID}}_1$ corresponds to the different power of (versions of) the axiom of choice in intuitionistic and classical logic.

$\widehat{\mathrm{ID}}_1^i$ is obtained from HA by adding for each strongly positive operator form $\Phi(P,x)$ (cf. pg. 3) a new predicate constant P_Φ and the axiom
$(\mathrm{FP}_\Phi)\quad \forall x(\,\Phi(P_\Phi,x) \leftrightarrow P_\Phi x\,)$; moreover the scheme of complete induction is extended to all formulas of the new language. We will prove that $\widehat{\mathrm{ID}}_1^i$ is interpretable in $\mathsf{HA}+\mathrm{CT}_0$, where CT_0 denotes the following schema (cf. [11] 3.2.14):
$(\mathrm{CT}_0)\ \forall x\exists y B(x,y) \to \exists u \forall x\, B(x,\{u\}(x))$.

Theorem 1
For each strongly positive operator form Φ there is an arithmetical formula $\mathbb{P}_\Phi(x)$ such that $\mathsf{HA}+\mathrm{CT}_0 \vdash \forall x(\,\Phi(\mathbb{P}_\Phi,x) \leftrightarrow \mathbb{P}_\Phi(x)\,)$.

Theorem 1 together with standard results on recursive realizability yields the announced conservative extension result.

Theorem 2
$\widehat{\mathrm{ID}}_1^i$ *is conservative over* HA *w.r.t. almost negative formulas.*

Proof:
For each formula A in the language of $\widehat{\mathrm{ID}}_1^i$ let A^* denote the result of replacing in A every subformula $P_\Phi t$ by $\mathbb{P}_\Phi(t)$. Then we have:
(1) $\widehat{\mathrm{ID}}_1^i \vdash A \;\Rightarrow\; \mathsf{HA}+\mathrm{CT}_0 \vdash A^*$ (Theorem 1),
(2) $\mathsf{HA}+\mathrm{CT}_0 \vdash B \;\Rightarrow\; \mathsf{HA} \vdash \exists u(u\mathbf{r}B)$ ([11] 3.2.18(ii)),
(3) $\mathsf{HA} \vdash \exists u(u\mathbf{r}B) \to B$, if B almost negative ([11] 3.2.11(i)).
Combining (1),(2),(3) yields the theorem.

Given a primitive recursive wellordering $\prec$, let $\mathrm{TI}(\prec\!\restriction)$ denote the scheme of transfinite induction over all proper initial segments of $\prec$. Then the above proof easily extends to $\widehat{\mathrm{ID}}_1^i+\mathrm{TI}(\prec\!\restriction)$, $\mathsf{HA}+\mathrm{TI}(\prec\!\restriction)$, (for step (2) cf. [11] 3.2.24(ii)), and one obtains

Theorem 2'
$\widehat{\mathrm{ID}}_1^i+\mathrm{TI}(\prec\!\restriction)$ *is conservative over* $\mathsf{HA}+\mathrm{TI}(\prec\!\restriction)$ *w.r.t. almost negative formulas.*

An interesting aspect of Theorem 2' is that $\widehat{\mathrm{ID}}_1^i$+TI($\prec\!\restriction$) is a canonical metatheory for carrying out cut-elimination in semiformal systems á la Schütte, and therefore it provides an elegant means to derive from a Schütte-style ordinal analysis for some formal theory *Th* (as can be found e.g. in [2], [3] Ch.VI, [6], [8], [9], [10] Ch.VIII) a result like *"every arithmetical sentence provable in Th is already provable in* PA+TI($\prec\!\restriction$)*"* (which of course can also be obtained by the classical method via coding of infinitary derivations by indices for recursive functions; cf. e.g. [3] pp. 306–330). Let us explain this in some more detail.

In essence a semiformal system á la Schütte is given by a derivability predicate $\mathcal{D}(\alpha,\rho,F)$ (*'F is derivable with order α and cut-rank ρ'*) defined by transfinite recursion on α as follows:

$$(\star)\quad \mathcal{D}(\alpha,\rho,F) \Leftrightarrow \begin{cases} \alpha \in field(\prec), \text{ and either } F \text{ is an axiom or } F \text{ is the con-} \\ \text{clusion of an inference with premises } (F_\iota)_{\iota\in I} \text{ such that} \\ \text{for every } \iota\in I \text{ there exists a } \beta_\iota \prec \alpha \text{ with } \mathcal{D}(\beta_\iota,\rho,F_\iota), \\ \text{and if the inference is a cut this has rank } \prec \rho. \end{cases}$$

Equivalently one can consider ($\star$) as a fixed-point axiom (available in $\widehat{\mathrm{ID}}_1^i$) which together with TI($\prec\!\restriction$) implicitely defines $\mathcal{D}$ as the least fixed point of the operator given by the righthand side of ($\star$).

Now let us assume that *Th* is a theory with classical logic (for intuitionistic theories the following argument is even simpler) and that we have established an ordinal analysis for *Th* using the well-ordering $\prec$. This includes that we have proved (via cut-elimination for a suitable semi-formal system):

$$\text{(X)}\ \ Th \vdash A \ \&\ A \text{ arithmetical } \Rightarrow \begin{cases} \text{there exists an } \alpha \in field(\prec) \text{ such that } A \text{ is} \\ \text{provable with order } \alpha \text{ in some cut-free} \\ \text{sequent calculus for classical } \omega\text{-arithmetic.} \end{cases}$$

For technical reasons let us assume that the sequent calculus mentioned in (X) is Tait-style, where sequents are finite sets of formulas and negation is defined via de Morgan's laws. By inspection of the proof of (X) (and by the above considerations concerning definability of semi-formal systems in $\widehat{\mathrm{ID}}_1^i$+TI($\prec\!\restriction$)) one easily sees that the meta-mathematical means used in that proof are all formalizable in $\widehat{\mathrm{ID}}_1^i$+TI($\prec\!\restriction$), and one immediately concludes:

$$\text{('X')}\ \ Th \vdash A \ \&\ A \text{ arithmetical } \Longrightarrow \widehat{\mathrm{ID}}_1^i\text{+TI}(\prec\!\restriction) \vdash P_\omega\langle \underline{\alpha}, \ulcorner\{A\}\urcorner\rangle,$$

where P_ω denotes a fixed-point constant of $\widehat{\mathrm{ID}}_1^i$ representing derivability in the cutfree Tait-calculus for classical ω-arithmetic.

By combining the (standard) technique of partial truth-predicates with the Gödel-Gentzen-translation $A \mapsto A^g$ one obtains

Theorem 3
$\widehat{\mathrm{ID}}_1^i$+TI($\prec\!\restriction$) $\vdash P_\omega\langle \underline{\alpha}, \ulcorner\{A\}\urcorner\rangle \to A^g$,
for each arithmetic sentence A, and each $\alpha \in field(\prec)$.

From ('X'), Theorem 2' and Theorem 3 one concludes that every arithmetical sentence provable in *Th* is already provable in PA+TI($\prec\!\restriction$).

In §1 below we will prove Theorem 1 in a somewhat stronger form, namely we will define a single arithmetical formula $\mathbb{P}(x,y)$ such that $\mathsf{HA} + \mathrm{CT}_0 \vdash \forall x(\,\Phi(\mathbb{P}_\Phi,x) \leftrightarrow \mathbb{P}_\Phi(x)\,)$ holds with $\mathbb{P}_\Phi(x) :\equiv \mathbb{P}(x,\ulcorner\Phi\urcorner)$ for each strongly positive operator form Φ. In §2 we sketch a proof of Theorem 3.

Remark

Having seen a preliminary version of this note Toshiyasu Arai was able to extend and strengthen our Theorem 2 as follows.

Theorem (Arai 1993)

$\widehat{\mathrm{ID}}^i_n$ is conservative over HA *(w.r.t. all arithmetic sentences) for each $n > 0$.*

Arai's proof runs as follows. First $\widehat{\mathrm{ID}}^i_n$ is interpreted in intuitionistic analysis EL+AC-NF. This is done by following Feferman's proof in [4]. Then by Goodman's theorem (cf. [5]) one can conclude the conservative extension result.

§1 Interpretation of $\widehat{\mathrm{ID}}^i_1$ in $\mathsf{HA}+\mathrm{CT}_0$

Let $\mathcal{L}_0$ be the language of arithmetic (with function symbols for all primitive recursive functions, but without second order variables). As usual $\mathrm{FV}(A)$ denotes the set of free variables of A. For each natural number n let $\underline{n}$ denote the corresponding numeral, i.e. the canonical $\mathcal{L}_0$-term $\mathsf{S}...\mathsf{S}0$ representing n. By TRUE_0 we denote the set of all true atomic $\mathcal{L}_0$-senetences. An $\mathcal{L}_0$-formula is said to be *almost negative* if it does not contain $\vee$, and contains $\exists$ only in front of an equation between terms (cf. [11] pg.193).

Let P be a new unary predicate symbol and let POS be the set of all $\mathcal{L}_0[P]$-formulas built up from formulas Py (y Variable) and atomic $\mathcal{L}_0$-formulas by means of $\wedge, \vee, \forall, \exists$.

$\mathrm{POS}_0 := \{A \in \mathrm{POS} : \mathrm{FV}(A) = \emptyset\}$

$\mathrm{POS}^* := \{\Phi \in \mathrm{POS} : \mathrm{FV}(\Phi) = \{x\}\}$ (where x is some fixed variable).

The formulas $\Phi \in \mathrm{POS}^*$ are called *strongly positive operator forms*.

Given $\Phi \in \mathrm{POS}^*$, a term s, and a formula $F(x)$ we denote by $\Phi(F, s)$ the result of replacing in Φ every free occurrence of x by s and every subformula Pt by $F(t)$.

The language $\mathcal{L}_1$ is obtained from $\mathcal{L}_0$ by adding a new unary predicate symbol P_Φ for each $\Phi \in \mathrm{POS}^*$.

The theory $\widehat{\mathrm{ID}}^i_1$ is HA (formulated in $\mathcal{L}_1$) extended by the fixed-point axioms:

(FP_Φ) $\forall x(\Phi(P_\Phi, x) \leftrightarrow P_\Phi x)$ $\quad (\Phi \in \mathrm{POS}^*)$

Proof of Theorem 1:

We fix an arbitrary $\Phi \in \mathrm{POS}^*$ and consider the following inference rules (with formulas from POS_0):

$$(\mathrm{Ax})\ \ \frac{}{A}\,,\ \text{for } A \in \mathrm{TRUE}_0 \qquad (\wedge)\ \ \frac{A_0 \quad A_1}{A_0 \wedge A_1}\,, \qquad (\vee)\ \ \frac{A_i}{A_0 \vee A_1}\,,$$

$$(\forall)^\infty\ \ \frac{\ldots A(\underline{n}) \ldots (n \in \mathbb{N})}{\forall x A(x)}\,, \qquad (\exists)\ \ \frac{A(\underline{k})}{\exists x A(x)}\,, \qquad (P)\ \ \frac{\Phi(P, \underline{n})}{P\underline{n}}$$

Then we define:

A *Φ-proof of A* is a (possibly non-wellfounded) tree of formulas from POS_0 which is locally correct w.r.t. the above inference rules and has endformula A.

$\vdash_\Phi A \ :\Leftrightarrow$ There exists a Φ-proof of A.

$\mathcal{P}_\Phi := \{n \in \mathbb{N} : \vdash_\Phi P\underline{n}\}$.

As one can easily verify, $\mathcal{P}_\Phi$ is a fixed-point of Φ,
i.e. $(\forall n)(n \in \mathcal{P}_\Phi \Leftrightarrow \mathbb{N} \models \Phi(\mathcal{P}_\Phi, n))$.

For example let $\Phi(P,x) \equiv x = 0 \vee \forall y \exists z(fxyz = 0 \wedge Pz)$. Then we have

$n \in \mathcal{P}_\Phi \Leftrightarrow$
$\Leftrightarrow \vdash_\Phi \Phi(P, \underline{n})$
$\Leftrightarrow \vdash_\Phi \underline{n} = 0$ or $\vdash_\Phi \forall y \exists z(f\underline{n}yz = 0 \wedge Pz)$
$\Leftrightarrow n = 0$ or $(\forall m) \vdash_\Phi \exists z(f\underline{n}\,\underline{m}\,z = 0 \wedge Pz)$
$\Leftrightarrow n = 0$ or $(\forall m)(\exists k) \vdash_\Phi f\underline{n}\,\underline{m}\,\underline{k} = 0 \wedge P\underline{k}$
$\Leftrightarrow n = 0$ or $(\forall m)(\exists k)(f(n,m,k) = 0 \,\&\, k \in \mathcal{P}_\Phi)$
$\Leftrightarrow \mathbb{N} \models \Phi(\mathcal{P}_\Phi, n)$.

Now we define an $\mathcal{L}_0$-formula $\mathbb{B}(e,a,q)$ such that (for $A \in \mathrm{POS}_0$ and $\Phi \in \mathrm{POS}^*$) $\mathbb{B}(e, \ulcorner A \urcorner, \ulcorner \Phi \urcorner)$ expresses that e codes a (recursive) Φ-proof with endformula A, i.e.

$\mathbb{B}(e,a,q) :\equiv$
$\forall v \exists z\big(\{e\}(0) \simeq a \wedge \{e\}(v) \simeq z \wedge$ '$z \in \mathrm{POS}_0$' $\wedge$ (
'$z \in \mathrm{TRUE}_0$' $\vee$
$[\,(z)_0 = \ulcorner \wedge \urcorner \wedge \{e\}(v * \langle 0 \rangle) \simeq (z)_1 \wedge \{e\}(v * \langle 1 \rangle) \simeq (z)_2\,] \vee$
$[\,(z)_0 = \ulcorner \vee \urcorner \wedge (\{e\}(v * \langle 0 \rangle) \simeq (z)_1 \vee \{e\}(v * \langle 0 \rangle) \simeq (z)_2)\,] \vee$
$[\,(z)_0 = \ulcorner \exists \urcorner \wedge \exists x(\{e\}(v * \langle 0 \rangle) \simeq \mathsf{sub}(z)_2(z)_1\, \nu x))\,] \vee$
$[\,(z)_0 = \ulcorner \forall \urcorner \wedge \forall x(\{e\}(v * \langle x \rangle) \simeq \mathsf{sub}(z)_2(z)_1\, \nu x)\,] \vee$
$[\,(z)_0 = \ulcorner P \urcorner \wedge \{e\}(v * \langle 0 \rangle) \simeq \mathsf{sub}\, q\, \ulcorner x \urcorner (z)_1\,]\big)\big)$.

Here ν denotes the prim. rec. function defined by $\nu(n) := \ulcorner \underline{n} \urcorner$. Concerning $\ulcorner\ \urcorner$ we stick to the following convention: in the meta-language $\ulcorner \theta \urcorner$ denotes 'the' Gödel number of θ (where θ is some string of symbols), while in the object language $\ulcorner \theta \urcorner$ denotes the numeral representing the Gödel number of θ, i.e. there $\ulcorner \theta \urcorner$ serves as abbreviation for $\underline{\ulcorner \theta \urcorner}$. All other notations used in the definition of $\mathbb{B}$ are standard or selfexplaining. In the sequel we also make use of the so-called 'dot notation': if A is a formula with free variables $x_1, ..., x_n$ then $\ulcorner A(\dot{x}_1, ..., \dot{x}_n) \urcorner$ is a prim. rec. term with free variables $x_1, ..., x_n$ representing the function $(k_1, ..., k_n) \mapsto \ulcorner A(\underline{k_1}, ..., \underline{k_n}) \urcorner$ from $\mathbb{N}^n$ to $\mathbb{N}$.

Definition: $\mathbb{P}(x,q) :\equiv \exists e\, \mathbb{B}(e, \ulcorner P\dot{x} \urcorner, q)$
and $\mathbb{P}_\Phi(x) :\equiv \mathbb{P}(x, \ulcorner \Phi \urcorner)$, $\mathbb{B}_\Phi(e,x) :\equiv \mathbb{B}(e, x, \ulcorner \Phi \urcorner)$.

The proof of $\mathbb{P}_\Phi(x) \leftrightarrow \Phi(\mathbb{P}_\Phi, x)$ in $\mathsf{HA} + \mathrm{CT}_0$ runs along the same lines as the above informal proof. We think that the crucial steps become sufficiently clear if we again consider the example $\Phi(P,x) \equiv x = 0 \vee \forall y \exists z(fxyz = 0 \wedge Pz)$ only. For that Φ we have

$\mathsf{HA} + \mathrm{CT}_0 \vdash$
$\mathbb{P}_\Phi(x) \leftrightarrow$
$\exists e\, \mathbb{B}_\Phi(e, \ulcorner \Phi(P, \dot{x}) \urcorner) \leftrightarrow$
$\exists e\, \mathbb{B}_\Phi(e, \ulcorner \dot{x} = 0 \urcorner) \vee \exists e\, \mathbb{B}_\Phi(e, \ulcorner \forall y \exists z(f\dot{x}yz = 0 \wedge Pz) \urcorner) \leftrightarrow$
'$\ulcorner \dot{x} = 0 \urcorner \in \mathrm{TRUE}_0$' $\vee \exists u \forall y\, \mathbb{B}_\Phi(\{u\}(y), \ulcorner \exists z(f\dot{x}\dot{y}z = 0 \wedge Pz) \urcorner) \leftrightarrow$
$x = 0 \vee \forall y \exists e\, \mathbb{B}_\Phi(e, \ulcorner \exists z(f\dot{x}\dot{y}z = 0 \wedge Pz) \urcorner) \leftrightarrow$
$x = 0 \vee \forall y \exists z \exists e\, \mathbb{B}_\Phi(e, \ulcorner f\dot{x}\dot{y}\dot{z} = 0 \wedge P\dot{z} \urcorner) \leftrightarrow$
$x = 0 \vee \forall y \exists z(fxyz = 0 \wedge \exists e\, \mathbb{B}_\Phi(e, \ulcorner P\dot{z} \urcorner)\,) \leftrightarrow$
$\Phi(\mathbb{P}_\Phi, x)$.

Here CT_0 is necessary to obtain the implication
$\forall y \exists e \mathbb{B}_\Phi(e, \ulcorner \exists z(f\dot{x}\dot{y}z = 0 \wedge Pz) \urcorner) \rightarrow \exists u \forall y \mathbb{B}_\Phi(\{u\}(y), \ulcorner \exists z(f\dot{x}\dot{y}z = 0 \wedge Pz) \urcorner)$.

§2 Partial truth-predicate and Gödel-Gentzen-translation

In this section we sketch a proof of Theorem 3, being aware that these things are more or less standard and wellknown. But on the other side it seems to us that the details are not completely obvious.

Definition

$$\mathrm{rk}(a) := \begin{cases} \max\{\mathrm{rk}((a)_1), \mathrm{rk}((a)_2)\} + 1 & \text{if } (a)_0 > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (a \in \mathbb{N})$$

$\mathrm{rk}(A) := \mathrm{rk}(\ulcorner A \urcorner)$.

Definition

$\mathcal{T}_0(z) :\equiv \neg\neg\mathrm{TRUE}_0(z)$,
where $\mathrm{TRUE}_0(z)$ is an $\mathcal{L}_0$-formula such that $\mathsf{HA} \vdash \mathrm{TRUE}_0(\ulcorner A \urcorner) \leftrightarrow A$, for each atomic $\mathcal{L}_0$-sentence A.

$\mathcal{T}_{n+1}(z) :\equiv$
$[\,(z)_0 = \ 0 \ \ \to \mathcal{T}_n(z)\,] \wedge$
$[\,(z)_0 = \ulcorner \neg \urcorner \to \neg\mathcal{T}_n((z)_1)\,] \wedge$
$[\,(z)_0 = \ulcorner \wedge \urcorner \to \mathcal{T}_n((z)_1) \wedge \mathcal{T}_n((z)_2)\,] \wedge$
$[\,(z)_0 = \ulcorner \to \urcorner \to \mathcal{T}_n((z)_1) \to \mathcal{T}_n((z)_2)\,] \wedge$
$[\,(z)_0 = \ulcorner \vee \urcorner \to \neg(\neg\mathcal{T}_n((z)_1) \wedge \neg\mathcal{T}_n((z)_2))\,] \wedge$
$[\,(z)_0 = \ulcorner \forall \urcorner \to \forall x\, \mathcal{T}_n(\mathsf{sub}(z)_2(z)_1\, \nu x)\,] \wedge$
$[\,(z)_0 = \ulcorner \exists \urcorner \to \neg\forall x \neg\mathcal{T}_n(\mathsf{sub}(z)_2(z)_1\, \nu x)\,]$.

Here we have assumed $0 < \ulcorner \neg \urcorner, \ulcorner \wedge \urcorner, \ulcorner \to \urcorner, ...$, and $(\ulcorner A \urcorner)_0 = 0$ for atomic A.

Lemma 1

a) $\mathsf{HA} \vdash \mathcal{T}_n(z) \leftrightarrow \neg\neg\mathcal{T}_n(z)$.
b) $\mathsf{HA} \vdash \mathrm{rk}(z) \leq \underline{n} \to (\mathcal{T}_n(z) \leftrightarrow \mathcal{T}_{n+1}(z))$.

Definition (Gödel-Gentzen-translation)

$A^g :\equiv \neg\neg A$, if A is atomic, $(\neg A)^g :\equiv \neg A^g$, $(A \wedge B)^g :\equiv A^g \wedge B^g$, $(A \to B)^g :\equiv A^g \to B^g$, $(A \vee B)^g :\equiv \neg(\neg A^g \wedge \neg B^g)$, $(\forall x A)^g :\equiv \forall x A^g$, $(\exists x A)^g :\equiv \neg\forall x \neg A^g$.

Lemma 2

If A is an $\mathcal{L}_0$-sentence with $\mathrm{rk}(A) \leq n$ then $\mathsf{HA} \vdash \mathcal{T}_n(\ulcorner A \urcorner) \leftrightarrow A^g$.

Notations:
For $M = \{a_0, ..., a_{n-1}\} \subseteq \mathbb{N}$ with $a_0 > ... > a_{n-1}$ let $M^\# := 2^{a_0} + ... + 2^{a_{n-1}}$.

If Γ is a finite set of formulas then $\ulcorner \Gamma \urcorner := \{\ulcorner A \urcorner : A \in \Gamma\}^\#$.

Let $\hat{\in}$ denote the prim. rec. relation $\{(a, M^\#) : M \subseteq \mathbb{N}$ finite and $a \in M\}$, and $\hat{\cup}$ the prim. rec. function defined by $\hat{\cup}(M^\#, b) := (M \cup \{b\})^\#$. More precisely, we assume that $\hat{\in}$ is a relation symbol and $\hat{\cup}$ a function symbol of $\mathcal{L}_0$ such that $\mathsf{HA} \vdash x \,\hat{\in}\, \hat{\cup} wy \leftrightarrow x \,\hat{\in}\, w \vee x = y$ and $\mathsf{HA} \vdash x \,\hat{\in}\, \ulcorner\{A\}\urcorner \leftrightarrow x = \ulcorner A \urcorner$.

Abbreviations: $\overline{\mathcal{T}_n}(w) :\equiv \forall x(x \,\hat{\in}\, w \to \neg\mathcal{T}_n(x))$, $\qquad \mathcal{T}_n^{seq}(w) :\equiv \neg\overline{\mathcal{T}_n}(w)$.

Lemma 3 $\quad \mathsf{HA} \vdash \mathcal{T}_n^{seq}(\hat{\cup} wy) \leftrightarrow (\overline{\mathcal{T}_n}(w) \to \mathcal{T}_n(y))$.

Proof:
We have $\vdash \forall x(x \,\hat{\in}\, \hat{\cup} wy \to \neg\mathcal{T}_n(x)) \leftrightarrow \forall x(x \,\hat{\in}\, w \to \neg\mathcal{T}_n(x)) \wedge \neg\mathcal{T}_n(y)$,
i.e. $\vdash \overline{\mathcal{T}_n}(\hat{\cup} wy) \leftrightarrow \overline{\mathcal{T}_n}(w) \wedge \neg\mathcal{T}_n(y)$.
This implies $\vdash \mathcal{T}_n^{seq}(\hat{\cup} wy) \leftrightarrow \neg(\overline{\mathcal{T}_n}(w) \wedge \neg\mathcal{T}_n(y)) \leftrightarrow (\overline{\mathcal{T}_n}(w) \to \neg\neg\mathcal{T}_n(y))$.
Hence $\vdash \mathcal{T}_n^{seq}(\hat{\cup} wy) \leftrightarrow (\overline{\mathcal{T}_n}(w) \to \mathcal{T}_n(y))$ by Lemma 1a.

Lemma 4 $\mathsf{HA} \vdash \mathrm{rk}(z) \leq \underline{n} \wedge C_n(w,z) \ \rightarrow \mathcal{T}_n^{seq}(\hat{\cup} w z)$, *where* $C_n(w,z)$ *abbreviates the disjunction of (i)–(iv) below:*
(i) $(z)_0 = \ulcorner \wedge \urcorner \wedge \mathcal{T}_n^{seq}(\hat{\cup} w (z)_1) \wedge \mathcal{T}_n^{seq}(\hat{\cup} w (z)_2)$
(ii) $(z)_0 = \ulcorner \vee \urcorner \wedge (\mathcal{T}_n^{seq}(\hat{\cup} w (z)_1) \vee \mathcal{T}_n^{seq}(\hat{\cup} w (z)_2))$
(iii) $(z)_0 = \ulcorner \forall \urcorner \wedge \forall x \, \mathcal{T}_n^{seq}(\hat{\cup} w \, \mathsf{sub}(z)_2 (z)_1 \, \nu x)$
(iv) $(z)_0 = \ulcorner \exists \urcorner \wedge \exists x \, \mathcal{T}_n^{seq}(\hat{\cup} w \, \mathsf{sub}(z)_2 (z)_1 \, \nu x)$.

Proof: We only consider (iv). The other cases are treated similarly.
Assume $\mathrm{rk}(z) \leq \underline{n} \wedge (z)_0 = \ulcorner \exists \urcorner \wedge \exists x \, \mathcal{T}_n^{seq}(\hat{\cup} w \, \mathsf{sub}(z)_2 (z)_1 \, \nu x)$.
By Lemma 1b and Lemma 3 this implies
$\mathcal{T}_n(z) \leftrightarrow \neg \forall x \neg \mathcal{T}_n(\mathsf{sub}(z)_2 (z)_1 \, \nu x)$ and $\exists x(\overline{\mathcal{T}_n}(w) \rightarrow \mathcal{T}_n(\mathsf{sub}(z)_2 (z)_1 \, \nu x) \,)$.
Since the scheme $\exists x(B \rightarrow A) \rightarrow (B \rightarrow \neg \forall x \neg A) \quad (x \notin \mathrm{FV}(B))$ is valid in intuitionistic logic, we get from the above $\overline{\mathcal{T}_n}(w) \rightarrow \mathcal{T}_n(z)$, and thus $\mathcal{T}_n^{seq}(\hat{\cup} w z)$ by Lemma 3.

Lemma 5 *For each* $\alpha \in field(\prec)$ *we have*
$\widehat{\mathrm{ID}}_1^i + \mathrm{TI}(\prec\restriction) \vdash u \preceq \underline{\alpha} \ \wedge \ P_\omega \langle u, w \rangle \ \wedge \ \forall x (x \,\hat{\in}\, w \rightarrow \mathrm{rk}(x) \leq \underline{n}) \rightarrow \mathcal{T}_n^{seq}(w)$.

Proof by (formal) $\prec$-induction on u.

Proof of Theorem 3:
Let $\alpha \in field(\prec)$ and $n := \mathrm{rk}(A)$. By Lemma 5 we get $\vdash P_\omega \langle \underline{\alpha}, \ulcorner \{A\} \urcorner \rangle \rightarrow \mathcal{T}_n^{seq}(\ulcorner \{A\} \urcorner)$. Obviously the above assumption on $\hat{\in}$ implies $\vdash \mathcal{T}_n^{seq}(\ulcorner \{A\} \urcorner) \leftrightarrow \neg\neg \mathcal{T}_n(\ulcorner A \urcorner)$. Now by Lemma 1a and Lemma 2 we get $\vdash P_\omega \langle \underline{\alpha}, \ulcorner \{A\} \urcorner \rangle \rightarrow A^g$.

References

1. Arai, T.: Remark on Buchholz' paper 'An intuitionistic fixed point theory'. Handwritten notes (1993)
2. Buchholz, W.: A simplified version of local predicativity. In: Aczel, Simmons, Wainer (eds.), Leeds Proof Theory 1990, Cambridge University Press 1992
3. Buchholz, W., Feferman, S., Pohlers, W., Sieg, W.: Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies. (Lect. Notes Math., vol. 897) Berlin Heidelberg New York: Springer 1981
4. Feferman, S.: Iterated fixed-point theories: application to Hancock's conjecture. In: G. Metakides (ed.) The Patras Symposium. Amsterdam: North-Holland 1982
5. Goodman, N.: Relativized realizability in intuitionistic arithmetic in all finite types. Journal of Symbolic Logic 43, 23–44 (1978)
6. Jäger, G., Pohlers, W.: Eine beweistheoretische Untersuchung von (Δ^1_2-CA)+(BI) und verwandter Systeme. Sitzungsber. d. Bayer. Akad. d. Wiss., Math.-Nat. Kl., 1–28 (1982).
7. Jäger, G., Primo, B.: About the proof-theoretic strength of weak fixed point theories. Journal of Symbolic Logic 57(3) (1992)
8. Rathjen, M.: Untersuchungen zu Teilsystemen der Zahlentheorie zweiter Stufe und der Mengenlehre mit einer zwischen (Δ^1_2-CA) und (Δ^1_2-CA)+(BI) liegenden Beweisstärke. Ph.D. thesis, Münster 1989
9. Rathjen, M.: Proof Theory of Reflection. Annals of Pure and Applied Logic 68, 181–224 (1994)

10. Schütte, K.: Proof Theory. Berlin Heidelberg New York: Springer 1977
11. Troelstra, A.: Metamathematical Investigation of Intuitionistic Arithmetic and Analysis. (Lect. Notes Math., vol. 344) Berlin Heidelberg New York: Springer 1973