An approach to variable binding using de Bruijn indices and named variables

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In these notes we present a method for representing syntax with variable binding which combines the advantages of the de Bruijn approach and the conventional approach via named bound variables. Unfortunately, after having produced a first version of these notes we had to notice that this method is not new; it was first introduced in

A. D. Gordon. A mechanisation of name-carrying syntax up to alpha-conversion. In J. J. Joyce and C.-J. H. Seger (Eds.), *Higher Order Logic Theorem Proving and its Applications. Proceedings, 1993*, Lecture Notes in Computer Science Vol. 780, pp. 414-426. Springer-Verlag, 1994.

We cite the abstract of this paper:

> *Abstract. We present a new strategy for representing syntax in a mechanised logic. We define an underlying type of de Bruijn terms, define an operation of named lambda-abstraction, and hence inductively define a set of conventional name-carrying terms. The result is a mechanisation of the practice of most authors studying formal calculi: to work with conventional name-carrying notation and substitution, but to identify terms up to alpha-conversion. This strategy falls between most previous works, which either treat bound variable names literally or dispense with them altogether. The theory has been implemented in the Cambridge HOL system and used in an experimental application.*

Our emphasis here is not so much on mechanised logics but rather on showing that the approach in question is very well suited for a neat representation of syntax in ordinary logic texts. To underpin this claim we develop in detail some basics of the untyped λ-calculus, the system $\lambda 2$ in Church style, and 1st order predicate logic.

§1 A general framework for variable binding and substitution

We assume the following pairwise disjoint sets of *basic symbols*.

Vars : infinite set of *variables*, denoted by $x, y, z, ...$;

$\{\circ_k : k \in \mathbb{N}\}$: set of *de Bruijn indices*;

$\mathcal{F}$: set of *function symbols*, denoted by f;

$\mathcal{B}$: set of *binding symbols (binders)*, denoted by $\flat$.

For every $f \in \mathcal{F}$ an *arity* $\#(f) \in \mathbb{N}$ is fixed; further we set $\#(\circ_k) := 0$ and $\#(\flat) := 1$.

$\mathcal{F}' := \{\circ_k : k \in \mathbb{N}\} \cup \mathcal{F} \cup \mathcal{B}$, $\mathcal{F}'_m := \{h \in \mathcal{F}' : \#(h) = m\}$, $\mathcal{F}_m := \mathcal{F}'_m \cap \mathcal{F}$.

Inductive Definition of the set $\mathcal{T}' = \mathcal{T}'(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ of quasiterms

1. $\mathsf{Vars} \subseteq \mathcal{T}'$;
2. $h \in \mathcal{F}'_m \;\&\; t_1, ..., t_m \in \mathcal{T}' \implies ht_1...t_m \in \mathcal{T}'$.

Notation: We use r, s, t to denote quasiterms.

Definition.

$\mathrm{FV}(t) :=$ set of all variables occurring in t,

$\mathrm{lh}(t) :=$ length of t as string of basic symbols.

Definition of $t_x[n] \in \mathcal{T}'$ for $t \in \mathcal{T}'$

1. For $t \in \mathsf{Vars} \cup \{\circ_k : k \in \mathbb{N}\}$: $t_x[n] := \begin{cases} \circ_n & \text{if } t = x \\ t & \text{otherwise} \end{cases}$;
2. $(ft_1...t_m)_x[n] := f(t_1)_x[n]...(t_m)_x[n]$;
3. $(\flat r)_x[n] := \flat r_x[n{+}1]$.

Definition. $\flat x.r := \flat r_x[0]$.

Corollary. (B0) $\mathrm{FV}(\flat x.r) = \mathrm{FV}(r) \setminus \{x\}$.

Proof: $\mathrm{FV}(\flat x.r) = \mathrm{FV}(\flat r_x[0]) = \mathrm{FV}(r_x[0]) = \mathrm{FV}(r) \setminus \{x\}$.

Inductive Definition of the set $\mathcal{T} = \mathcal{T}(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ of terms

1. $\mathsf{Vars} \subseteq \mathcal{T}$;
2. $f \in \mathcal{F}_m \;\&\; t_1, ..., t_m \in \mathcal{T} \implies ft_1...t_m \in \mathcal{T}$;
3. $\flat \in \mathcal{B} \;\&\; r \in \mathcal{T} \implies \flat x.r \in \mathcal{T}$.

Definition

A *substitution* is a mapping $\theta : \mathcal{T}' \to \mathcal{T}'$, $t \mapsto t\theta$ such that

(i) $x\theta \in \mathcal{T}$ for all $x \in \mathsf{Vars}$,

(ii) $(ht_1...t_m)\theta = h(t_1\theta)...(t_m\theta)$ for all $ht_1...t_m \in \mathcal{T}' \setminus \mathsf{Vars}$.

SUB := set of all substitutions. $\epsilon := \mathrm{id}_{\mathcal{T}'}$.

We use θ, θ' to denote substitutions.

Lemma 1.1.

(S0) $\forall t \in \mathcal{T}(t\theta \in \mathcal{T})$;

(S1) $\forall x \in \mathrm{FV}(t)(x\theta = x\theta') \iff t\theta = t\theta'$;

(S2) $\epsilon \in \mathrm{SUB}$;

(S3) $\theta, \theta' \in \mathrm{SUB} \implies \theta \circ \theta' \in \mathrm{SUB}$.

(S4) For every $\theta \in \mathrm{SUB}$, $x \in \mathsf{Vars}$, $s \in \mathcal{T}$ there is a unique $\theta^s_x \in \mathrm{SUB}$ with $y\theta^s_x := \begin{cases} s & \text{if } y = x \\ y\theta & \text{otherwise} \end{cases}$.

The proof of (S0) will be given below. The other statements are easily seen, where for (S3) one uses (S0).

Remark. $\forall x \in \mathsf{Vars}(x\theta = x\theta') \implies \theta = \theta'$. [cf. (S1)]

Notation: $t_x(s) := t(x/s) := t\,\epsilon^s_x$

Remark. $x \notin \mathrm{FV}(t) \Rightarrow t_x(s) = t$. [cf. (S1),(S2)]

Lemma 1.2.

(a) $x \neq y \Rightarrow (x \in \mathrm{FV}(t) \Leftrightarrow t\epsilon^y_x \neq t)$.

(b) $\mathrm{FV}(t\theta) = \bigcup_{z \in \mathrm{FV}(t)} \mathrm{FV}(z\theta)$.

(c) $y \notin (\flat x.r)\theta \implies r\theta^y_x\epsilon^s_y = r\theta^s_x$.

(d) $x \in \mathrm{FV}(t) \implies \mathrm{FV}(t_x(s)) = (\mathrm{FV}(t) \setminus \{x\}) \cup \mathrm{FV}(s)$.

(e) $r_x(s)\theta = r\theta^{s\theta}_x$.

(f) $y \notin \mathrm{FV}(\flat x.r)\theta \implies r_x(s)\theta = (r\theta^y_x)_y(s\theta)$.

Proof:

(a) $x \notin \mathrm{FV}(t) \Leftrightarrow \forall z \in \mathrm{FV}(t)(z \neq x) \overset{\mathrm{Def}.\epsilon^y_x}{\Leftrightarrow} \forall z \in \mathrm{FV}(t)(z\epsilon^y_x = z) \overset{(\mathrm{S1}),(\mathrm{S2})}{\Leftrightarrow} t\epsilon^y_x = t.$

(b) Let $x \neq y$. Then: $x \notin \mathrm{FV}(t\theta) \overset{(\mathrm{a})}{\Leftrightarrow} t\theta\epsilon^y_x = t\theta \overset{(\mathrm{S1}),(\mathrm{S3})}{\Leftrightarrow} \forall z \in \mathrm{FV}(t)(z\theta\epsilon^y_x = z\theta) \overset{(\mathrm{a})}{\Leftrightarrow} \forall z \in \mathrm{FV}(t)(x \notin \mathrm{FV}(z\theta)).$

(c) 1. $x\theta^y_x\epsilon^s_y = y\epsilon^s_y = s = x\theta^s_x$.

2. $x \neq z \in \mathrm{FV}(r) \overset{(\mathrm{B0})}{\Rightarrow} z \in \mathrm{FV}(\flat x.r) \overset{(\mathrm{b})}{\Rightarrow} y \notin \mathrm{FV}(z\theta) \Rightarrow z\theta^y_x\epsilon^s_y = z\theta\epsilon^s_y \overset{(\mathrm{S1}),(\mathrm{S2})}{=} z\theta = z\theta^s_x$.

Now the claim follows by (S1),(S3).

(d) $\mathrm{FV}(t_x(s)) \overset{(\mathrm{b})}{=} \bigcup_{z\in\mathrm{FV}(t)} \mathrm{FV}(z_x(s)) = \bigcup_{z\in\mathrm{FV}(t)\setminus\{x\}}\{z\} \cup \mathrm{FV}(s) = (\mathrm{FV}(t) \setminus \{x\}) \cup \mathrm{FV}(s)$.

(e) 1. $x_x(s)\theta = s\theta = x\theta^{s\theta}_x$. 2. $y \neq x \Rightarrow y_x(s)\theta = y\theta = y\theta^{s\theta}_x$. Now the claim follows by (S1),(S3).

(f) $r_x(s)\theta \overset{(\mathrm{e})}{=} r\theta^{s\theta}_x \overset{(\mathrm{c})}{=} r\theta^y_x\epsilon^{s\theta}_y$.

Lemma 1.3. For $r, r' \in \mathcal{T}$ the following holds

(B1) $\flat x.r = \flat x.r' \implies r = r'$.

(B2) $y \notin \mathrm{FV}((\flat x.r)\theta) \implies (\flat x.r)\theta = \flat y.r\theta^y_x$.

The proof will be given below.

Lemma 1.4. For $r, r' \in \mathcal{T}$ we have

(a) $y \notin \mathrm{FV}(\flat x.r) \implies \flat x.r = \flat y.r_x(y)$.

(b) $\flat x.r = \flat y.r' \implies r' = r_x(y)$.

(c) $\flat x.r = \flat y.r' \iff \forall s \in \mathcal{T}(r_x(s) = r'_y(s))$.

Proof:

(a) follows from (B2) with $\theta := \epsilon$.

(b) $\flat x.r = \flat y.r' \overset{(\mathrm{B0})}{\Rightarrow} y \notin \mathrm{FV}(\flat x.r) \overset{(\mathrm{a})}{\Rightarrow} \flat y.r' = \flat x.r = \flat y.r_x(y) \overset{(\mathrm{B1})}{\Rightarrow} r' = r_x(y)$.

(c) "$\Rightarrow$": $\flat x.r = \flat y.r' \overset{(\mathrm{B0}),(\mathrm{b})}{\Rightarrow} y \notin (\flat x.r)\ \&\ r_x(y) = r' \overset{\mathrm{L.1.2c}}{\Rightarrow} r_x(s) = r_x(y)_y(s) = r'_y(s)$.

"$\Leftarrow$": $r = r_x(x) = r'_y(x) \overset{\mathrm{L.1.2d}}{\Rightarrow} \mathrm{FV}(r) = \mathrm{FV}(r'_y(x)) \subseteq (\mathrm{FV}(r') \setminus \{y\}) \cup \{x\} \Rightarrow$

$\Rightarrow y \notin \mathrm{FV}(r) \setminus \{x\} \overset{(\mathrm{B0})}{=} \mathrm{FV}(\flat x.r) \overset{(\mathrm{a})}{\Rightarrow} \flat x.r = \flat y.r_x(y) = \flat y.r'_y(y) = \flat y.r'$.

Definition.

$\flat\mathcal{T} := \{\flat x.r : x \in \mathsf{Vars}\ \&\ r \in \mathcal{T}\}$

$\beta : \flat\mathcal{T}\times\mathcal{T} \to \mathcal{T}$, $\beta(\flat x.r, s) := r_x(s)$ (due to Lemma 1.4c, β is well defined)

Lemma 1.5.

(a) $t \in \mathcal{T} \Rightarrow t\theta \in \mathcal{T}$ and $(\flat x.t)\theta \in \flat\mathcal{T}$.

(b) $r, s \in \mathcal{T} \Rightarrow \beta((\flat x.r)\theta, s) = r\theta^s_x$.

(c) $t \in \flat\mathcal{T}\ \&\ y \notin \mathrm{FV}(t) \Rightarrow t = \flat y.\beta(t, y)$.

(d) $t \in \flat\mathcal{T}\ \&\ s \in \mathcal{T} \Rightarrow \beta(t, s)\theta = \beta(t\theta, s\theta)$.

Proof:

(a) Proof of $t\theta \in \mathcal{T}$ by induction on $\mathrm{lh}(t)$. Assume $t = \flat x.r$ with $r \in \mathcal{T}$. Take $y \notin \mathrm{FV}(t\theta)$. Then by I.H. $r\theta^y_x \in \mathcal{T}$ and thus $t\theta \overset{(\mathrm{B2})}{=} \flat y.r\theta^y_x \in \mathcal{T}$.

(b) Let $y \notin \mathrm{FV}((\flat x.r)\theta)$. Then $\beta((\flat x.r)\theta, s) \overset{(\mathrm{B2})}{=} \beta(\flat y.r\theta^y_x, s) = (r\theta^y_x)_y(s) \overset{\mathrm{L1.2c}}{=} r\theta^s_x$.

(c) Let $t = \flat x.r$. Then $t = \flat y.r_x(y) = \flat y.\beta(t, y)$.

(d) Let $t = \flat x.r$. $\beta((\flat x.r)\theta, s\theta) \overset{(\mathrm{b})}{=} r\theta^{s\theta}_x \overset{1.2\mathrm{e}}{=} r_x(s)\theta = \beta(\flat x.r, s)\theta$.

Proof of (S0), (B1), (B2)

Inductive definition of sets $\mathcal{T}_n = \mathcal{T}_n(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ of quasiterms

1. $\mathsf{Vars} \cup \{\circ_k : k < n\} \subseteq \mathcal{T}_n$;

2. $f \in \mathcal{F}_m \;\&\; t_1, ..., t_m \in \mathcal{T}_n \implies ft_1...t_m \in \mathcal{T}_n$;

3. $r \in \mathcal{T}_{n+1} \;\&\; \flat \in \mathcal{B} \;\Rightarrow\; \flat r \in \mathcal{T}_n$.

Remark. $n < m \;\Rightarrow\; \mathcal{T}_n \subseteq \mathcal{T}_m$. $\quad \mathcal{T}' = \bigcup_{n\in\mathbb{N}} \mathcal{T}_n$.

Lemma 1.6.

(a) $t \in \mathcal{T}_n \;\Rightarrow\; t_x[n] \in \mathcal{T}_{n+1}$.

(b) $\mathcal{T} = \mathcal{T}_0$.

(c) $t \in \mathcal{T}_n \;\Rightarrow\; t\theta \in \mathcal{T}_n$.

(d) $t, t' \in \mathcal{T}_n \;\&\; t_x[n] = t'_x[n] \;\Rightarrow\; t = t'$.

(e) $t \in \mathcal{T}_n \;\&\; y = x\theta \notin \bigcup_{z\in \mathrm{FV}(t)\setminus\{x\}} \mathrm{FV}(z\theta) \;\Rightarrow\; t_x[n]\theta = (t\theta)_y[n]$.

Proof:

(a) obvious.

(b) "$\mathcal{T} \subseteq \mathcal{T}_0$": $r \in \mathcal{T}_0 \overset{\text{(a)}}{\Rightarrow} r_x[0] \in \mathcal{T}_1 \;\Rightarrow\; \flat x.r = \flat r_x[0] \in \mathcal{T}_0$.

"$\mathcal{T}_0 \subseteq \mathcal{T}$": One easily proves: (*) $t' \in \mathcal{T}_{n+1} \;\&\; x \notin \mathrm{FV}(t') \;\Rightarrow\; \exists t \in \mathcal{T}_n(t' = t_x[n])$.

Now by induction on $\mathrm{lh}(t)$ on proves $(t \in \mathcal{T}_0 \Rightarrow t \in \mathcal{T})$:

Let $t = \flat r'$ with $r' \in \mathcal{T}_1$. Take $x \notin \mathrm{FV}(r')$. Then by (*) there is an $r \in \mathcal{T}_0$ with $r' = r_x[0]$.

Now $\mathrm{lh}(r) = \mathrm{lh}(r') < \mathrm{lh}(t)$ and therefore by I.H. $r \in \mathcal{T}$ and thus $t = \flat r_x[0] = \flat x.r \in \mathcal{T}$.

(c) follows from (b) by induction on $\mathcal{T}_n$.

(d) 1. $t \in \mathsf{Vars} \cup \{\circ_k : k < n\}$: Then also $t' \in \mathsf{Vars} \cup \{\circ_k : k < n\}$.

1.1. $t = x$: $t'_x[n] = t_x[n] = \circ_n \;\Rightarrow\; t' = x$.

1.2. $t \neq x$: $t'_x[n] = t_x[n] = t \neq \circ_n \;\Rightarrow\; t' = t'_x[n] = t$.

2. $t = \flat r$ with $r \in \mathcal{T}_{n+1}$: Then $t' = \flat r'$ with $r' \in \mathcal{T}_{n+1}$ and $\flat r_x[n{+}1] = t_x[n] = t'_x[n] = \flat r'_x[n{+}1]$.

Hence $r_x[n{+}1] = r'_x[n{+}1]$ and by I.H. $r = r'$ which yields $t = t'$.

(e) 1. $t = x$: $t_x[n]\theta = \circ_n = y_y[n] = (x\theta)_y[n]$.

2. $x \neq t \in \mathsf{Vars} \cup \{\circ_k : k < n\}$: $t_x[n]\theta = t\theta = (t\theta)_y[n]$, since $y \notin \mathrm{FV}(t\theta)$.

3. $t = \flat r$ with $r \in \mathcal{T}_{n+1}$: $t_x[n]\theta = \flat r_x[n{+}1]\theta \overset{\text{IH}}{=} \flat\,(r\theta)_y[n{+}1] = (\flat\,(r\theta))_y[n] = (t\theta)_y[n]$.

(S0) $t \in \mathcal{T} \overset{1.6b}{\Rightarrow} t \in \mathcal{T}_0 \overset{1.6c}{\Rightarrow} t\theta \in \mathcal{T}_0 \overset{1.6b}{\Rightarrow} t\theta \in \mathcal{T}$.

(B1) $r, r' \in \mathcal{T} \;\&\; \flat x.r = \flat x.r' \overset{1.6b}{\Rightarrow} r, r' \in \mathcal{T}_0 \;\&\; r_x[0] = r'_x[0] \overset{1.6d}{\Rightarrow} r = r'$.

(B2) $y \notin \mathrm{FV}((\flat x.r)\theta) \overset{\text{L.1.2b}}{=} \bigcup_{z\in\mathrm{FV}(\flat x.r)} \mathrm{FV}(z\theta) \overset{\text{(B0)}}{=} \bigcup_{z\in\mathrm{FV}(r)\setminus\{x\}} \mathrm{FV}(z\theta) \;\Rightarrow$

$\Rightarrow\; (\flat x.r)\theta \overset{\text{(B0),(S1)}}{=} (\flat x.r)\theta^y_x = \flat\, r_x[0]\theta^y_x \overset{1.6b,e}{=} \flat\,(r\theta^y_x)_y[0] = \flat y.r\theta^y_x$.

Remark.

Under computational aspects it is useful to have a direct algorithm for the operation $\beta : \flat\mathcal{T} \times \mathcal{T} \to \mathcal{T}$. This is achieved as follows.

Definition of $t_n[s]$ for $t \in \mathcal{T}_{n+1}$, $s \in \mathcal{T}_0$

1. For $t \in \mathsf{Vars} \cup \{\circ_k : k \leq n\}$: $t_n[s] := \begin{cases} s & \text{if } t = \circ_n \\ t & \text{otherwise} \end{cases}$;
2. $(ft_1 ... t_m)_n[s] := f(t_1)_n[s] \ldots (t_m)_n[s]$;
3. $(\flat r)_n[s] := \flat r_{n+1}[s]$.

Lemma 1.7

(a) $t \in \mathcal{T}_n \;\Rightarrow\; t_x[n]_n[s] = t_x(s)$.

(b) $\flat\tilde{r} \in \mathcal{T}_0 \;\Rightarrow\; \beta(\flat\tilde{r}, s) = \tilde{r}_0[s]$.

Proof:

(a) 1. $t = x$: $t_x[n]_n[s] = (\circ_n)_n[s] = s = t_x(s)$.

2. $x \neq t \in \mathsf{Vars} \cup \{\circ_k : k < n\}$: $t_x[n]_n[s] = t_n[s] = t = t_x(s)$.

3. $t = \flat r$ with $r \in \mathcal{T}_{n+1}$: $t_x[n]_n[s] = (\flat r_x[n{+}1])_n[s] = \flat r_x[n{+}1]_{n+1}[s] \overset{\text{IH}}{=} \flat r_x(s) = t_x(s)$.

(b) Since $\mathcal{T}_0 = \mathcal{T}$, we have $\flat\tilde{r} = \flat x.r = \flat r_x[0]$ for some $x \in \mathsf{Vars}$, $r \in \mathcal{T}_0$.
Hence $\beta(\flat\tilde{r}, s) = r_x(s) \overset{\text{(a)}}{=} r_x[0]_0[s] = \tilde{r}_0[s]$.

Remark.

The above introduced set of terms $\mathcal{T}(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ can be seen as a concrete realization of a more general kind of structure $(\mathcal{T}, \mathsf{Vars}, \mathrm{FV}, \mathrm{SUB}, \mathcal{B})$ where

$\mathcal{T}$ is a set,
Vars is an infinite subset of $\mathcal{T}$,
for each $t \in \mathcal{T}$, $\mathrm{FV}(t) \subseteq \mathsf{Vars}$,
SUB is a set of mappings $\theta : \mathcal{T} \to \mathcal{T}$, $t \mapsto t\theta$,
for each $\flat \in \mathcal{B}$, $(x, r) \mapsto \flat x.r$ is a mapping from $\mathsf{Vars} \times \mathcal{T}$ to $\mathcal{T}$.

From the proofs of Lemmata 1.2, 1.4, 1.5 one can derive the following

Theorem. If $(\mathcal{T}, \mathsf{Vars}, \mathrm{FV}, \mathrm{SUB}, \mathcal{B})$ is a structure as described above which satisfies the axioms (S1)-(S4),(B1),(B2) then also (B0) and the Lemmata 1.2, 1.4, 1.5 hold, where in 1.5b one has to add the premise $\mathsf{Vars} \setminus \mathrm{FV}((\flat x.r)\theta) \neq \emptyset$.

Proof of (B0):

Note that by L.1.2a, $\mathrm{FV}(y) = \{y\}$ for each $y \in \mathsf{Vars}$.

1. We choose some $z \neq x$.
L.1.2b $\Rightarrow$ $x \notin \mathrm{FV}((\flat x.r)\epsilon_x^z) \overset{\text{(B2)}}{\Rightarrow} (\flat x.r)\epsilon_x^z = \flat x.r(\epsilon_x^z)_x^x = \flat x.r \overset{\text{1.2a}}{\Rightarrow} x \notin \mathrm{FV}(\flat x.r)$.

2. Let $y \neq x$. We choose some $z \notin \{x, y\}$. By 1. we have $x \notin \mathrm{FV}(\flat x.r)$.
Hence (by L.1.2b) $x \notin \mathrm{FV}((\flat x.r)_y(z))$ which by (B2) yields $(\flat x.r)_y(z) = \flat x.r_y(z)$. Now we get:
$y \notin \mathrm{FV}(\flat x.r) \overset{\text{1.2a}}{\Leftrightarrow} \flat x.r = (\flat x.r)_y(z) \;\Leftrightarrow\; \flat x.r = \flat x.r_y(z) \overset{\text{(B1)}}{\Leftrightarrow} r = r_y(z) \overset{\text{1.2a}}{\Leftrightarrow} y \notin \mathrm{FV}(r)$.

§2 Untyped λ-calculus

Let Vars be an infinite set of *variables* (denoted by $x, y, z, ...$) and App, λ constructor symbols.

Inductive definition of the set Λ

1. $\mathsf{Vars} \subseteq \Lambda$;
2. $r, s \in \Lambda \implies \mathsf{App}\, rs \in \Lambda$;
3. $r \in \Lambda \implies \lambda x.r \in \Lambda$.

This definition is to be understood in the sense of §1,
i.e. $\Lambda := \mathcal{T}(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ with $\mathcal{F} = \{\mathsf{App}\}$, $\mathcal{B} = \{\lambda\}$ and $\#(\mathsf{App}) = 2$.

We set $\flat\Lambda := \{\lambda x.r : x \in \mathsf{Vars} \;\&\; r \in \Lambda\}$.

In the following $r, s, t, r', ...$ always denote elements of Λ.

As usual we write rs or (rs) for App, and $rs_1...s_n$ for $(...((rs_1)s_2)...)$.

Inductive definition of $t \to_\beta t'$

$(\to_\beta 1)$ $rs \to_\beta \beta(r, s)$, if $r \in \flat\Lambda$;

$(\to_\beta 2)$ $r \to_\beta r' \implies rs \to_\beta r's$,
$s \to_\beta s' \implies rs \to_\beta rs'$;

$(\to_\beta 3)$ $r \to_\beta r' \implies \lambda x.r \to_\beta \lambda x.r'$.

Remark. (i) $t \to_\beta t' \implies \mathrm{FV}(t') \subseteq \mathrm{FV}(t)$; (ii) $t \to_\beta t' \;\&\; t \in \flat\Lambda \implies t' \in \flat\Lambda$.

Definition. $\to_\beta^*$ denotes the reflexive and transitive closure of $\to_\beta$.

Lemma 2.1.

(a) $t \to_\beta t' \implies t\theta \to_\beta t'\theta$.

(b) If $\lambda x.r \to_\beta t'$ then $t' = \lambda x.r'$ with $r \to_\beta r'$.

(c) $\forall x \in \mathrm{FV}(t)(x\theta \to_\beta^* x\theta') \implies t\theta \to_\beta^* t\theta'$.

(d) $r \in \flat\Lambda \;\&\; r \to_\beta r' \;\&\; s \to_\beta s' \implies \beta(r, s) \to_\beta^* \beta(r', s')$.

Proof:

(a) Induction on $\to_\beta$:

$(\to_\beta 1)$ $t = rs \;\&\; r \in \flat\Lambda \;\&\; \;\&\; t' = \beta(r, s) \;\Rightarrow\; t\theta = (r\theta)(s\theta) \;\&\; r\theta \in \flat\Lambda \;\&\; t'\theta = \beta(r\theta, s\theta)$.

$(\to_\beta 2)$ immediate by I.H.

$(\to_\beta 3)$ Assume $t = \lambda x.r$ and $t' = \lambda x.r'$ with $r \to_\beta r'$. Take some $y \notin \mathrm{FV}(t\theta, t'\theta)$.

Then by I.H. $r\theta^y_x \to_\beta r'\theta^y_x$ and thus $t\theta = \lambda y.r\theta^y_x \to_\beta \lambda y.r'\theta^y_x = t'\theta$.

(b) We have $\lambda x.r = \lambda y.\tilde{r}$ and $t' = \lambda y.\tilde{r}'$ with $\tilde{r} \to_\beta \tilde{r}'$.

Now by (a) we obtain $r = \tilde{r}_y(x) \to_\beta \tilde{r}'_y(x) =: r'$ and $t' = \lambda y.\tilde{r}' \overset{x \notin \mathrm{FV}(t')}{=} \lambda x.r'$.

(c) Induction on $\mathrm{lh}(t)$:

1. $t \in \mathsf{Vars}$: trivial.
2. $t = (rs)$: By I.H. $r\theta \to_\beta^* r\theta'$ and $s\theta \to_\beta^* s\theta'$. Hence $t\theta \to_\beta^* r\theta' s\theta \to_\beta^* r\theta' s\theta' = t\theta'$.
3. $t = \lambda x.r$: Let $y \notin \mathrm{FV}(t\theta, t\theta')$. Then by I.H. $r\theta^y_x \to_\beta^* r\theta'^y_x$. Hence $t\theta = \lambda y.r\theta^y_x \to_\beta^* \lambda y.r\theta'^y_x = t\theta'$.

(d) We have $r = \lambda x.\tilde{r}$ and $r' = \lambda x.\tilde{r}'$ with $\tilde{r} \to_\beta \tilde{r}'$.

Hence by (a) and (c) $\beta(r,s) = \tilde{r}_x(s) \to_\beta \tilde{r}'_x(s) \to_\beta^* \tilde{r}'_x(s') = \beta(r',s')$.

Inductive Definition of Λ_{nf}

($\Lambda_{nf}1$) $s_1, ..., s_n \in \Lambda_{nf} \implies xs_1...s_n \in \Lambda_{nf}$;

($\Lambda_{nf}2$) $r \in \Lambda_{nf} \implies \lambda x.r \in \Lambda_{nf}$.

Lemma 2.2.

(a) $t \in \Lambda_{nf}$ & $\forall x \in \mathrm{FV}(t)(x\theta \in \mathsf{Vars}) \implies t\theta \in \Lambda_{nf}$;

(b) $\lambda x.r \in \Lambda_{nf} \implies r \in \Lambda_{nf}$.

Proof as for 2.1a,b:

(a) Assume $t = \lambda x.r$ with $r \in \Lambda_{nf}$, and let $y \notin \mathrm{FV}(t\theta)$. Then by I.H. $r\theta^y_x \in \Lambda_{nf}$ and thus $t\theta = \lambda y.r\theta^y_x \in \Lambda_{nf}$.

(b) We have $\lambda x.r = \lambda y.\tilde{r}$ with $\tilde{r} \in \Lambda_{nf}$. Now by (a) we obtain $r = \tilde{r}_y(x) \in \Lambda_{nf}$.

Lemma 2.3. $t \notin \Lambda_{nf} \iff \exists t'(t \to_\beta t')$.

Proof:

1. $t \in \flat\Lambda$:

"$\Rightarrow$": $t = \lambda x.r \notin \Lambda_{nf} \Rightarrow r \notin \Lambda_{nf} \overset{\text{IH}}{\Rightarrow} r \to_\beta r' \Rightarrow t \to_\beta \lambda x.r'$.

"$\Leftarrow$": $t = \lambda x.r$ and $t' = \lambda x.r'$ with $r \to_\beta r' \overset{\text{IH}}{\Rightarrow} r \notin \Lambda_{nf} \overset{\text{L.2.2b}}{\Rightarrow} t = \lambda x.r \notin \Lambda_{nf}$.

2. $t = (\lambda x.r)s\vec{s}$: Then $t \notin \Lambda_{nf}$ and $t \to_\beta r_x(s)\vec{s}$.

3. $t = xs_1...s_k$: $t \notin \Lambda_{nf} \Leftrightarrow \exists i(s_i \notin \Lambda_{nf}) \overset{\text{IH}}{\Leftrightarrow} \exists i, s'(s_i \to_\beta s') \Leftrightarrow \exists t'(t \to_\beta t')$.

Inductive Definition of $t \to_p t'$

$(\to_p 1)$ $r \to_p r'$ & $s \to_p s'$ & $r \in \flat\Lambda \implies rs \to_p \beta(r', s')$;

$(\to_p 2)$ $r \to_p r'$ & $s \to_p s' \implies rs \to_p r's'$;

$(\to_p 3)$ $r \to_p r' \Rightarrow \lambda x.r \to_p \lambda x.r'$;

$(\to_p 4)$ $x \to_p x$.

Remark.

(a) $t \to_p t'$ & $t \in \flat\Lambda \Rightarrow t' \in \flat\Lambda$.

(b) $t \to_p t' \Rightarrow \mathrm{FV}(t') \subseteq \mathrm{FV}(t)$.

(c) $t \to_p t$.

(d) $t \to_\beta t' \implies t \to_p t'$.

(e) $t \to_p t' \implies t \to_\beta^* t'$. [cf. Lemma 2.1d]

Lemma 2.4.

(a) $t \to_p t'$ & $\forall x \in \mathrm{FV}(t)(x\theta \to_p x\theta') \implies t\theta \to_p t'\theta'$.

(b) If $\lambda x.r \to_p t'$ then $t' = \lambda x.r'$ with $r \to_p r'$.

(c) $t \to_p t'$ & $s \to_p s'$ & $t \in \flat\Lambda \implies \beta(t, s) \to_p \beta(t', s')$.

Proof:

(a) Induction on $\to_p$:

$(\to_p 1)$ $r \to_p r'$ & $s \to_p s'$ & $r \in \flat\Lambda \overset{\text{IH}}{\implies} r\theta \to_p r'\theta'$ & $s\theta \to_p s'\theta'$ & $r\theta \in \flat\Lambda \implies$

$\implies (rs)\theta = (r\theta)(s\theta) \to_p \beta(r'\theta', s'\theta') = \beta(r', s')\theta'$.

$(\to_p 2)$ Immediate by I.H.

$(\to_p 3)$ As in the proof of 2.1a.

$(\to_p 4)$ trivial.

(b) As 2.1b.

(c) We have $t = \lambda x.r$ & $t' = \lambda x.r'$ with $r \to_p r'$. Hence by (a) $\beta(t, s) = r_x(s) \to_p r'_x(s') = \beta(t', s')$.

Lemma 2.5 $t \to_p t'$ & $t \to_p t'' \implies \exists \tilde{t}(t' \to_p \tilde{t}$ & $t'' \to_p \tilde{t})$.

Proof by induction on $\mathrm{lh}(t)$:

1. $t = rs$, $t' = \beta(r', s')$ with $r \in \flat\Lambda$, $r \to_p r'$ and $s \to_p s'$:

Then $t'' = \beta(r'', s'')$ [or $t'' = r''s''$] with $r \to_p r''$ and $s \to_p s''$.

By I.H. there are $\tilde{r}, \tilde{s}$ such that $r' \to_p \tilde{r}$, $r'' \to_p \tilde{r}$, $s' \to_p \tilde{s}$, $s'' \to_p \tilde{s}$. Hence by Lemma 2.4c

$t' = \beta(r', s') \to_p \beta(\tilde{r}, \tilde{s})$ and $t'' = \beta(r'', s'') \to_p \beta(\tilde{r}, \tilde{s})$ [or $t'' = r''s'' \to_p \beta(\tilde{r}, \tilde{s})$].

2. $t = rs$, $t' = r's'$, $t'' = r''s''$: immediate by I.H.

3. $t = \lambda x.r$, $t' = \lambda x.r'$ with $r \to_p r'$: By Lemma 2.4b we then have $t'' = \lambda x.r''$ with $r \to_p r''$, and the claim follows immediately by the I.H.

4. $t = x$: trivial.

If one wants to define a function on Λ by recursion over the inductive definition of Λ one is confronted with the problem that for $t = \lambda x.r$ one cannot refer to x and r, since these data are not determined by t. To overcome this difficulty we introduce some function $\mathsf{v} : \Lambda \to \mathsf{Vars}$ such that $\mathsf{v}(t) \notin \mathrm{FV}(t)$ for each t. Then every $t \in \flat\Lambda$ can be uniquely written as $t = \lambda x.r$ with $x := \mathsf{v}(t)$.

As an example we consider the operation $t \mapsto t^\ell$ $(t \notin \Lambda_{nf})$ where t^ℓ results from t by carrying out the leftmost β-reduction in t.

Definition of t^ℓ for $t \notin \Lambda_{nf}$

$((\lambda x.r)s\vec{s})^\ell := r_x(s)\vec{s} = \beta(\lambda x.r, s)\vec{s}$,

$(\lambda x.r)^\ell := \lambda x.r^\ell$ if $x = \mathsf{v}(\lambda x.r)$,

$(xs_1...s_k) := xs_1...s_{i-1}s_i^\ell s_{i+1}...s_k$ where i is minimal such that $s_i \notin \Lambda_{nf}$.

Remark: $\mathrm{FV}(t^\ell) \subseteq \mathrm{FV}(t)$.

Lemma 2.6.

(a) $\forall x \in \mathrm{FV}(t)(x\theta \in \mathsf{Vars}) \implies (t\theta)^\ell = t^\ell\theta$.

(b) $(\lambda x.r)^\ell = \lambda x.r^\ell$.

Proof:

(a) 1. $t = rs\vec{s}$ with $r \in \flat\Lambda$: $(t\theta)^\ell = (r\theta s\theta\vec{s}\theta)^\ell = \beta(r\theta, s\theta)\vec{s}\theta = (\beta(r,s)\vec{s})\theta = t^\ell\theta$.

2. $t = \lambda x.r$ with $x := \mathsf{v}(t)$: Then $t\theta = \lambda y.r\theta^y_x$ with $y := \mathsf{v}(t\theta)$. Hence $(t\theta)^\ell = \lambda y.(r\theta^y_x)^\ell \overset{\mathrm{IH}}{=} \lambda y.r^\ell\theta^y_x \overset{(*)}{=} (\lambda x.r^\ell)\theta = t^\ell\theta$. $(*)$ $\mathrm{FV}(t^\ell) \subseteq \mathrm{FV}(t) \Rightarrow \mathrm{FV}(t^\ell\theta) \subseteq \mathrm{FV}(t\theta) \Rightarrow y \notin \mathrm{FV}(t^\ell\theta)$.

3. $t = xs_1...s_k$ and $1 \leq i \leq k$ with $s_i \notin \Lambda_{nf}$ and $s_1, ..., s_{i-1} \in \Lambda_{nf}$:

Let $z := z\theta$. Then $t\theta = z(s_1\theta)...(s_k\theta)$ and $s_i\theta \notin \Lambda_{nf}$ (by L.2.1a,2.3) and $s_1\theta, ..., s_{i-1}\theta \in \Lambda_{nf}$ (by L.2.2).

Hence $(t\theta)^\ell = z(s_1\theta)...(s_{i-1}\theta)(s_i\theta)^\ell(s_{i+1}\theta)...(s_k\theta) \overset{\mathrm{IH}}{=} z(s_1\theta)...(s_{i-1}\theta)(s_i^\ell)\theta(s_{i+1}\theta)...(s_k\theta) =$

$= (xs_1...s_{i-1}s_i^\ell s_{i+1}...s_k)\theta = t^\ell\theta$.

(b) Let $y := \mathsf{v}(\lambda x.r)$. Then $(\lambda x.r)^\ell = (\lambda y.r_x(y))^\ell = \lambda y.r_x(y)^\ell \overset{(a)}{=} \lambda y.(r^\ell)_x(y) = \lambda x.r^\ell$.

§3 The system $\lambda 2$

Typevars: $\alpha, \beta, ...$; Termvars: $x, y, z, ...$

$\mathsf{Vars} := \mathsf{Typevars} \cup \mathsf{Termvars}$, $\mathcal{B} := \{\Lambda, \forall, \boldsymbol{l}\}$, $\mathcal{F} := \{\mathsf{App}, \to, \lambda^*\}$ (where $\mathsf{App}, \to, \lambda^*$ all have arity 2).

Inductive Definition of Types

1. $\mathsf{Typevars} \subseteq \mathsf{Types}$;
2. $A, B \in \mathsf{Types} \implies {\to}AB \in \mathsf{Types}$;
3. $A \in \mathsf{Types} \implies \forall\alpha.A \in \mathsf{Types}$.

This is to be understood in the sense that $\mathsf{Types} = \mathcal{T}(\mathsf{Typevars}; \{\to\}; \{\forall\}) \subseteq \mathcal{T}(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$.

Types are denoted by $A, B, ...$; we write $(A{\to}B)$ for ${\to}AB$.

Abbreviation: $\lambda x^A.r := \lambda^* A\, \boldsymbol{l}x.r$, $(rs) := \mathsf{App}\, rs$

r, s, t range over elements of $\mathcal{T}(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$.

θ ranges over substitutions such that $\forall\alpha \in \mathsf{Typevars}(\alpha\theta \in \mathsf{Types})$*; then also* $\forall A \in \mathsf{Types}(A\theta \in \mathsf{Types})$.

We extend the operation β by: $\beta(\lambda x^A.r, s) := \beta(\boldsymbol{l}x.r, s) = r_x(s)$.

Then $\beta((\lambda x^A.r)\theta, s\theta) = \beta(\lambda^* A\theta\, (\lambda x.r)\theta, s\theta) = \beta((\boldsymbol{l}x.r)\theta, s\theta) = \beta(\boldsymbol{l}x.r, s)\theta = \beta(\lambda x^A.r, s)\theta$.

Proposition.

(a) $\mathrm{FV}(\lambda x^A.r) = \mathrm{FV}(A) \cup (\mathrm{FV}(r) \setminus \{x\})$.

(b) $\mathrm{FV}((\lambda x^A.r)\theta) = \mathrm{FV}(A\theta) \cup \mathrm{FV}((\boldsymbol{l}x.r)\theta)$.

(c) $y \notin \mathrm{FV}((\lambda x^A.r)\theta) \;\Rightarrow\; (\lambda x^A.r)\theta = \lambda y^{A\theta}.r\theta^y_x$.

Proof:

(b) $y \notin \mathrm{FV}((\lambda x^A.r)\theta) \;\Rightarrow\; y \notin \mathrm{FV}((\boldsymbol{l}x.r)\theta) \;\Rightarrow\; (\lambda x^A.r)\theta = \lambda^* A\theta\, (\boldsymbol{l}x.r)\theta = \lambda^* A\theta\, \boldsymbol{l}y.r\theta^y_x = \lambda y^{A\theta}.r\theta^y_x$.

A *context* is a finite set of pairs $x_1 : A_1, \ldots, x_n : A_n$ where the term variables $x_1, ..., x_n$ are all distinct.

Contexts are denoted by $\Gamma, \Gamma', ...$

Inductive Definition of $\Gamma \vdash t : C$

0. $x : A \in \Gamma \implies \Gamma \vdash x : A$;
1. $\Gamma \vdash r : A{\to}B \;\&\; \Gamma \vdash s : A \implies \Gamma \vdash (rs) : B$;
2. $\Gamma, x : A \vdash r : B \implies \Gamma \vdash \lambda x^A.r : A{\to}B$;
3. $\Gamma \vdash r : \forall\alpha.A \implies \Gamma \vdash (rB) : A_\alpha(B)$;
4. $\Gamma \vdash r : A \;\&\; \alpha \notin \mathrm{FV}(\Gamma) \implies \Gamma \vdash \Lambda\alpha.r : \forall\alpha.A$.

In 2. it is understood that $x{:}A \notin \Gamma$.

Corollary.

(1) $\Gamma \vdash t : C \;\Rightarrow\; t \notin \mathsf{Types}$.

(2) $\Gamma \vdash t : C \;\&\; \Gamma \vdash t : C' \implies C = C'$.

Lemma 3.1.

$$\left.\begin{array}{l} x_1 : A_1, ..., x_m : A_m \vdash t : C \ \ \& \\ \Delta' \vdash x_i\theta : A_i\theta \text{ for } i = 1, ..., m \text{ and all } \Delta' \supseteq \Delta \end{array}\right\} \Longrightarrow \ \Delta \vdash t\theta : C\theta.$$

Proof:

Let $\Gamma := x_1 : A_1, ..., x_m : A_m$.

0. $t = x_i$, $C = A_i$: By assumption $\Delta \vdash x_i\theta : A_i\theta$.

1. $\Delta \vdash r\theta : A\theta{\to}C\theta \ \ \& \ \ \Delta \vdash s\theta : A\theta \implies \Delta \vdash (rs)\theta : C\theta$.

2. $t = \lambda x^A.r$, $C = A{\to}B$, and $\Gamma, x : A \vdash r : B$:

Choose $y \notin \mathrm{FV}(t\theta) \cup \mathrm{FV}(\Delta)$; then $t\theta = \lambda y^{A\theta}.r\theta^y_x$ and,

$\Delta' \vdash x_i\theta^y_x : A_i\theta$ and $\Delta' \vdash x\theta^y_x : A\theta$, for all $\Delta' \supseteq \Delta, y : A\theta$.

Hence, by I.H., $\Delta, y : A\theta \vdash r\theta^y_x : B\theta^y_x$; this yields $\Delta \vdash \lambda y^{A\theta}.r\theta^y_x : A\theta{\to}B\theta$, i.e., $\Delta \vdash t\theta : C\theta$.

3. $t = (rB)$ with $\Gamma \vdash r : A$, and $C = \beta(A, B)$:

IH $\implies \Delta \vdash r\theta : A\theta \ \Rightarrow \ \Delta \vdash (r\theta)(B\theta) : \beta(A\theta, B\theta) \ \Rightarrow \ \Delta \vdash (rB)\theta : \beta(A, B)\theta$.

4. $t = \Lambda\alpha.r$ with $\Gamma \vdash r : A$, $\alpha \notin \mathrm{FV}(\Gamma)$, and $C = \forall\alpha.A$: Choose $\beta \notin \mathrm{FV}(t\theta) \cup \mathrm{FV}(C\theta) \cup \mathrm{FV}(\Delta)$.

Then $t\theta = \Lambda\beta.r\theta^\beta_\alpha$, $C\theta = \forall\beta.A\theta^\beta_\alpha$, and $\Delta' \vdash x_i\theta^\beta_\alpha : A_i\theta^\beta_\alpha$ (since $\alpha \notin \mathrm{FV}(\Gamma)$).

I.H. $\implies \Delta \vdash r\theta^\beta_\alpha : A\theta^\beta_\alpha \ \Rightarrow \ \Delta \vdash \Lambda\beta.r\theta^\beta_\alpha : \forall\beta.A\theta^\beta_\alpha$, i.e., $\Delta \vdash t\theta : C\theta$.

Corollary 3.2. $\Gamma \vdash t : C \ \& \ \Gamma \subseteq \Delta \implies \Delta \vdash t : C$.

Proof: Let $\Gamma = x_1 : A_1, ..., x_m : A_m$.

$\Gamma \subseteq \Delta \implies \Delta' \vdash x_i : A_i$ for $i = 1, ..., m$ and all $\Delta' \supseteq \Delta \implies \Delta \vdash t : C$.

Corollary 3.3.

$x_1 : A_1, ..., x_m : A_m \vdash t : C \ \ \& \ \ \Delta \vdash x_i\theta : A_i\theta$ for $i = 1, ..., m \implies \Delta \vdash t\theta : C\theta$.

Proof: Immediate from Theorem and Corollary 3.2.

Corollary 3.4. $\Gamma \vdash t : C \ \ \& \ \ \forall x \in \mathsf{Termvars}(x\theta = x) \implies \Gamma\theta \vdash t\theta : C\theta$.

Proof:

Let $\Gamma = x_1 : A_1, ..., x_m : A_m$. Then $\Gamma\theta \vdash x_i\theta : A_i\theta$ for $i = 1, ..., m$. Hence $\Gamma\theta \vdash t\theta : C\theta$ by Corollary 3.3.

β-reduction

Definition of $\to_\beta$

1.1. $(\lambda x^A.r)s \to_\beta r_x(s) \ \ [= \beta(\lambda x^A.r, s) \ ]$

1.2. $(\Lambda\alpha.r)B \to_\beta r_\alpha(B) \ \ [= \beta(\Lambda\alpha.r, B) \ ]$

2.1. $r \to_\beta r' \ \Rightarrow \ (rs) \to_\beta (r's)$

2.2. $s \to_\beta s' \ \Rightarrow \ (rs) \to_\beta (rs')$

3.1. $r \to_\beta r' \ \Rightarrow \ \lambda x^A.r \to_\beta \lambda x^A.r'$

3.2. $r \to_\beta r' \ \Rightarrow \ \Lambda\alpha.r \to_\beta \Lambda\alpha.r'$.

Corollary. $t \to_\beta t' \ \Rightarrow \ \mathrm{FV}(t') \subseteq \mathrm{FV}(t)$.

Lemma 3.5.

(a) $t \to_\beta t' \implies t\theta \to_\beta t'\theta$ if θ correct;

(b) (i) $\lambda x^A.r \to_\beta t' \implies t' = \lambda x^A.r'$ with $r \to_\beta r'$;

(ii) $\Lambda\alpha.r \to_\beta t' \implies t' = \Lambda\alpha.r'$ with $r \to_\beta r'$.

Proof: As for Lemma 2.1a,b.

(a) Induction on $\to_\beta$:

1.1. $t = (\lambda x^A.r)s$ & $t' = \beta(\lambda x^A.r, s) \Rightarrow t\theta = ((\lambda x^A.r)\theta)(s\theta) \to_\beta \beta((\lambda x^A.r)\theta, s\theta) = t'\theta$.

1.2. $t = (\Lambda\alpha.r)B$ & $t' = \beta(l\alpha.r, B) \Rightarrow t\theta = ((\Lambda\alpha.r)\theta)(B\theta) \to_\beta \beta((\Lambda\alpha.r)\theta, B\theta) = t'\theta$.

2.1. and 2.2.: immediate by I.H.

3.1. $t = \lambda x^A.r$ and $t' = \lambda x^A.r'$ with $r \to_\beta r'$: Take some $y \notin \mathrm{FV}(t\theta, t'\theta)$.

Then by I.H. $r\theta^y_x \to_\beta r'\theta^y_x$ and thus $t\theta = \lambda y^{A\theta}.r\theta^y_x \to_\beta \lambda y^{A\theta}.r'\theta^y_x = t'\theta$.

3.2. $t = \Lambda\alpha.r$ and $t' = \Lambda\alpha.r'$ with $r \to_\beta r'$: analogous to 3.1.

(b) (i) We have $\lambda x^A.r = \lambda y^A.\tilde{r}$ and $t' = \lambda y^A.\tilde{r}'$ with $\tilde{r} \to_\beta \tilde{r}'$.

Now by (a) we obtain $r = \tilde{r}_y(x) \to_\beta \tilde{r}'_y(x) =: r'$ and $t' = \lambda y^A.\tilde{r}' \overset{x \notin \mathrm{FV}(t')}{=} \lambda x^A.r'$.

(ii): analogous to (i).

Lemma 3.6 (Subject reduction). $\Gamma \vdash t : C$ & $t \to_\beta t' \implies \Gamma \vdash t' : C$.

Proof by induction on $\mathrm{lh}(t)$:

1. $t = (\tilde{t}s)$ and $t' = \beta(\tilde{t}, s)$:

1.1. $\tilde{t} = \lambda x^A.r$ with $\Gamma, x : A \vdash r : C$ and $\Gamma \vdash s : A$:

By Corollary 3.3 (with $\Delta := \Gamma$ and $\theta := \epsilon^s_x$) we obtain $\Gamma \vdash r_x(s) : C$, i.e. $\Gamma \vdash t' : C$.

1.2. $\tilde{t} = \Lambda\alpha.r$, $s \in \mathsf{Types}$, $C = \beta(\forall\alpha.A, s)$ and $\Gamma \vdash r : A$ with $\alpha \notin \mathrm{FV}(\Gamma)$:

By Corollary 4.4 (with $\theta := \epsilon^s_\alpha$) we obtain $\Gamma \vdash r_\alpha(s) : A_\alpha(s)$, i.e. $\Gamma \vdash t' : C$.

2.1. $t = (rs)$ and $t' = (r's)$ with $r \to_\beta r'$: Then one of the following two cases holds.

(i) $\Gamma \vdash r : A{\to}C$ and $\Gamma \vdash s : A$; (ii) $\Gamma \vdash r : \forall\alpha.A$, $s \in \mathsf{Types}$ and $C = A_\alpha(s)$.

In both cases the claim follows immediately from the I.H.

2.2. $t = (rs)$ and $t' = (rs')$ with $s \to_\beta s'$:

Then $\Gamma \vdash r : A{\to}C$ & $\Gamma \vdash s : A$, and the claim follows immediately from the I.H.

3.1. $t \to_\beta t'$ holds by 3.1 in the definition of $\to_\beta$:

Then $t = \lambda x^A.r$ with $\Gamma, x : A \vdash r : B$ and $C = A{\to}B$.

By Lemma 3.5b we obtain $t' = \lambda x^A.r'$ with $r \to_\beta r'$.

Hence by I.H., $\Gamma, x : A \vdash r' : B$ which implies $\Gamma \vdash \lambda x^A.r' : A{\to}B$.

3.2. $t \to_\beta t'$ holds by 3.2 in the definition of $\to_\beta$: analogous to 3.1.

§4 1st order predicate logic

Let $\mathcal{L}$ be a 1st order language, i.e., a set of function and relation symbols where each symbol $p \in \mathcal{L}$ has a fixed arity $\#(p) \in \mathbb{N}$.

Inductive Definition of $\mathcal{L}$-terms

1. Every $x \in \mathsf{Vars}$ is a $\mathcal{L}$-term.
2. If $h \in \mathcal{L}$ is an n-ary function symbol and $t_1, ..., t_n$ are $\mathcal{L}$-terms then $ht_1...t_n$ is an $\mathcal{L}$-term.

Inductive Definition of $\mathcal{L}$-formulas

1. If $p \in \mathcal{L}$ is an n-ary relation symbol, and $t_1, ..., t_n$ are $\mathcal{L}$-terms then $pt_1...t_n$ is an $\mathcal{L}$-formula.
2. If A, B are $\mathcal{L}$-formulas then $\neg A$ and $\vee AB$ are $\mathcal{L}$-formulas.
3. If A is an $\mathcal{L}$-formula and $x \in \mathsf{Vars}$ then $\forall x.A$ and $\exists x.A$ are $\mathcal{L}$-formulas.

These definitions are to be understood so that $\mathcal{L}$-terms and $\mathcal{L}$-formulas are elements of $\mathcal{T}(\mathsf{Vars}; \mathcal{F}; \mathcal{B})$ with $\mathcal{F} := \mathcal{L} \cup \{\neg, \vee\}$ and $\mathcal{B} := \{\forall, \exists\}$.

In the following, θ ranges over substitutions having the property that $x\theta$ is an $\mathcal{L}$-term for each $x \in \mathsf{Vars}$.

Lemma 4.1.
(a) If t is an $\mathcal{L}$-term then $t\theta$ is an $\mathcal{L}$-term.
(b) If C is an $\mathcal{L}$-formula then $C\theta$ is an $\mathcal{L}$-formula.

Proof of (b) by induction on the inductive definition of $\mathcal{L}$-formulas (or by induction on $\mathrm{lh}(C)$):

1. For $C = pt_1...t_n$ the claim follows from (a).
2. Let $C = \forall x.A$. Choose $y \notin \mathrm{FV}(C\theta)$. Then $C\theta = \forall y.A\theta^y_x$, and by I.H. $A\theta^y_x$ is an $\mathcal{L}$-formula. This yields the claim.

In the following, let $\mathcal{M}$ be an $\mathcal{L}$-structure with universe M, and let ξ, η range over $\mathcal{M}$-assignments, i.e., functions $\xi : \mathsf{Vars} \to M$.

For each assignment ξ, the *value* $[\![t]\!]_\xi \in M$ of an $\mathcal{L}$-term t and the *truth value* $[\![A]\!]_\xi \in \{0, 1\}$ of an $\mathcal{L}$-formula are define as usual. Only the quantifier case requires some additional care; here (as at the end of §2) make use of some previously fixed function v which assigns to each formula C a variable $\mathsf{v}(C) \notin \mathrm{FV}(C)$:

Def.: *If $C = \overset{\forall}{\exists} x.A$ with $x = \mathsf{v}(C)$ then* $[\![C]\!]^{\mathcal{M}}_\xi := \min\limits_{\max} \{[\![A]\!]_{\xi^a_x} : a \in M\}$.

Lemma 4.2. $\forall z \in \mathrm{FV}(C)(\xi(z) = \eta(z)) \implies [\![C]\!]_\xi = [\![C]\!]_\eta$.

Proof:
$[\![\forall x.A]\!]_\xi = \min\{[\![A]\!]_{\xi^a_x} : a \in M\} \overset{\text{IH}}{=} \min\{[\![A]\!]_{\eta^a_x} : a \in M\} = [\![\forall x.A]\!]_\eta$.

Lemma 4.3. $\forall z \in \mathrm{FV}(C)([\![z\theta]\!]_\xi = [\![z]\!]_\eta) \implies [\![C\theta]\!]_\xi = [\![C]\!]_\eta$.

Proof:

Let $C = \forall x.A$ with $x = \mathsf{v}(C)$; then $C\theta = \forall y.A\theta^y_x$ with $y = \mathsf{v}(C\theta)$.

$[\![C\theta]\!]_\xi = [\![\forall y.A\theta^y_x]\!]_\xi = \min\{[\![A\theta^y_x]\!]_{\xi^a_y} : a \in M\} \overset{\mathrm{IH}+(*)}{=} \min\{[\![A]\!]_{\eta^a_x} : a \in M\} = [\![C]\!]_\eta$.

$(*)$ 1. $[\![x\theta^y_x]\!]_{\xi^a_y} = a = [\![x]\!]_{\eta^a_x}$.

2. If $z \in \mathrm{FV}(A) \setminus \{x\} = \mathrm{FV}(C)$ then $y \notin \mathrm{FV}(z\theta)$ (since $y \notin \mathrm{FV}(C\theta)$) and thus

$[\![z\theta^y_x]\!]_{\xi^a_y} = [\![z\theta]\!]_{\xi^a_y} \overset{\mathrm{L.4.2}}{=} [\![z\theta]\!]_\xi = [\![z]\!]_\eta = [\![z]\!]_{\eta^a_x}$.

Lemma 4.4. $[\![\forall x.A]\!]_\xi = \min\{[\![A]\!]_{\xi^a_x} : a \in M\}$

Proof:

Let $y := \mathsf{v}(\forall x.A)$. Then $\forall x.A = \forall y.A_x(y)$, and thus

$[\![\forall x.A]\!]_\xi = \min\{[\![A_x(y)]\!]_{\xi^a_y} : a \in M\} \overset{\mathrm{L.4.3}+(*)}{=} \min\{[\![A]\!]_{\xi^a_x} : a \in M\}$.

$(*)$: cf. $(*)$ with $\theta := \epsilon$ in the proof of 4.3.