# A Uniform Approach to Fundamental Sequences and Hierarchies

Wilfried Buchholz, *
Adam Cichon†
and Andreas Weiermann ‡

**Abstract**

In this article we give a unifying approach to the theory of fundamental sequences and their related Hardy hierarchies of number-theoretic functions and we show the equivalence of the new approach with the classical one.

## Introduction

A fascinating result of ( Gentzen-style) proof theory is the characterization of the provably total functions of Peano-arithmetic in terms of Kreisel's ordinal recursive functions (Kreisel 1952), or alternatively, in terms of the $\prec$-descent recursive functions (cf. Smith 1985, Takeuti 1987 or Friedman & Sheard 1993), where $\prec$ denotes a standard representation of $\varepsilon_0$ in the natural numbers. This class of functions can also be characterized by hierarchies of number-theoretic functions which are defined relative to the system of standard fundamental sequences for the ordinals less than $\varepsilon_0$. Examples are here the Hardy-hierarchy, the extended Grzegorczyk hierarchy and a hierarchy which is based on iterated enumeration (Schwichtenberg 1971, Wainer 1972). A generalization of the latter concepts would still seem to be problematic. There were some results concerning $\Gamma_0$, the proof-theoretic ordinal of predicative analysis, or $\eta_\nu$, the proof-theoretic ordinal of $ID_\nu$ (cf. Zemke 1977, Buchholz 1987). But the larger the countable ordinal in question, the harder becomes the problem of assigning an appropriate (Bachmann) system of fundamental sequences. In his article "Termination

*Mathematisches Institut der Universität München, Theresienstr. 39, D-80333 München, Germany. e-mail: buchholz@rz.mathematik.uni-muenchen.de
†INRIA-Lorraine–CNRS-CRIN, BP 101 F-54602 Villers-lès-Nancy, France.
e-mail: cichon@loria.crin.fr
‡Institut für mathematische Logik und Grundlagenforschung der Westfälischen Wilhelms-Universität Münster, Einsteinstraße 62, D-48149 Münster, Germany.
e-mail: weierma@math.uni-muenster.de

orderings and complexity characterizations" Cichon proposes implicitly a very simple and general method for approaching this problem. His approach is based on the interplay between Bachmann systems of fundamental sequences and a term-complexity-function (which we call *norm* from now on). The importance of a norm function is already implicit in the literature (cf. Zemke 1977 and Smith 1985). The new idea is to define a Bachmann system of fundamental sequences $(\alpha[n]_C)_{n\in\mathbb{N}}$ in terms of the norm instead of defining it by refering to some normal form representation of the respective ordinals. For example, if $N\colon \varepsilon_0 \to \mathbb{N}$ is a norm function such that $N0 = 0$, $N(\alpha+1) \leq N\alpha + 1$ for all $\alpha < \varepsilon_0$ and $card\{\beta < \alpha\colon N\beta < d\}$ is finite for all $d \in \mathbb{N}$ and $\alpha < \varepsilon_0$, then we put (cf. Cichon 1992):

$$\alpha[n]_C := \max\{\beta < \alpha\colon N\beta \leq N\alpha + n\}.^1$$

Moreover it turns out that the Hardy-hierarchies can be defined in terms of the norm without any reference to fundamental sequences at all. If we put

$$H_\alpha(n) := \max\big(\{n\} \cup \{H_\beta(n+1)\colon \beta < \alpha \;\&\; N\beta \leq N\alpha + n\}.$$

then this hierarchy coincides with (a slight variant of) the usual Hardy hierarchy with respect to $\cdot[\cdot]_C$, namely for $\alpha > 0$ one has $H_\alpha(n) = H_{\alpha[n]_C}(n+1)$.
An immediate consequence of this definition which is very useful for applications is the following effective majorization property:

$$\text{(EMP)} \qquad \alpha < \beta \;\&\; N\alpha \leq N\beta + n \;\Rightarrow\; H_\alpha(n) < H_\beta(n).$$

Of course (EMP) yields the usual majorization property of the Hardy hierarchy, namely $\alpha < \beta \;\Rightarrow\; (\exists m)(\forall n \geq m)[H_\alpha(n) < H_\beta(n)]$. But, in additition, (EMP) gives an useful effective criterion how to compute a natural number $m$ such that $H_\alpha(n) < H_\beta(n)$ holds for all $n \geq m$. Simply pick $m$ such that $N\alpha \leq N\beta + m$! In the more traditional approach the verification of assertions like (EMP) (especially for proof-theoretic ordinals larger than $\varepsilon_0$) is not always immediate (cf. Zemke 1977).

In this article we investigate the consequences of this new approach to the theory of Hardy-hierarchies and we will compare this approach with the usual one. It turns out that under some natural assumptions the new approach is equivalent to the old one. The new approach has proven useful in (Weiermann 1993) where a comparatively simple and straightforward characterization of the provably total functions of Peano-arithmetic is given in terms of a Cichon-style Hardy hierarchy.[2]

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[1]A similar (but not equivalent) definition is contained in Friedman & Sheard (1993) in Lemma 1.31

[2]The new approach has recently also been proved useful for bounding derivation lengths of rewrite systems with slow growing functions (cf. Weiermann 1993a) and for investigations on slow versus fast growing for proof-theoretic ordinals larger than the first subrecursive ordinal (cf. Weiermann 1993b).

In Section 1 we develop a general theory of normed Bachmann systems and their related Hardy hierarchies. In Section 2 following [Wainer 1970] we compare the Hardy hierarchies with certain hierarchies of primitive recursively norm bounded descent functions. This also provides means for comparing Hardy hierarchies belonging to different normed Bachmann systems. In Section 3 we relate Cichon's approach to the more traditional theory of normed Bachmann systems presented in Section 1. In Section 4 we give some applications to $\varepsilon_0$ and other proof-theoretic ordinals. Moreover we reformulate the main result of Section 2 in a way which avoids any reference to fundamental sequences.

# 1 Fundamental Sequences and Hardy Hierarchies

Let $\omega$ be the least infinite ordinal and let $\tau$ be fixed such that $\exists \tau_0 > 0(\tau = \omega^\omega \cdot \tau_0)$. In the following $\alpha, \beta, \gamma, \xi$ range over ordinals less than $\tau$, and $i, j, k, l, m, n$ over natural numbers (finite ordinals). The set of natural numbers is denoted by $\mathbb{N}$ and the set of limit ordinal less than $\tau$ is denoted by $Lim$.

**Definition**
Let $\cdot[\cdot]\colon \tau \times \mathbb{N} \to \tau$ and $N : \tau \to \mathbb{N}$.

1. $\cdot[\cdot]$ is called a *system of fundamental sequences* or an *assignment of fundamental sequences* if

(B1) $(\forall \alpha, n)\big[0[n] = 0 \,\&\, (\alpha+1)[n] = \alpha \,\&\, (\alpha \in Lim \Rightarrow \alpha[n] < \alpha[n+1] < \alpha)\big]$.

2. The *Hardy hierarchy* $(H_\alpha)_{\alpha<\tau}$ for $\langle \tau, \cdot[\cdot]\rangle$ is defined by
$H_0(n) := n$, and $H_\alpha(n) := H_{\alpha[n]}(n+1)$ for $\alpha > 0$.

Now let $\cdot[\cdot]$ be an assignment of fundamental sequences.

3. $\langle \tau, \cdot[\cdot]\rangle$ is called a *Bachmann system* if

(B2) $(\forall \alpha, \beta, n)\big[\alpha[n] < \beta < \alpha \;\Rightarrow\; \alpha[n] \leq \beta[0]\big]$.

4. We say that $\cdot[\cdot]$ is *compatible with* $N$ and call $\langle \tau, \cdot[\cdot], N\rangle$ a *normed Bachmann system* if

(B3) $(\forall \alpha, \beta, n)\big[\alpha[n] < \beta < \alpha \;\Rightarrow\; N\alpha[n] < N\beta\big]$,

(B4) $(\forall \alpha \in Lim)\big[N\alpha \leq N\alpha[0] + 1\big]$.

5. We call $\langle \tau, \cdot[\cdot], N\rangle$ a *regular Bachmann system* if $(\forall \beta < \alpha)[\beta \leq \alpha[N\beta])$.

6. $N$ is called a *norm* on $\tau$ if $(\forall \alpha)(\forall n)[card\{\beta < \alpha : N\beta \leq n\} < \omega]$.

*Remarks:* In Lemma 3 we shall prove that $\sup\{\alpha[n] : n \in \mathbb{N}\} = \alpha$ for Bachmann systems and limit ordinals $\alpha$. Since we shall concentrate mainly on Bachmann systems in this article we do not demand this additional property in (B1). Our definition of the Hardy hierarchy is a slight modification of the usual Hardy hierarchy which is defined by $H_0(n) := n$, $H_{\alpha+1}(n) := H_\alpha(n+1)$, and $H_\alpha(n) := H_{\alpha[n]}(n)$ for limit ordinals $\alpha$. Our choice has some technical advantages but our approach can be carried out in the same way also for the classical Hardy hierachy. Condition (B2) is the so-called Bachmann property (cf.

Schmidt 1976). The usefulness of this condition in investigations on subrecursive hierarchies is discussed for example in (Rose 1984). The idea of considering normed systems of fundamental sequences is already contained in Zemke (1977). Compared with his definition our approach is more restrictive (see Lemma 1 below). The concept of a norm is contained in Smith (1985). A slightly different concept of Bachmann system, namely the concept of a structured tree ordinal has been investigated in (Wainer 1991).

**Lemma 1**
If $\langle \tau, \cdot[\cdot], N \rangle$ is a normed Bachmann system, then
a) $\langle \tau, \cdot[\cdot] \rangle$ is a Bachmann system,
b) $N$ is a norm on $\tau$,
c) $\langle \tau, \cdot[\cdot], N \rangle$ is a regular Bachmann system.

*Proof:*
a) Let $\alpha[n] < \beta < \alpha$ and assume $\beta[0] < \alpha[n]$. Then by (B1),(B3),(B4) we get $N\alpha[n] < N\beta \leq N\beta[0] + 1 \leq N\alpha[n]$. Contradiction.
b) $(*)$ $\{\beta < \alpha : N\beta \leq n\} \subseteq \{\beta \leq \alpha[n] : N\beta \leq n\}$.
Proof: For $\alpha \notin Lim$ this is trivial. If $\alpha \in Lim$ then by (B3) we have $(\forall n)\big[N\alpha[n] < N\alpha[n+1]\big]$ and therefore $n \leq N\alpha[n]$. (B3) also yields $(\forall \beta < \alpha)\big[N\beta \leq N\alpha[n] \Rightarrow \beta \leq \alpha[n]\big]$. – ¿From $(*)$ it follows by induction on $\alpha$ that $N$ is a norm on $\tau$.
c) Let $\alpha \in Lim$. Then by (B3) we have $(\forall n)\big[n \leq N\alpha[n]\big]$, so, in particular, $N\beta \leq N\,\alpha[N\beta]$. The latter together with (B3) yields $(\beta < \alpha \Rightarrow \beta \leq \alpha[N\beta])$.

*Remark:* Lemma 1 c) yields that normed Bachmann systems are regulated in the sense of Zemke (1977). It can be shown that the conclusion of Lemma 1 c) can be sharpened to $(\forall \beta {<} \alpha)\big[\beta \leq \alpha[N\beta \dot{-} (N\alpha \dot{-} 1)]\big]$.

In the following, we write $\alpha[0]^i$ to denote $\alpha\underbrace{[0]\ldots[0]}_{i\ times}$.

**Lemma 2**
Let $\langle \tau, \cdot[\cdot] \rangle$ be a system of fundamental sequences and let $G\colon \tau \to \mathbb{N}$ be defined by $G\alpha := \min\{i : \alpha[0]^i = 0\}$. Then:
a) If $\langle \tau, \cdot[\cdot] \rangle$ is a Bachmann system then $\langle \tau, \cdot[\cdot], G \rangle$ is a normed Bachmann system and $(\forall \alpha)[card\{\beta\colon \beta[0] \leq \alpha < \beta\} < \omega]$.
b) If $(\forall \alpha)[card\{\beta\colon \beta[0] = \alpha\} < \omega]$, then $G$ is a norm.

*Proof:*
a) If $\alpha[n] < \beta < \alpha$ then $\alpha[n] = \beta[0]^i$ for some $i > 0$, and thus $G\alpha[n] < G\beta$. Obviously $G\alpha = G\alpha[0] + 1$ for each $\alpha > 0$. Thus $\langle \tau, \cdot[\cdot], G \rangle$ is a normed Bachmann system. By Lemma 1 b) $G$ is a norm and therefore $(\forall \alpha)[card\{\beta\colon G\beta \leq G\alpha + 1\} < \omega]$. Since $\langle \tau, \cdot[\cdot], G \rangle$ is a normed Bachmann system, we also have $\{\beta\colon \beta[0] \leq \alpha < \beta\} \subseteq \{\beta\colon G\beta \leq G\alpha + 1\}$.
b) Assume that there are $\alpha$ and $d < \omega$ such that $\{\beta < \alpha\colon G\beta < d\}$ is infinite. Then $X := \{\beta < \alpha\colon G\beta = k\}$ is infinite for some minimal $k < d$. Then $k > 0$, since $\beta[0]^0 = \beta > 0$, if $\beta > 0$. Put $Y := \{\beta[0]\colon \beta \in X\}$. The minimality of $k$

yields that $Y$ is finite. So there is a $\gamma$ such that $\gamma = \beta[0]$ for infinitely many $\beta \in X$. This contradicts the assumption.

For the remainder of this section let $\cdot[\cdot] : \tau \times \mathbb{N} \to \tau$ be fixed such that $\langle \tau, \cdot[\cdot], N \rangle$ is a regular Bachmann system, and let $(H_\alpha)_{\alpha<\tau}$ be the Hardy hierarchy for $\langle \tau, \cdot[\cdot] \rangle$.

**Lemma 3**
a) $H_\alpha(n) < H_\alpha(n+1)$,
b) $\beta[m] < \alpha < \beta \;\Rightarrow\; H_{\beta[m]}(n) < H_\alpha(n)$,
c) $0 < \alpha \,\&\, m \le n \;\Rightarrow\; H_{\alpha[m]}(n+1) \le H_\alpha(n)$,
d) $\forall \beta < \alpha (N\beta \le n \Rightarrow H_\beta(n+1) \le H_\alpha(n))$,
e) $\alpha \in Lim \;\Rightarrow\; \alpha = \sup\{\alpha[n] : n \in \mathbb{N}\}$,
f) $H_\alpha(n) = \min\{k \ge n : \alpha[n][n+1]\ldots[k-1] = 0\} = \min\{k : \alpha[n:k] = 0\}$,
with $\alpha[n:k] := \alpha + (n \dot{-} k)$, for $k \le n$, and $\alpha[n : k+1] := (\alpha[n:k])[k]$, for $k \ge n$.

*Proof:* (Compare the proof of theorem 2, Cichon (1983).)
a) and b) are proved simultaneously by induction on $\alpha$: Let $\alpha > 0$.
a) $H_\alpha(n) = H_{\alpha[n]}(n+1) < H_{\alpha[n]}(n+2) < H_{\alpha[n+1]}(n+2) = H_\alpha(n+1)$.
b) $\beta[m] < \alpha < \beta \Rightarrow \beta[m] \le \alpha[n] < \beta \Rightarrow H_{\beta[m]}(n) \le H_{\alpha[n]}(n) < H_{\alpha[n]}(n+1)$.
c) follows from b).
d) Induction on $\alpha$: Let $\beta < \alpha \,\&\, N\beta \le n$. Then $\beta \le \alpha[N\beta] \le \alpha[n]$. If $\beta = \alpha[n]$ then $H_\beta(n+1) = H_\alpha(n)$. If $\beta < \alpha[n]$ then [by I.H. and a)] $H_\beta(n+1) < H_\beta(n+2) \le H_{\alpha[n]}(n+1) = H_\alpha(n)$.
e) Suppose $\beta < \alpha \,\&\, (\forall m)[\alpha[m] < \beta]$. Then by b) we have $(\forall m)\big[H_{\alpha[m]}(0) < H_{\alpha[m+1]}(0) < H_\beta(0)\big]$. Contradiction.
f) Let $m := \min\{k : \alpha[n:k] = 0\}$. Then $m \ge n$ and, by definition, $H_{\alpha[n:i]}(i) = H_{\alpha[n:i+1]}(i+1)$, for $n \le i < m$. Hence $H_\alpha(n) = H_{\alpha[n:n]}(n) = H_{\alpha[n:m]}(m) = H_0(m) = m$.

**Definition**
$NF(\alpha,\beta) \;:\Leftrightarrow\; \alpha, \beta > 0 \;\&\; (\exists \alpha_0, ..., \alpha_m, \beta_0, ..., \beta_n)\big[\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_m} \,\&\, \beta = \omega^{\beta_0} + ... + \omega^{\beta_n} \,\&\, \alpha_0 \ge ... \ge \alpha_m \ge \beta_0 \ge ... \ge \beta_n\big]$.

**Lemma 4**
Assume
(B5) $(\forall \alpha, \beta)\big[NF(\alpha,\beta) \;\Rightarrow\; \alpha + \beta[n] \le (\alpha+\beta)[n]\big]$
(B6) $(\forall m, n)\big[\omega^m \cdot (n+1) \le \omega^{m+1}[n]\big]$.
Then
a) $NF(\alpha,\beta) \;\Rightarrow\; H_\alpha(H_\beta(n)) \le H_{\alpha+\beta}(n)$,
b) $(H_{\omega^m})^{(n+1)}(n+1) \le H_{\omega^{m+1}}(n)$,
c) For each primitive recursive function $f$ there exists $m$ such that $(\forall \vec{x})\big[f(\vec{x}) < H_{\omega^m}(\max\{\vec{x}\})\big]$.

*Proof:*

a) From (B5) it follows that $H_{\alpha+\beta}(n) = H_{\alpha+\gamma}(n+1)$ with $\beta[n] \leq \gamma < \beta$. By Lemma 3a,b and I.H. we obtain $H_\alpha(H_\beta(n)) \leq H_\alpha(H_\gamma(n+1)) \leq H_{\alpha+\gamma}(n+1)$.
b) From (B6) it follows that $H_{\omega^{m+1}}(n) = H_{\omega^m\cdot(n+1)+\gamma}(n+1)$ for some $\gamma < \omega^{m+1}$. By a) we have $H_{\omega^m}^{(n+1)}(n+1) \leq H_{\omega^m\cdot(n+1)}(n+1) \leq H_{\omega^m\cdot(n+1)+\gamma}(n+1)$.
c) follows from a) and b).

**Definition**
PR$^*$ := set of all primitive recursive functions which are strictly increasing in each argument.

**Lemma 5**
If $N$ satisfies
(N1) $(\exists h \in \mathrm{PR}^*)(\forall n, \alpha_1, \ldots, \alpha_n)\big[\alpha_1 \geq \ldots \geq \alpha_n \;\Rightarrow\; \max\{n, N\omega^{\alpha_1}, \ldots, N\omega^{\alpha_n}\} \leq N(\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}) \leq h(\max\{n, N\omega^{\alpha_1}, \ldots, N\omega^{\alpha_n}\})\big]$,
(N2) $(\exists h \in \mathrm{PR}^*)(\forall m)[m < N\omega^m \leq h(m)]$,
then there exists a $g \in \mathrm{PR}^*$ such that $\forall \alpha, l\,(\,N(\omega^l \cdot (\alpha+1)) \leq g(N\alpha, l))$.

*Proof:*
Let $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_k} + \omega^{m_1} + \cdots \omega^{m_n}$, where $\alpha_1 \geq \ldots \geq \alpha_k \geq \omega > m_1 \geq \ldots \geq m_n$. Then $\omega^l \cdot (\alpha+1) = \omega^{\alpha_1} + \cdots + \omega^{\alpha_k} + \omega^{l+m_1} + \cdots \omega^{l+m_n} + \omega^l$. Thus $N(\omega^l \cdot (\alpha+1)) \leq h(\max\{k+n+1, N\omega^{\alpha_1}, \ldots, N\omega^{\alpha_k}, N\omega^{l+m_1}, \ldots, N\omega^{l+m_n}, N\omega^l\}) \leq$
$\leq h(\max\{k+n+1, N\omega^{\alpha_1}, \ldots, N\omega^{\alpha_k}, h(l+m_1), \ldots, h(l+m_n), h(l)\}) \leq h(h(l + N\alpha+1))$.

## 2 Descent Functions and Hardy Hierarchies

**Definition**
$\mathcal{R}(\tau, N)$ := the set of all functions $f : \mathbb{N}^n \to \mathbb{N}$, $f(\vec{x}) = \min\{k : \delta(\vec{x}, k+1) \not< \delta(\vec{x}, k)\}$
with $\delta : \mathbb{N}^{n+1} \to \tau$ such that $(\exists\alpha)(\forall\vec{x} \in \mathbb{N}^n)[\delta(\vec{x}, 0) \leq \alpha]$ and
$(\exists q \in \mathrm{PR}^*)(\forall\vec{x}, k)[N\delta(\vec{x}, k) \leq q(\vec{x}, k)]$.

**Definition**
If $\mathcal{F}$ is a set of functions $f : \mathbb{N} \to \mathbb{N}$ then $Cl(\mathcal{F})$ is the set of all functions $g : \mathbb{N}^n \to \mathbb{N}$ such that $(\exists f \in \mathcal{F})(\forall\vec{x})\big[g(\vec{x}) \leq f(\max\{\vec{x}\})\big]$.

The proof of the following Theorem has been extracted from (Wainer 1970).

**Theorem 1**
If $\langle \tau, \cdot[\cdot], N \rangle$ is a normed Bachmann system satisfying (B5),(B6),(N1),(N2) then $\mathcal{R}(\tau, N) \subseteq Cl(\{H_\alpha : \alpha < \tau\})$.

*Proof:*
Let $\delta : \mathbb{N}^2 \to \tau,\ \alpha < \tau,\ g \in \mathrm{PR}^*$ such that $(\forall n)[\delta(n, 0) \leq \alpha]$ and $(\forall n, k)[N\delta(n, k) \leq g(n, k)]$. (This case can be assumed without loss of generality.)
By Lemma 5 we may assume that also $N(\omega^l \cdot (\xi+1)) \leq g(N\xi, l)$ for all $\xi, l$.

By Lemma 4c there is an $m \geq 1$ such that $g(g(n, k{+}2), l) < H_{\omega^m}(\max\{l, n, k\})$, for all $l, k, n$.
Abbreviations: $\alpha(n,k) := \omega^m \cdot \delta(n,k)$, $\quad f(n,k) := g(g(n,k+1),m)$.
Then we have
(1) $N(\alpha(n,k+1) + \omega^m) = N(\omega^m \cdot (\delta(n,k+1)+1)) \leq g(N\delta(n,k+1),m) \leq g(g(n,k+1),m) = f(n,k)$,
and
(1') $\alpha(n,0) \leq \omega^m \cdot \alpha$ & $N\alpha(n,0) \leq f(n,0)$.
(2) $f(n,k+1) < H_{\omega^m}(\max\{m,n,k\})$.
¿From (2) we get
(3) $f(n,k+1) < H_{\omega^m}(f(n,k))$.
Now we are going to prove
(4) $\delta(n,k+1) < \delta(n,k) \;\Rightarrow\; H_{\alpha(n,k+1)}(f(n,k+1)) < H_{\alpha(n,k)}(f(n,k))$.
Proof: The premise yields $\alpha(n,k+1) + \omega^m \leq \alpha(n,k)$ and thus, by (1) and Lemma 3a,d,
$H_{\alpha(n,k+1)+\omega^m}(f(n,k)) \leq H_{\alpha(n,k)}(f(n,k))$.
By (3), Lemma 3a and Lemma 4a we get
$H_{\alpha(n,k+1)}(f(n,k{+}1)) < H_{\alpha(n,k+1)}H_{\omega^m}(f(n,k)) \leq H_{\alpha(n,k+1)+\omega^m}(f(n,k))$. This proves (4).

¿From (4) it follows that $\min\{k : \delta(n,k+1) \not< \delta(n,k)\} \leq H_{\alpha(n,0)}(f(n,0))$.
By (1'), (2), Lemma 4a we obtain
$H_{\alpha(n,0)}(f(n,0)) \leq H_{\omega^m \cdot \alpha}(f(n,0)) \leq H_{\omega^m \cdot (\alpha+1)}(\max\{m,n\}) \leq H_{\omega^m \cdot (\alpha+1)+m}(n)$.

**Lemma 6**
If there is an $h \in \mathrm{PR}^*$ such that $(\forall\alpha)[N(\alpha+1) \leq h(N\alpha)]$ & $(\forall\alpha,n)[N(\alpha[n]) \leq h(\max\{N\alpha,n\})]$, then $\{H_\alpha : \alpha < \tau\} \subseteq \mathcal{R}(\tau, N)$.
$\forall n,k(N\delta(n,k) \leq h^{(n+k)}(N\alpha))$.

*Proof:*
By Lemma 3f we have $H_\alpha(n) = \min\{k : \alpha[n:k+1] \not< \alpha[n:k]\}$ with $\alpha[n:k] := \alpha + (n \dot- k)$, for $k \leq n$, and $\alpha[n:k+1] := (\alpha[n:k])[k]$, for $k \geq n$. The premises of Lemma 6 yield $(\forall n)\alpha[n:0] < \alpha + \omega$ and $(\forall n,k)N\alpha[n:k] \leq h^{(n+k)}(N\alpha)$. Hence $H_\alpha \in \mathcal{R}(\tau,N)$.

*Remark*
Theorem 1 and Lemma 6 provide useful criteria for comparing the growth rates of Hardy hierarchies belonging to different normed Bachmann systems. Suppose that $\langle\tau, \cdot[\cdot], N\rangle$ and $\langle\tau, \cdot[\cdot]', N'\rangle$ satisfy the assumptions of Theorem 1, Lemma 6 respectively, and that there is a function $q \in \mathrm{PR}^*$ with $\forall\alpha < \tau(N\alpha \leq q(N'\alpha))$. Then $\{H'_\alpha : \alpha < \tau\} \subseteq Cl(\{H_\alpha : \alpha < \tau\}$ holds for the respective Hardy hierarchies.

**Definition**
Let $\langle\mathcal{A}, \prec\rangle$ be a primitive recursive well-order ($\mathcal{A} \subseteq \mathbb{N}$, $\prec \subseteq \mathbb{N} \times \mathbb{N}$) of order

type $\tau$, and let $ord\colon \mathbb{N} \to \tau$ and $(\cdot)^*: \tau \to \mathcal{A}$, such that $(\forall\alpha \in \tau)[ord(\alpha^*) = \alpha]$ and $(\forall a, b \in \mathcal{A})[b \prec a \iff ord(b) < ord(a)]$.

Let $\mathcal{R}(\mathcal{A}, \prec)$ be the set of all functions $f : \mathbb{N}^n \to \mathbb{N}$, $f(\vec{x}) = \min\{k : \Theta(\vec{x}, k+1) \not\prec \Theta(\vec{x}, k)\}$ with $\Theta : \mathbb{N}^{n+1} \to \mathcal{A}$ primitive recursive and $(\exists a \in \mathcal{A})(\forall\vec{x})[\Theta(\vec{x}, 0) \preceq a]$. For each set $X$ of functions $f\colon \mathbb{N}^n \to \mathbb{N}$ let $PR[X]$ be the set of all functions which are primitive recursive in $X$.

**Theorem 2**
Let $\langle\tau, \cdot[\cdot]\rangle$ be a Bachmann system and assume
(1) $\langle\tau, \cdot[\cdot]\rangle$ satisfies (B5),(B6),
(2) $\exists q \in \mathrm{PR}^* \forall\alpha, \beta(\beta < \alpha \Rightarrow \beta \leq \alpha[q(\beta^*)]$,
(3) $N := \lambda\alpha.\alpha^*$ satisfies (N1),(N2).
Then $\mathcal{R}(\mathcal{A}, \prec) \subseteq \bigcup_{\alpha<\tau} PR[H_\alpha]$.

*Proof:*
Let $N'\alpha := q(\alpha^*)$, with $q$ from assumption (2).
Obviously $N'$ also satisfies (N1),(N2).
So $\langle\tau, \cdot[\cdot], N'\rangle$ is a regular Bachmann system satisfying (B5),(B6),(N1),(N2) and therefore, by Theorem 1, $\mathcal{R}(\tau, N') \subseteq Cl(\{H_\alpha : \alpha < \tau\})$.

Now let $f(x) = \min\{k : \Theta(x, k+1) \not\prec \Theta(x, k)\}$ . We set $\delta(x, k) := ord(\Theta(x, k))$. Then $f(x) = \min\{k : \delta(x, k+1) \not< \delta(x, k)\}$ and $N'\delta(x, k) = q(\Theta(x, k))$. Hence $f \in \mathcal{R}(\tau, N') \subseteq Cl(\{H_\alpha : \alpha < \tau\})$ and therefore $\forall x(f(x) \leq H_\alpha(x))$, for some $\alpha < \tau$. It follows $f(x) = \min\{k \leq H_\alpha(x) : \Theta(x, k+1) \not\prec \Theta(x, k)\}$ and thus $f \in PR[H_\alpha]$.

**Theorem 3**
Let $\langle\tau, \cdot[\cdot]\rangle$ be a Bachmann system satisfying
(B5)' $(\forall\alpha, \beta, n)[NF(\alpha, \beta) \Rightarrow \alpha + \beta[n] = (\alpha + \beta)[n]]$,
(B6)' $(\forall m, n)[\omega^m \cdot (n+1) = (\omega^{m+1})[n]]$,
(X) $\exists q \in \mathrm{PR}^* \forall\alpha[G\alpha := \min\{i : \alpha[0]^i = 0\} \leq q(\alpha^*)]$.
Then $\mathcal{R}(\mathcal{A}, \prec) \subseteq \mathcal{R}(\tau, G) \subseteq \bigcup_{\alpha<\tau} PR[H_\alpha]$.

*Proof:*
By (B5') we have ( $NF(\alpha, \beta) \Rightarrow G(\alpha + \beta) = G\alpha + G\beta$ ), and (B6') yields $G\omega^{m+1} = G\omega^m + 1$, hence $G\omega^m = m + 1$. It follows that $G$ satisfies (N1),(N2). Hence, by Lemma 2, $\langle\tau, \cdot[\cdot], G\rangle$ is a regular Bachmann system, and ¿from Theorem 1 we obtain $\mathcal{R}(\tau, G) \subseteq Cl\{H_\alpha : \alpha < \tau\}$.

Now let $f(x) = \min\{k : \Theta(x, k+1) \not\prec \Theta(x, k)\}$ . We set $\delta(x, k) := ord(\Theta(x, k))$. Then $f(x) = \min\{k : \delta(x, k+1) \not< \delta(x, k)\}$ and $G\delta(x, k) \leq q(\delta(x, k)^*) = q(\Theta(x, k))$, i.e. $f \in \mathcal{R}(\tau, G)$.

*Remark* (**)
The conditions of the last theorem are satisfied by the standard coding $(\mathcal{E}, \prec_\mathcal{E})$ of $\varepsilon_0$ and the standard system of fundamental sequences $\cdot[\cdot]_s$.

**Lemma 7**
If $\langle\tau, \cdot[\cdot]\rangle$ is a Bachmann system such that the function $s(a, n) := (ord(a)[n])^*$

is primitive recursive then $\{H_\alpha : \alpha < \tau\} \subseteq PR[\mathcal{R}(\mathcal{A}, \prec)]$.

*Proof:*

Let $\alpha < \tau$ and $a := \alpha^*$. From Lemma 3f it follows that $H_\alpha(n) = n + \min\{k : \Theta(n, k+1) \not\prec \Theta(n,k)\}$ with $\Theta(n,0) := a,\ \Theta(n,k+1) := s(\Theta(n,k), n+k)$.

# 3 Cichon's Fundamental Sequences and Normed Bachmann Systems

In this section we relate (a slightly generalized version of) Cichon's definition $\alpha[n]_C := \max\{\beta < \alpha : N\beta \leq N\alpha + n\}$ to the concept of a normed Bachmann system and its corresponding Hardy hierarchy. Actually in Theorem 4 and Theorem 5 we show that under certain rather weak assumptions both approaches are interchangeable.

**Theorem 4**

Let $N : \tau \to \mathbb{N}$ be a norm on $\tau$ with $(\forall\alpha)[N0 \leq N\alpha]$ and $(\forall\alpha)[N(\alpha+1) \leq N\alpha + 1]$, and let $p : \tau \to \mathbb{N}$ with $(\forall\alpha)[N\alpha \leq p(\alpha) + 1 \leq p(\alpha+1)]$.

We define $\cdot[\cdot] : \tau \times \mathbb{N} \to \tau$ by $0[n] := 0$ and $\alpha[n] := \max\{\beta < \alpha : N\beta \leq p(\alpha + n)\}$, for $\alpha > 0$.

Let $(H_\alpha)_{\alpha<\tau}$ be the Hardy hierarchy belonging to $\cdot[\cdot]$.

Then

a) $\langle \tau, \cdot[\cdot], N\rangle$ is a normed Bachmann system,

b) $p(\lambda + n) = N\lambda[n]$, for $\lambda \in Lim$,

c) $H_\alpha(n) = \max\{H_\beta(n+1) : \beta < \alpha \,\&\, N\beta \leq p(\alpha+n)\}$, for $\alpha > 0$.

*Proof*:

b) Let $\lambda \in Lim$. Then $\lambda[n] < \lambda$ & $N\lambda[n] \leq p(\lambda + n)$ and thus $(N\lambda[n] \neq p(\lambda+n) \Rightarrow \lambda[n]+1 < \lambda \,\&\, N(\lambda[n]+1) \leq p(\lambda+n))$. By maximality of $\lambda[n]$ the latter yields $N\lambda[n] = p(\lambda+n)$.

a) For $\alpha = \alpha_0 + 1$ we have $\alpha[n] = \max\{\beta \leq \alpha_0 : N\beta \leq p(\alpha_0 + n + 1)\} = \alpha_0$, since $N\alpha_0 \leq p(\alpha_0 + n + 1)$. Now let $\alpha \in Lim$:

(B1): $N(\alpha[n]+1) = N\alpha[n] + 1 \leq p(\alpha+n) + 1 \leq p(\alpha+n+1) \Rightarrow \alpha[n]+1 \leq \alpha[n+1]$.

(B3): $\alpha[n] < \beta < \alpha \Rightarrow N\alpha[n] = p(\alpha+n) < N\beta$.

(B4): $N\alpha \leq p(\alpha) + 1 = N\alpha[0] + 1$.

c) By definition we have $N\alpha[n] \leq p(\alpha+n)$. Hence $H_\alpha(n) = H_{\alpha[n]}(n+1) \leq \max\{H_\beta(n+1) : \beta < \alpha \text{ \& } N\beta \leq p(\alpha+n)\}$.

For the reverse direction we prove by induction on $\alpha$:

$(*)\quad \beta < \alpha \text{ \& } N\beta \leq p(\alpha+n) \Rightarrow H_\beta(n+1) \leq H_\alpha(n)$.

Proof:

¿From the premise we get $H_\alpha(n) = H_{\alpha[n]}(n+1)$ and $\beta \leq \alpha[n]$. If $\beta = \alpha[n]$ then $H_\beta(n+1) = H_\alpha(n)$. Now let $\beta < \alpha[n]$. Using $(\alpha_0+1)[n] = \alpha_0$ and b) one easily verifies that $p(\alpha+n) \leq p(\alpha[n]+n+1)$. ¿From $N\beta \leq p(\alpha+n) \leq p(\alpha[n]+n+1)$ by I.H. one obtains $H_\beta(n+1) < H_\beta(n+2) \leq H_{\alpha[n]}(n+1) = H_\alpha(n)$.

Now we show that the roles of $p$ and $\cdot[\cdot]$ in the last theorem can be reversed in some sense.

**Theorem 5**
Let $\langle\tau, \cdot[\cdot], N\rangle$ be a normed Bachmann system with $(\forall\alpha)[N0 \le N\alpha]$ and $(\forall\alpha)[N(\alpha+1) \le N\alpha+1]$, and let $(H_\alpha)_{\alpha<\tau}$ be the corresponding Hardy hierarchy. We define $p : \tau \to \mathbb{N}$ by $p(n) := N(n)$, and $p(\lambda+n) := N\lambda[n]$, for $\lambda \in Lim$.

Then the following holds for all $\alpha$:
a) $N\alpha \le p(\alpha) + 1 \le p(\alpha+1)$,
b) $\alpha[n] = \max\{\beta < \alpha : N\beta \le p(\alpha+n)\}$, for $\alpha > 0$,
c) $H_\alpha(n) = \max\{H_\beta(n+1) : \beta < \alpha \,\&\, N\beta \le p(\alpha+n)\}$, for $\alpha > 0$.

*Proof:*
a) Let $\alpha = \lambda + n$ with $\lambda \in Lim$. Then $N\alpha = N\lambda + n \le N\lambda[0] + n + 1 \le N\lambda[n] + 1 = p(\alpha) + 1 \le N\lambda[n+1] = p(\alpha+1)$.
b) If $\alpha = \beta + 1$ then $\alpha[n] = \beta$, $N\beta \le p(\beta+1) \le p(\alpha+n)$. If $\alpha \in Lim$ then $N\alpha[n] = p(\alpha+n)$ and $\forall\beta < \alpha(N\beta \le N\alpha[n] \Rightarrow \beta \le \alpha[n])$.
c) This follows from a), b) and Theorem 4.

**Theorem 6**
There exists $N : \tau \to \mathbb{N}$ such that $N0 = 0$, $N(\alpha+1) = N\alpha + 1$ and $(\forall d \in \mathbb{N})[card\{\beta < \alpha : N\beta \le d\} < \omega]$.

*Proof:*
Let $N' : \tau \cap Lim \to \mathbb{N}$ be an arbitrary injective function and put $N(\alpha+n) := N'(\alpha) + n$ for $\alpha \in Lim$.

This last theorem yields for example that for every countable ordinal $\tau$ there is an assignment of fundamental sequences which has the Bachmann property (B2). The standard proof of this fact given in Rose (1984) is based on a non immediate transfinite recursion!

# 4 Applications

In a first step we concentrate on the ordinal $\varepsilon_0 := \min\{\xi : \xi = \omega^\xi\}$, the proof-theoretic ordinal of $PA$. We assume a standard coding $\langle\mathcal{E}, \prec_\mathcal{E}\rangle$ of $\varepsilon_0$ in the natural numbers (see, for example, Rose (1984) for a definition) such that especially $\mathcal{E}$ and $\prec_\mathcal{E}$ are primitive recursive. The standard system of fundamental sequences $\cdot[\cdot]$ is given by the following definition. If $\alpha$ is a limit, then

$$(\omega^\alpha \cdot (\beta+1))[x] := \omega^\alpha \cdot \beta + \omega^{\alpha[x]}$$

and if $\alpha = \alpha_0 + 1$, then

$$(\omega^\alpha \cdot (\beta+1))[x] := \omega^\alpha \cdot \beta + \omega^{\alpha_0} \cdot (x+1).$$

Let $(H_\alpha)_{\alpha<\tau}$ be the corresponding Hardy hierarchy. One easily verifies that $\langle\epsilon_0, \cdot[\cdot]\rangle$ is a Bachmann system.

Let $G : \varepsilon_0 \to \mathbb{N}$, $G(\alpha) := \min\{k : \alpha[0]^k = 0\}$. Then $\langle \varepsilon_0, \cdot[\cdot], G\rangle$ is a normed Bachmann system (cf. Lemma 2). Moreover the "axioms" (B5),(B6) are satisfied, and $\forall \alpha, \beta > 0(\max\{G\alpha, G\beta\} < G(\alpha \# \beta) = G\alpha + G\beta)$ and $\forall m(G\omega^m = m + 1)$. Hence by Theorem 1
(1) $\mathcal{R}(\tau, G) \subseteq Cl(\{H_\alpha : \alpha < \tau\})$.
Now let $N : \epsilon_0 \to \mathbb{N}$ be a norm with $N0 = 0$ and $\forall \alpha(N(\alpha+1) = N\alpha + 1)$. Let $q \in \mathrm{PR}^*$ and $p := q \circ N$. Let $H_0^{N,p}(n) := n$, and $H_\alpha^{N,p}(n) := \max\{H_\beta^{N,p}(n+1) : \beta < \alpha \,\&\, N\beta \le p(\alpha+n)\}$, for $\alpha > 0$. Then by Lemma 6 and Theorem 4 we have
(2) $\{H_\alpha^{N,p} : \alpha < \epsilon_0\} \subseteq \mathcal{R}(\tau, N)$.
¿From (1) and (2) we get
(3) $\exists q \in \mathrm{PR}^* \forall \alpha(G\alpha \le q(N\alpha)) \;\Rightarrow\; \{H_\alpha^{N,p} : \alpha < \epsilon_0\} \subseteq Cl(\{H_\alpha : \alpha < \tau\})$.
Now put $p(n) := N(n)$ and $p(\lambda + n) := N(\lambda[n])$. A transfinite induction with the use of Theorem 4 and 5 yields $H_\alpha(n) = H_\alpha^{N,p}$. So the classical Hardy hierarchy appears as a special case of our approach. We arrive at the following classical result.

**Corollary**
If the Hardy-functions are defined with respect to $\langle \varepsilon_0, \cdot[\cdot]\rangle$, then

$$\bigcup\{PR[\{H_\alpha\}] : \alpha < \varepsilon_0\} = PR[\mathcal{R}(\mathcal{E}, \prec_{\mathcal{E}})].$$

Now we look at three concrete examples of norms. The first norm, $N_1$ is given by the depth or rank of the ordinal term when the ordinals less than $\varepsilon_0$ are represented by the "constant" 0 and the binary function $\lambda\xi\eta.\omega^\xi + \eta$. More precisely, let $N_1 0 := 0$ and $N_1\alpha := \max\{N_1\alpha_1, N_1\alpha_2\} + 1$ if $\alpha = \omega^{\alpha_1} + \alpha_2 > \alpha_1, \alpha_2$.

The second norm (cf. Cichon 1992) is given by the depth of the ordinal term when the ordinals less than $\varepsilon_0$ are represented by 0, and the varyadic function $\lambda\xi_1 \dots \xi_m.\omega^{\xi_1} + \cdots + \omega^{\xi_m}$. Let $N_2 0 := 0$ and $N_2\alpha := \max\{1 + N_2\alpha_1, \dots, 1 + N_2\alpha_m, m\}$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m} > \alpha_1 \ge \dots \ge \alpha_m$.

Finally we consider a norm which is given by the number of symbols (except 0 and +) which occur in an ordinal term (cf. Weiermann 1993). Let $N_3(0) := 0$ and $N_3\alpha := 1 + N_3\alpha_1 + \cdots + 1 + N_3\alpha_m$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m} > \alpha_1 \ge \dots \ge \alpha_m$. (It can be seen easily that $N_3$ equals the function $G$ from the discussion above.)

Let $h(x) := 3^{x+1}$, $p_i := h \circ N_i$ and $\alpha[n]_i := \max\{\beta < \alpha : N_i\beta \le p_i(\alpha + n)\}$. Then for $i = 1, 2, 3$ the "axioms" $(N1)_i$ and $(N2)_2$ are satisfied for $h$ and $N_i$. Moreover the axioms $(B5)_i$ and $(B6)_i$ are satisfied for $\cdot[\cdot]_i$ and $i \in \{1, 2, 3\}$. Therefore $\{H_\alpha^{N_i,p_i} : \alpha < \varepsilon_0\} \subseteq \mathcal{R}(\varepsilon_0, N_i) \subseteq Cl(\{H_\alpha^{N_i,p_i} : \alpha < \varepsilon_0\})$. Furthermore the systems $\langle \varepsilon_0, \cdot[\cdot]_i, N_i\rangle$ are normed Bachmann systems. (In Weiermann (1993) it is shown that every provably total function of $PA$ is bounded by $H_\alpha^{N_3,p_3}$ for some $\alpha < \varepsilon_0$.)

**Corollary**
If $i \in \{1, 2, 3\}$, then

$$\bigcup\{PR[\{H_\alpha^{N_i,p_i}\}] : \alpha < \varepsilon_0\} = PR[\mathcal{R}(\mathcal{E}, \prec_{\mathcal{E}})].$$

Now we turn to a strong generalization of this example. Let $\mathcal{T}(M)$ and $< \subseteq \mathcal{T}(M) \times \mathcal{T}(M)$ be defined as in Rathjen (1991). This ordinal notation system is characteristic for the theory $KPM$ formalizing a recursively Mahlo universe. Let $\langle \mathcal{M}, \prec_{\mathcal{M}} \rangle$ be a standard coding of $\langle \mathcal{T}(M), < \rangle$ in the natural numbers. Define $N \colon \mathcal{T}(M) \to \omega$ as follows:

1. $N0$
2. $NM := 1$,
3. $N \oplus (\alpha_1, \ldots, \alpha_n) := N\alpha_1 + \cdots + N\alpha_n$,
4. $N\overline{\omega}^{\alpha} := N\alpha + 1$,
5. $N\overline{\varphi}\alpha\beta := 1 + N\alpha + N\beta$,
6. $N\psi\kappa\alpha := 1 + N\kappa + N\alpha$,
7. $NZ\alpha := N\alpha + 1$.

For $\alpha \in \mathcal{T}(M)$ let $\alpha[x]_M := \max\{\beta < \alpha : N\beta \leq 3^{3^{N\alpha+x+1}}\}$. Then we get the following corollary.

**Corollary**
If the Hardy-functions are defined with respect to $\langle \mathcal{T}(M), \cdot[\cdot]_M \rangle$, then

$$\bigcup \{PR[\{H_\alpha\}] \colon \alpha \in \mathcal{T}(M)\} = PR[\mathcal{R}(\mathcal{M}, \prec_{\mathcal{M}})].$$

Furthermore, for every function $f : \omega \to \omega$ which is provably total in $KPM$ there is an $\alpha \in \mathcal{T}(M)$ such that $f(x) < H_\alpha(x)$ for all $x < \omega$.

We think that our method also applies to arbitrary so-called "natural well-orderings". But since so far a precise definition of this notion has not been given we can only give the construction of fundamental sequences for notation systems which satisfy certain not too restrictive "naturalness"-assumptions. (These assumptions are satisfied by all notation systems used in proof-theory so far.) We assume that we are given an inductively defined primitive recursive set of terms $T$, where the underlying set of function symbols is finite, together with a primitive recursive well-ordering $<_T$ on $T$. We assume furthermore, that there is a zero-constant $0^*$ in $T$ which denotes the $<_T$-minimal element of $T$ and that (among the function symbols in question) there is a binary function $+^*$ (respectively varyadic function $+^*$ such that the order type of $t_1 +^* \cdots +^* t_n$ with respect to $<_T$ is the sum of the order types of $t_1, \ldots, t_n$ with respect to $<_T$. Then we define a norm $N$ of an ordinal term $t$ by taking the number of occurrences of the "non+"-function symbols and nonzero-constants which occur in $T$. As an assumption on $T$ we additionally demand that $N1^* = 1$ where $1^*$

denotes the $<_T$-successor of $0^*$. Then $N$ satisfies the assumptions of Theorem 4, and therefore by using Cichon's method for defining assignments of fundamental sequences we automatically obtain a normed Bachmann system $\langle \tau, \cdot[\cdot], N \rangle$.

As shown in Theorems 4 and 5 the theory of Hardy hierarchies could as well be developed without any reference to fundamental sequences (using instead the auxiliary functions $p$). For the reader's convenience we present one of our main results (namely the comparison between descent functions and the Hardy hierarchy) in a formulation which does not refer to fundamental sequences at all.

**THEOREM**

Let $N : \tau \to \mathbb{N}$ be a norm on $\tau$ with

(1) $\forall \alpha > 0 (N\alpha > N0)$,

(2) $N\omega^m = m + 1$,

(3) $\alpha, \beta > 0 \Rightarrow N(\alpha \# \beta) = N\alpha + N\beta$.

Let $p : \tau \to \mathbb{N}$ such that

(4) $p(\alpha) < p(\alpha + 1)$,

(5) $N\alpha \le p(\alpha) + 1 \le h(N\alpha)$, for some $h \in \mathrm{PR}^*$,

(6) $N(\omega^m \cdot n) \le p(\omega^{m+1} + n)$,

(7) $NF(\alpha, \beta) \Rightarrow N\alpha + p(\beta) \le p(\alpha + \beta)$.

Let $H_0^{N,p}(n) := n$ and $H_\alpha^{N,p}(n) := \max\{H_\beta^{N,p}(n+1) : \beta < \alpha \,\&\, N\beta \le p(\alpha + n)\}$, for $\alpha > 0$.

Then $Cl(\mathcal{R}(\tau, N)) = Cl(\{H_\alpha^{N,p} : \alpha < \tau\})$.

*Proof:*

Let $0[n] := 0$ and $\alpha[n] := \max\{\beta < \alpha : N\beta \le p(\alpha + n)\}$, for $\alpha > 0$. By Theorem 4 $\langle \tau, \cdot[\cdot], N \rangle$ is a normed Bachmann system and $(H_\alpha^{N,p})_{\alpha < \tau}$ is the corresponding Hardy hierarchy.

By definition and (2),(3),(5) we have $N\alpha[n] \le p(\alpha + n) \le h(N\alpha + n)$. Therefore Lemma 6 yields $\{H_\alpha : \alpha < \tau\} \subseteq \mathcal{R}(\tau, N)$.

According to Theorem 1 it remains to verify (B5),(B6),(N1),(N2).

(N1) follows from (3) and (1).

(N2) follows from (2).

(B6) follows from (6).

(B5) follows from (7). [ $N(\alpha + \beta[n]) = N\alpha + N(\beta[n]) \le N\alpha + p(\beta + n) \le p(\alpha + \beta + n)$.]

*Remark:*

(4)–(7) are satisfied if for example $p(\alpha) := f(2^{N\alpha})$ with $f \in \mathrm{PR}^*$.

ad (6): $N(\omega^m \cdot n) = (N\omega^m) \cdot n < 2^{N\omega^{m+1} + n} \le f(2^{N(\omega^{m+1} + n)}) = p(\omega^{m+1} + n)$.

ad (7): $N\alpha + p(\beta) = N\alpha + f(2^{N\beta}) \le f(2^{N\alpha + N\beta}) = p(\alpha + \beta)$.

Finally assume additionally that the norm $N$ is primitive recursive and that the computation of $\max\{s <_T t : Ns \le p(t +^* n)\}$ (with respect to $<_T$) can be done primitive recursively in the arguments $t$ and $n$. (These assumptions are also satisfied by all notation systems used in proof-theory so far.) Then the system $\langle\langle T, <_T\rangle, \cdot[\cdot], N\rangle$ is p.r.-regulated in the sense of Zemke (1977). Therefore, as an immediate corollary of the main theorem of Zemke (1977) we obtain the hierarchy equivalence property for these ordinal notation systems!

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