A term calculus for (co-)recursive definitions on streamlike data-structures

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Introduction

We consider recursion equations $(*)$ $FX = t(F, X)$ where X ranges over streams (i.e., elements of $\mathbb{S} := \mathbb{N} \to \mathbb{N}$), F is of type $\mathsf{stream} \to \mathsf{stream}$, and the term t is build up from F, X and previously introduced function symbols. The question is whether such an equation has a (total) solution. It is wellknown and explicated at many places in the literature (e.g. [2], [4], [6], [7]) that under certain conditions a (unique) solution for $(*)$ is provided by Banach's Fixed Point Theorem. There is a canonical way to endow the space $\mathbb{S} \to \mathbb{S}$ with a complete (ultra)metric d. So, if the operator $\Phi : (\mathbb{S} \to \mathbb{S}) \to (\mathbb{S} \to \mathbb{S})$ is contracting w.r.t. d then according to Banach's FPT, Φ has a unique fixed point F, i.e. $(*)$ has a unique solution F. In the present paper we establish certain syntactic criteria for $t(F, X)$ which guarantee that Φ is contracting, but on the other side are sufficiently liberal to cover a rather large variaty of equations.

In §1 we present some prerequisites on ultrametric spaces. As shown in [7], the notion of an ultrametric space is in a strong sense equivalent to that of a set X endowed with a *separating family* $(\approx_l)_{l\in\mathbb{N}}$ of equivalence relations (cf. also [2] and [4]). The latter will be called a U-space here.

In §2 we introduce sets T^τ of typed terms. A type τ is either a ground type or a function type $\sigma \overset{\varphi}{\to} \rho$ where φ is a modulus (i.e. a weakly increasing function from $\mathbb{N}$ into $\mathbb{N}$). To each basic type ι we assign a complete U-space $\mathcal{U}^\iota$, and then extend this to all types by setting $\mathcal{U}^{\sigma \overset{\varphi}{\to} \rho} := \mathcal{U}^\sigma \overset{\varphi}{\to} \mathcal{U}^\rho := \{f \in \mathcal{U}^\sigma \to \mathcal{U}^\rho : \forall a, a' \in \mathcal{U}^\sigma (a \approx_{\varphi(l)} a' \Rightarrow fa \approx_l fa')\}$ and $f \approx_l f' :\Leftrightarrow \forall a \in \mathcal{U}^\sigma (fa \approx_l f'a)$. Terms are generated from typed variables and constants by application, (suitably restricted) λ-abstraction, and fixed point formation: $t \in \mathsf{T}^{\tau \overset{\ominus}{\to} \tau}$ & $\mathrm{FV}(t) = \emptyset \Rightarrow \mathsf{Y}t \in \mathsf{T}^\tau$. For each term $t \in \mathsf{T}^\tau$ and variable assignment ξ an interpretation $[\![t]\!]_\xi \in \mathcal{U}^\tau$ is defined in the canonical way, especially, $[\![\mathsf{Y}t]\!]$ is taken to be the unique fixed point of $[\![t]\!]$. This makes sense, since $\mathcal{U}^\tau$ is a complete metric space and $\mathcal{U}^{\tau \overset{\ominus}{\to} \tau}$ is the set of contracting mappings $f : \mathcal{U}^\tau \to \mathcal{U}^\tau$. The only critical point in the definition of $[\![t]\!]_\xi$ is the case $t = \lambda y^\sigma.r \in \mathsf{T}^{\sigma \overset{\varphi}{\to} \tau}$. There one defines $[\![t]\!]_\xi$ to be the mapping $a \mapsto [\![r]\!]_{\xi_y^a}$ from $\mathcal{U}^\sigma$ into $\mathcal{U}^\tau$ (as expected), and has to prove that this mapping actually is in $\mathcal{U}^{\sigma \overset{\varphi}{\to} \tau}$. The formation of terms $\lambda y.r \in \mathsf{T}^{\sigma \overset{\varphi}{\to} \tau}$ is restricted just in such a way (by a side condition "$\mathsf{m}_y(r) \le \varphi$") that this holds.

In the second part of §2 we define for each term t a sequence of *approximating* terms $t^{(n)}$ $(n \in \mathbb{N})$ which are Y-*free* (i.e., contain no occurrences of Y). Roughly speaking, $t^{(n)}$ is obtained from t by replacing every subterm $\mathsf{Y}s$ by $s(\ldots(s(s0^\tau))\ldots)$. By recursion on the build up of t one can find a modulus $\boldsymbol{\varphi}_t$ (primitive recursive in the moduli "occurring" in t) such that $\forall l \forall n \ge \boldsymbol{\varphi}_t(l)([\![t]\!]_\xi \approx_l [\![t^{(n)}]\!]_\xi)$.

In §3 we concentrate on a special system $\textsc{Tree} = (\mathsf{T}^\tau)_\tau$ where the only ground types are nat, tree (with $\mathcal{U}^{\mathsf{nat}} := \mathbb{N}$, $\mathcal{U}^{\mathsf{tree}} := \mathbb{N}^{<\omega} \to \mathbb{N}$), and the only constants are $0^{\mathsf{nat}} : \mathsf{nat}$, $0^{\mathsf{tree}} : \mathsf{tree}$, $\mathsf{tl} : \mathsf{tree} \overset{\oplus}{\to} \mathsf{nat} \overset{\mathrm{id}}{\to} \mathsf{tree}$,

$\mathsf{hd} : \mathsf{tree} \overset{\mathrm{id}}{\to} \mathsf{nat}$, $\mathsf{cons} : \mathsf{nat} \overset{\mathrm{id}}{\to} (\mathsf{nat} \overset{\mathrm{id}}{\to} \mathsf{tree}) \overset{\ominus}{\to} \mathsf{tree}$, and symbols for primitive recursive functions. Using the approximating sequences $(t^{(n)})_{n\in\mathbb{N}}$ we show that for every term $t(x^{\mathsf{tree}}) \in \mathsf{T}^{\mathsf{tree}}$ the functional $(\alpha, \nu) \mapsto [\![t(\alpha)]\!](\nu)$ is primitive recursive.

In §4 we introduce an especially nice subsystem $\widehat{\text{TREE}} = (\widehat{\mathsf{T}}^\tau)_{\tau\ simple}$ of TREE which has the advantage that one can rather easily decide whether a given term belongs to it or not. On the other side the system is comprehensive enough to cover most of the examples occurring in "practice".

In §5 we use the results of §2 and §4 to define (a version of) Mint's continuous cut-elimination operator R_1 for ω-arithmetic (cf. [5]). Actually our approach is slightly more general insofar as instead of ω-arithmetic we treat infinitary propositional logic where formulas may have transfinite (even non-wellfounded) rank.

§1 Prerequisites

In a slightly different form the material of this section can be found (e.g.) in Chapter 8 of [7]. But for the readers convenience we prefer to develop all things we need from scratch here.

Definition.
By a *U-space* we mean a set X together with a family $(\approx_l)_{l\in\mathbb{N}}$ of equivalence relations $\approx_l$ on X such that for all $x, x' \in X$ the following holds

(i) $x \approx_0 x'$, (ii) $x \approx_{l+1} x' \Rightarrow x \approx_l x'$, (iii) $\forall l(x \approx_l x') \Rightarrow x = x'$.

Definition.
A *modulus* is a weakly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ with $\varphi(0) := 0$. Moduli are denoted by φ, ψ.

Abbreviation. $\psi \le \varphi :\Leftrightarrow \forall l(\psi(l) \le \varphi(l))$.

Some special moduli:
$\mathsf{0}(l) := 0$, $\mathrm{id}(l) := l$, $\oplus(l) := \begin{cases} 0 & \text{if } l = 0 \\ l{+}1 & \text{otherwise} \end{cases}$, $\ominus(l) := l \dot{-} 1 := \begin{cases} 0 & \text{if } l = 0 \\ l{-}1 & \text{otherwise} \end{cases}$.
$\max\{\varphi, \psi\}(l) := \max\{\varphi(l), \psi(l)\}$.

Definitions. Let $(X, (\approx_l)_{l\in\mathbb{N}})$ be a U-space, $x \in X$, and $(x_n)_{n\in\mathbb{N}}$ a sequence in X.
$\lim_n x_n = x :\Leftrightarrow \forall l \exists N \forall n \ge N(x_n \approx_l x)$.
$(x_n)_{n\in\mathbb{N}}$ is a *Cauchy-sequence* $:\Leftrightarrow \forall l \exists N \forall m, n \ge N(x_m \approx_l x_n)$.
$(X, (\approx_l)_{l\in\mathbb{N}})$ is *complete* iff for every Cauchy sequence in X there exists an $x \in X$ with $\lim_n x_n = x$.

Lemma 1.1. In each U-space the following holds:
(a) $\lim_n x_n = x$ & $\lim_n x_n = x' \implies x = x'$.
(b) $\lim_n x_n = x$ & $\forall n \ge n_0(x_n \approx_l a) \implies x \approx_l a$.

Proof :
(a) $\forall l \exists N \forall n \ge N(x_n \approx_l x)$ & $\forall l \exists N \forall n \ge N(x_n \approx_l x') \Rightarrow \forall l \exists N(x \approx_l x_N \approx_l x') \Rightarrow \forall l(x \approx_l x') \Rightarrow x = x'$.
(b) $\exists N \forall n \ge N(x_n \approx_l x)$ & $\forall n \ge n_0(x_n \approx_l a) \Rightarrow \exists n(x \approx_l x_n \approx_l a) \Rightarrow x \approx_l a$.

Definition.
Let $(X, (\approx_l^X)_{l\in\mathbb{N}})$, $(Y, (\approx_l^Y)_{l\in\mathbb{N}})$ be U-spaces and $f : X \to Y$.

f is *φ-continuous* $:\iff \quad \forall l \forall x, x' \in X (x \approx_{\varphi(l)}^X x' \Rightarrow f(x) \approx_l^Y f(x'))$.

$f : X \to X$ is *contracting* $:\iff \quad f$ is $\ominus$-continuous.

Definition.
If X, Y are sets then $X \to Y :=$ set of all functions $f : X \to Y$.

If $X = (X, \approx_l^X)$, $Y = (Y, \approx_l^Y)$ are U-spaces then $X \overset{\varphi}{\to} Y := \{f \in X \to Y : f$ is φ-continuous$\}$;
in addition, $X \overset{\varphi}{\to} Y$ denotes the space $(X \overset{\varphi}{\to} Y, \approx_l^{X\to Y})$ with $f \approx_l^{X\to Y} f' :\Leftrightarrow \forall x \in X (f(x) \approx_l^Y f'(x))$.

Remark. $f \in X \overset{\varphi}{\to} Y$ & $f' \in X \to Y$ & $x, x' \in X$ & $f \approx_l f'$ & $x \approx_{\varphi(l)} x' \implies f(x) \approx_l f'(x')$.

Theorem 1.2. If X, Y are U-spaces (and Y is complete) then $X \overset{\varphi}{\to} Y$ is a (complete) U-space.

Proof of completeness:
Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $X \overset{\varphi}{\to} Y$. Then $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y for each $x \in X$. Let $f(x) := \lim_n f_n(x)$. Since (f_n) is Cauchy, we have $\forall l \exists N \forall x \forall m, n \geq N (f_n(x) \approx_l f_m(x))$. By Lemma 1.1b from this we get $\forall l \exists N \forall x \forall n \geq N (f(x) \approx_l f_n(x))$, i.e., $\forall l \exists N \forall n \geq N (f \approx_l f_n)$. So we have $\lim_n f_n = f$, and it remains to prove $f \in X \overset{\varphi}{\to} Y$. But this follows immediately from $\forall l \exists N \forall n \geq N \forall x (f(x) \approx_l f_n(x))$ and $\forall l \forall x, x' \in X (x \approx_{\varphi(l)} x' \Rightarrow f_n(x) \approx_l f_n(x'))$.

Theorem 1.3 (Banach's Fixed-Point Theorem).
If X is a complete U-space then every $f \in X \overset{\ominus}{\to} X$ has a unique fixed point $\mathsf{fp}(f) \in X$, namely $\mathsf{fp}(f) = \lim_n x_n$ where $x_0 \in X$ is arbitrary and $x_{n+1} := f(x_n)$.

Moreover we have $\forall l \forall n \geq l (\mathsf{fp}(f) \approx_l x_n)$.

Proof :
Uniqueness: $f(x) = x$ & $f(y) = y \Rightarrow \forall l (x \approx_l y \;\Rightarrow\; x = f(x) \approx_{l+1} f(y) = y) \overset{\text{Induction}}{\Rightarrow} \forall l (x \approx_l y) \;\Rightarrow\; x = y$.
Existence: By induction on l we get $(*)$ $\forall n, m \geq l (x_m \approx_l x_n)$:
IH $\Rightarrow \;\forall m, n \geq l{+}1 (x_{m-1} \approx_l x_{n-1}) \;\Rightarrow\; \forall m, n \geq l{+}1 (x_m = f(x_{m-1}) \approx_{l+1} f(x_{n-1}) = x_n)$.

By $(*)$, $x := \lim_n x_n$ exists, and (by Lemma 1.1b) we have $\forall l \forall n \geq l (x \approx_l x_n)$.
From this we conclude $\forall l \forall n \geq l (f(x) \approx_{l+1} f(x_n) = x_{n+1})$ and so $f(x) = \lim_n x_n = x$.

Theorem 1.4.
If X is a complete U-space then the mapping
$\mathsf{fp} : (X \overset{\ominus}{\to} X) \to X$, $f \mapsto \mathsf{fp}(f)$ is id-continuous, i.e. $\mathsf{fp} \in (X \overset{\ominus}{\to} X) \overset{\text{id}}{\to} X$.

Proof :
To prove: $\forall l \forall f, f' \in X \overset{\ominus}{\to} X (f \approx_l f' \Rightarrow \mathsf{fp}(f) \approx_l \mathsf{fp}(f'))$.
So, assume $f, f' \in X \overset{\ominus}{\to} X$ & $\forall x \in X (f(x) \approx_l f'(x))$:
We have $\mathsf{fp}(f) \approx_l x_l$ & $\mathsf{fp}(f') \approx_l x'_l$, where $\forall n (x_{n+1} = f(x_n)$ & $x'_{n+1} = f'(x'_n))$. Further
$\forall n < l (x_n \approx_n x'_n \Rightarrow x_{n+1} = f(x_n) \approx_{n+1} f(x'_n) \approx_{n+1} f'(x'_n) = x'_{n+1})$, and so $\mathsf{fp}(f) \approx_l x_l \approx_l x'_l \approx_l \mathsf{fp}(f')$.

§2 λ-Terms

We introduce a system of typed λ-terms together with a canonic interpretation in complete U-spaces. A special feature of our system is that every function type is decorated with some modulus φ. This corresponds to the formation of the function spaces $X \overset{\varphi}{\to} Y$ in §1.

Types

1. There are certain *(ground types)* $\iota_0, \iota_1, ...$;
2. If σ, τ are types, and φ is a modulus then $\sigma \overset{\varphi}{\to} \tau$ is a type.

Abbreviations. $\sigma \to \tau := \sigma \overset{\mathrm{id}}{\to} \tau$, $\sigma^n \to \tau := \underbrace{\sigma \to \ldots \to \sigma}_{n} \to \tau$.

Atomic terms (or **atoms** for short) are typed *variables* $x, y, z, ...$, and typed *constants* $\mathsf{c}, ...$

We write $\mathsf{a} : \tau$ to express that atom a has type τ.

Further we use x^τ, y^τ, z^τ, (c^τ, *resp.*) to denote variables (constants, resp.) of type τ.

As usual, $\mathrm{FV}(t)$ denotes the set of free variables of t.

Inductive Definition of term sets T^τ

Simultaneously we define a modulus $\mathsf{m}_x(t)$ for each $t \in \mathsf{T}^\tau$ and each variable x.

1. x^τ, $\mathsf{c}^\tau \in \mathsf{T}^\tau$;
2. $r \in \mathsf{T}^{\sigma \overset{\varphi}{\to} \tau}$ & $s \in \mathsf{T}^\sigma \implies (rs) \in \mathsf{T}^\tau$;
3. $r \in \mathsf{T}^\tau$ & $\mathsf{m}_y(r) \le \varphi$ & $y : \sigma \implies \lambda^\varphi y.r \in \mathsf{T}^{\sigma \overset{\varphi}{\to} \tau}$;
4. $t \in \mathsf{T}^{\tau \overset{\ominus}{\to} \tau}$ & $\mathrm{FV}(t) = \emptyset \implies (\mathsf{Y}t) \in \mathsf{T}^\tau$.

Definition of the moduli $\mathsf{m}_x(t)$

1. $\mathsf{m}_x(x) := \mathrm{id}$, and $\mathsf{m}_x(t) = \mathsf{0}$ for $x \notin \mathrm{FV}(t)$;
2. $\mathsf{m}_x(rs) := \max\{\mathsf{m}_x(r)\,,\, \mathsf{m}_x(s) \circ \varphi\}$ where $r \in \mathsf{T}^{\sigma \overset{\varphi}{\to} \tau}$;
3. $\mathsf{m}_x(\lambda^\varphi y.r) := \mathsf{m}_x(r)$ $(x \neq y)$.

Abbreviation. $\lambda x.r := \lambda^{\mathrm{id}} x.r$.

As usual we identify α-equivalent terms, and save parenthesis by writing $rs_1...s_n$ for $(\ldots((rs_1)s_2)\ldots s_n)$ (provided r, s_i have types $\sigma_1 \overset{\varphi_1}{\to} \ldots \to \sigma_n \overset{\varphi_n}{\to} \tau$, σ_i, respectively).

Interpretation

We assume that to each ground type ι there is assigned a complete U-space $\mathcal{U}^\iota \neq \emptyset$. For function types we define (inductively) $\mathcal{U}^{\sigma \overset{\varphi}{\to} \tau} := \mathcal{U}^\sigma \overset{\varphi}{\to} \mathcal{U}^\tau$. Then from Theorem 1.2 it follows that every $\mathcal{U}^\tau$ is a complete U-space. We also assume that for each constant c^τ an interpretation $[\![\mathsf{c}^\tau]\!] \in \mathcal{U}^\tau$ is fixed.

An *assignment* is a mapping ξ which assigns to each variable x^τ an object $\xi(x^\tau) \in \mathcal{U}^\tau$.
We use ξ, η to denote assignments.

Definition of $[\![t]\!]_\xi \in \mathcal{U}^\tau$ for $t \in \mathsf{T}^\tau$.

1. $[\![x]\!]_\xi := \xi(x)$, $[\![\mathsf{c}]\!]_\xi := [\![\mathsf{c}]\!]$;
2. $[\![rs]\!]_\xi := [\![r]\!]_\xi [\![s]\!]_\xi$;

3. $[\![\lambda^\varphi y^\sigma.r^\rho]\!]_\xi :=$ the mapping $\mathcal{U}^\sigma \ni a \mapsto [\![r]\!]_{\xi_y^a} \in \mathcal{U}^\rho$.

4. $[\![\mathsf{Y}t]\!]_\xi := \mathsf{fp}([\![t]\!]_\xi)$ (cf. Theorem 1.3).

Simultaneously with this definition one proves that indeed $[\![t^\tau]\!]_\xi \in \mathcal{U}^\tau$. This is obvious in all cases except in "3. $t = \lambda^\varphi y^\sigma.r^\rho$ " which will be taken care of in the following theorem.

Theorem 2.1.

For $t \in \mathsf{T}^\tau$ the following holds:

(i) $[\![t]\!]_\xi \in \mathcal{U}^\tau$;

(ii) $\forall x \in \mathrm{FV}(t)(\,\xi(x) \approx_{\mathsf{m}_x(t)(l)} \eta(x)\,) \implies [\![t]\!]_\xi \approx_l [\![t]\!]_\eta$.

Proof by induction on t (simultaneously for (i) and (ii)):

(i) We only consider the case $t = \lambda^\varphi y.r$; in all other cases the claim is trivial or follows immediately from the I.H. So, let $t = \lambda^\varphi y.r$ with $r \in \mathsf{T}^\rho$ & $\mathsf{m}_y(r) \le \varphi$ & $y : \sigma$. Then $\tau = \sigma \xrightarrow{\varphi} \rho$, and by the I.H. we have (i) $\forall a \in \mathcal{U}^\sigma([\![r]\!]_{\xi_y^a} \in \mathcal{U}^\rho)$ and (ii) $\forall l \forall a, b \in \mathcal{U}^\sigma(a \approx_{\mathsf{m}_y(r)(l)} b \;\Rightarrow\; [\![r]\!]_{\xi_y^a} \approx_l [\![r]\!]_{\xi_y^b})$. Hence $[\![t]\!]_\xi \in \mathcal{U}^\sigma \xrightarrow{\varphi} U^\rho = \mathcal{U}^\tau$.

(ii) Assume $\forall x \in \mathrm{FV}(t)(\xi(x) \approx_{\mathsf{m}_x(t)(l)} \eta(x))$ $(*)$.

1.1. $t = x$: Then $\mathsf{m}_x(t) = \mathrm{id}$ and thus $[\![t]\!]_\xi = \xi(x) \approx_l \eta(x) = [\![t]\!]_\eta$.

1.2. $\mathrm{FV}(t) = \emptyset$: trivial.

2. $t = rs$ with $r \in \mathsf{T}^{\sigma \xrightarrow{\varphi} \tau}$ and $s \in \mathsf{T}^\sigma$: By definition of $\mathsf{m}_x(rs)$ from $(*)$ we get
$\forall x \in \mathrm{FV}(r)(\xi(x) \approx_{\mathsf{m}_x(r)(l)} \eta(x))$ & $\forall x \in \mathrm{FV}(s)(\xi(x) \approx_{\mathsf{m}_x(s)(\varphi(l))} \eta(x))$ $(**)$.
From this by I.H. we conclude $[\![r]\!]_\xi \approx_l [\![r]\!]_\eta$ & $[\![s]\!]_\xi \approx_{\varphi(l)} [\![s]\!]_\eta$.
By I.H. we also have $[\![r]\!]_\xi \in \mathsf{T}^{\sigma \xrightarrow{\varphi} \tau}$ which together with $(**)$ yields $[\![rs]\!]_\xi \approx_l [\![rs]\!]_\eta$.

3. $t = \lambda^\varphi y.r$ with $r \in \mathsf{T}^\rho$ & $\mathsf{m}_y(r) \le \varphi$: Since $\forall x \in \mathrm{FV}(t)(\mathsf{m}_x(r) = \mathsf{m}_x(t))$, from $(*)$ we get
$\forall x \in \mathrm{FV}(r)(\xi_y^a(x) \approx_{\mathsf{m}_x(r)(l)} \eta_y^a(x))$. Now the I.H. yields $\forall a \in \mathcal{U}^\sigma([\![r]\!]_{\xi_y^a} \approx_l [\![r]\!]_{\eta_y^a})$, i.e. $[\![t]\!]_\xi \approx_l [\![t]\!]_\eta$.

In the next step for each $t \in \mathsf{T}^\tau$ we define a modulus $\boldsymbol{\varphi}_t$ and a sequence of approximating terms $t^{(n)} \in \mathsf{T}^\tau$ $(n \in \mathbb{N})$ such that $\forall l \forall n \ge \boldsymbol{\varphi}_t(l)(\,[\![t]\!]_\xi \approx_l [\![t^{(n)}]\!]_\xi\,)$.

We assume that for each ground type ι there is a special constant $\mathsf{0}^\iota \in \mathsf{T}^\iota$.
For $\tau = \sigma \xrightarrow{\varphi} \rho$ we set $\mathsf{0}^\tau := \lambda^\varphi x^\sigma.\mathsf{0}^\rho$.

Definition of t^+

1. $x^+ := x$, $\mathsf{c}^+ := \mathsf{c}$; 2. $(rs)^+ := r^+ s^+$; 3. $(\lambda^\varphi x.r)^+ := \lambda^\varphi x.r^+$; 4. $(\mathsf{Y}t)^+ := t^+(\mathsf{Y}t)$.

Definition of $t^{(n)}$

1.1. $x^{(0)} := x$, $\mathsf{c}^{(0)} := \mathsf{c}$; 1.2. $(rs)^{(0)} := r^{(0)} s^{(0)}$; 1.3. $(\lambda^\varphi x.r)^{(0)} := \lambda^\varphi x.r^{(0)}$; 1.4. $(\mathsf{Y}t)^{(0)} := \mathsf{0}^\tau$.

2. $t^{(n+1)} := (t^+)^{(n)}$.

Lemma 2.2. $(rs)^{(n)} = r^{(n)} s^{(n)}$ and $(\lambda^\varphi x.r)^{(n)} = \lambda^\varphi x.r^{(n)}$.

Definition of $\boldsymbol{\varphi}_t$

1. $\boldsymbol{\varphi}_x := \boldsymbol{\varphi}_{\mathsf{c}} := \mathsf{0}$;

2. $\boldsymbol{\varphi}_{rs} := \max\{\boldsymbol{\varphi}_r\,,\, \boldsymbol{\varphi}_s \circ \psi\}$ where $r \in \mathsf{T}^{\sigma \xrightarrow{\psi} \tau}$;

3. $\boldsymbol{\varphi}_{\lambda^{\psi} y.r} := \boldsymbol{\varphi}_r$;

4. If $t = \mathsf{Y}s$ then $\boldsymbol{\varphi}_t(0) := 0$, $\boldsymbol{\varphi}_t(l{+}1) := \max\{\boldsymbol{\varphi}_t(l){+}1, \boldsymbol{\varphi}_s(l{+}1)\}$.

Remark. If $t = \mathsf{Y}s$ with $\boldsymbol{\varphi}_s \leq \mathrm{id}$ then $\boldsymbol{\varphi}_t = \mathrm{id}$.

Theorem 2.3. $t \in \mathsf{T}^\tau \implies \forall l \forall n \geq \boldsymbol{\varphi}_t(l)(\, [\![t]\!]_\xi \approx_l [\![t^{(n)}]\!]_\xi\,)$.

Proof by induction on t:

1. t atomic: trivial.

2. $t = rs$ with $r \in \mathsf{T}^{\sigma \overset{\psi}{\to} \tau}$:

$n \geq \boldsymbol{\varphi}_t(l) \;\Rightarrow\; n \geq \boldsymbol{\varphi}_r(l)\ \&\ n \geq \boldsymbol{\varphi}_s(\psi(l)) \overset{\text{IH}}{\Rightarrow} [\![r]\!]_\xi \approx_l [\![r^{(n)}]\!]_\xi\ \&\ [\![s]\!]_\xi \approx_{\psi(l)} [\![s^{(n)}]\!]_\xi \;\Rightarrow\; [\![t]\!]_\xi \approx_l [\![t^{(n)}]\!]_\xi$.

3. $t = \lambda^\varphi y.r$:

$n \geq \boldsymbol{\varphi}_t(l) \;\Rightarrow\; n \geq \boldsymbol{\varphi}_r(l) \overset{\text{IH}}{\Rightarrow} \forall a([\![r]\!]_{\xi^a_y} \approx_l [\![r^{(n)}]\!]_{\xi^a_y}) \;\Rightarrow\; [\![t]\!]_\xi = [\![\lambda^\varphi y.r]\!]_\xi \approx_l [\![\lambda^\varphi y.r^{(n)}]\!]_\xi = [\![t^{(n)}]\!]_\xi$.

4. $t = \mathsf{Y}s$ with $s \in \mathsf{T}^{\tau \overset{\ominus}{\to} \tau}$: First note that $t^{(n+1)} = (t^+)^{(n)} = (s^+t)^{(n)} = (s^+)^{(n)} t^{(n)} = s^{(n+1)} t^{(n)}$ $(*)$.

We prove $\forall n \geq \boldsymbol{\varphi}_t(l)([\![t]\!]_\xi \approx_l [\![t^{(n)}]\!]_\xi)$ by a subsidiary induction on l:

The case $l = 0$ is trivial. Assume now that $n \geq \boldsymbol{\varphi}_t(l{+}1)$. Then $n > 0$, $n{-}1 \geq \boldsymbol{\varphi}_t(l)$, and $n \geq \varphi_s(l{+}1)$.

Hence $[\![t]\!]_\xi \approx_l [\![t^{(n-1)}]\!]_\xi$ (by S.I.H.) and $[\![s]\!]_\xi \approx_{l+1} [\![s^{(n)}]\!]_\xi$ (by I.H.).

Together with $[\![s]\!] \in \mathcal{U}^{\tau \overset{\ominus}{\to} \tau}$ this implies $[\![t]\!]_\xi = [\![s]\!]_\xi [\![t]\!]_\xi \approx_{l+1} [\![s]\!]_\xi [\![t^{(n-1)}]\!]_\xi \approx_{l+1} [\![s^{(n)}]\!]_\xi [\![t^{(n-1)}]\!]_\xi \overset{(*)}{=} [\![t^{(n)}]\!]_\xi$.

§3 The system Tree

In this section we introduce a special instance Tree of our generic system $(\mathsf{T}^\tau)_\tau$ and prove that every Tree-definable functional $F : (\mathbb{N}^{<\omega} \to \mathbb{N}) \to \mathbb{N}^{<\omega} \to \mathbb{N}$ is primitive recursive in the modulus of its defining term (cf. Theorem 3.4).

Tree has the ground types nat, tree and the constants 0^{nat}, 0^{tree}, $\mathsf{hd} : \mathsf{tree} {\to} \mathsf{nat}$, $\mathsf{tl} : \mathsf{tree} \overset{\oplus}{\to} \mathsf{nat} {\to} \mathsf{tree}$, $\mathsf{cons} : \mathsf{nat} {\to} (\mathsf{nat} {\to} \mathsf{tree}) \overset{\ominus}{\to} \mathsf{tree}$, $\mathsf{f}^n_i : \mathsf{nat}^n \to \mathsf{nat}$.

Semantics of Tree.

We assume a bijective coding $\mathbb{N}^{<\omega} \to \mathbb{N}$, $(a_0, ..., a_{n-1}) \mapsto \langle a_0, ..., a_{n-1} \rangle$ of finite sequences of natural numbers.

We use ν, ν' to denote natural numbers considered as codes for finite sequences. As usual we define $\mathrm{lh}(\langle a_0, ..., a_{n-1} \rangle) := n$, $\langle a_0, ..., a_{n-1} \rangle * \langle b_0, ..., b_{m-1} \rangle := \langle a_0, ..., a_{n-1}, b_0, ..., b_{n-1} \rangle$, and $(\langle a_0, ..., a_{n-1} \rangle)_i :=$ if $i < n$ then a_i else 0 . We assume $\langle\rangle = 0$ and that the funcions $a \mapsto \mathrm{lh}(a)$, $(a, b) \mapsto a{*}b$, $(a, i) \mapsto (a)_i$, and (for each fixed n) $(a_0, ..., a_n) \mapsto \langle a_0, ..., a_n \rangle$ are primitive recursive.

In consideraton of this coding, the elements of $\mathbb{S} := \mathbb{N} \to \mathbb{N}$ are called *trees*.

If α is a tree then $\alpha(0)$ is the content of its *root*, and $\alpha[n] := \lambda\nu.\alpha(\langle n \rangle {*} \nu)$ is its *n-th immediate subtree.*

The ground types and constants of Tree are interpreted as follows

$\mathcal{U}^{\mathsf{nat}} := (\mathbb{N}, \approx^{\mathsf{nat}}_l)$ with $a \approx^{\mathsf{nat}}_l a' :\Leftrightarrow a, a' \in \mathbb{N}\ \&\ (1 \leq l \Rightarrow a = a')$.

$\mathcal{U}^{\mathsf{tree}} := (\mathbb{S}, \approx^{\mathsf{tree}}_l)$ with $\alpha \approx^{\mathsf{tree}}_l \alpha' :\Leftrightarrow \alpha, \alpha' \in \mathbb{S}\ \&\ \forall \nu(\mathrm{lh}(\nu) < l \Rightarrow \alpha(\nu) = \alpha'(\nu))$.

We assume that to each constant f_i^n there is assigned some fixed primitive recursive function $[\![\mathsf{f}_i^n]\!] : \mathbb{N}^n \to \mathbb{N}$, so that the (universal) mapping $(\ulcorner \mathsf{f}_i^n \urcorner, k) \mapsto [\![\mathsf{f}_i^n]\!](k)_0 ... (k)_{n-1}$ is also primitive recursive.

Especially $[\![\mathsf{f}_0^0]\!] = 0$ and $[\![\mathsf{f}_0^1]\!](a) = a{+}1$.

Abbreviations. $\mathsf{S} := \mathsf{f}_0^1$, $\underline{0} := \mathsf{0}^{\mathsf{nat}} := \mathsf{f}_0^0$, $\underline{n{+}1} := \mathsf{S}\,\underline{n}$.

The interpretation of the other constants is given by

$[\![\mathsf{0}^{\mathsf{nat}}]\!] := 0$, $[\![\mathsf{0}^{\mathsf{tree}}]\!] := \lambda\nu.0$;

$[\![\mathsf{hd}]\!] : \mathbb{S} \to \mathbb{N}$, $[\![\mathsf{hd}]\!]\alpha := \alpha(0)$;

$[\![\mathsf{tl}]\!] : \mathbb{S} \to \mathbb{N} \to \mathbb{S}$, $([\![\mathsf{tl}]\!]\alpha n) := \alpha[n] = \lambda\nu.\alpha(\langle n\rangle * \nu)$;

$[\![\mathsf{cons}]\!] : \mathbb{N} \to (\mathbb{N} \to \mathbb{S}) \to \mathbb{S}$, $([\![\mathsf{cons}]\!]af)(\nu) := \begin{cases} a & \text{if } \nu = \langle\rangle \\ (fn)(\nu') & \text{if } \nu = \langle n\rangle * \nu' \end{cases}$.

One easily shows that $[\![\mathsf{c}^\tau]\!] \in \mathcal{U}^\tau$ holds for each of these constants c^τ.

What follows are preparations for the proof of the above-mentioned Theorem 3.4.

Abbreviation.

For $t \in \mathsf{T}^{\mathsf{tree}}$ and $r_1, ..., r_n \in \mathsf{T}^{\mathsf{nat}}$ let $\mathsf{tl}_{r_1 \ldots r_n} t := \begin{cases} t & \text{if } n = 0 \\ \mathsf{tl}(\mathsf{tl}_{r_1 \ldots r_{n-1}} t) r_n & \text{otherwise} \end{cases}$.

For $\nu = \langle k_1, ..., k_n\rangle$ let $\mathsf{tl}_\nu\, t := \mathsf{tl}_{\underline{k_1} \ldots \underline{k_n}}\, t$.

Lemma 3.1. For each $t \in \mathsf{T}^{\mathsf{tree}}$ the following holds:

(a) $\boldsymbol{\varphi}_{\mathsf{tl}_\nu t}(l) = \boldsymbol{\varphi}_t(\mathrm{lh}(\nu){+}l)$.

(b) $[\![t]\!]_\xi(\nu) = [\![\mathsf{tl}_\nu\, t^{(m)}]\!]_\xi(0)$ for all $m \geq \boldsymbol{\varphi}_t(\mathrm{lh}(\nu){+}1)$.

Proof :

(a) $\boldsymbol{\varphi}_{\mathsf{tl}_0 t} = \boldsymbol{\varphi}_t$, $\boldsymbol{\varphi}_{\mathsf{tl}_{\nu * \langle n\rangle}}(l) = \boldsymbol{\varphi}_{\mathsf{tl}(\mathsf{tl}_\nu t)\underline{n}}(l) = \boldsymbol{\varphi}_{\mathsf{tl}(\mathsf{tl}_\nu t)}(l) = \boldsymbol{\varphi}_{\mathsf{tl}_\nu t}(l{+}1) \overset{\mathrm{IH}}{=} \boldsymbol{\varphi}_t(l{+}1{+}\mathrm{lh}(\nu))$.

(b) For $m \geq \boldsymbol{\varphi}_t(\mathrm{lh}(\nu){+}1)$ we have $[\![t]\!]_\xi \approx_{\mathrm{lh}(\nu)+1} [\![t^{(m)}]\!]$ and thus $[\![\mathsf{tl}_\nu\, t]\!]_\xi \approx_1 [\![\mathsf{tl}_\nu\, t^{(m)}]\!]_\xi$.

Hence $[\![t]\!]_\xi(\nu) = [\![\mathsf{tl}_\nu\, t]\!]_\xi(0) = [\![\mathsf{tl}_\nu\, t^{(m)}]\!]_\xi(0)$.

Definition of $t \triangleright t'$

$t \triangleright t'$ if, and only if, $t, t' \in \mathsf{T}^\tau$ and one of the following cases holds

($\triangleright$1) $t = (\lambda x.r)s$ and $t' = r_x(s)$,

($\triangleright$2) $t = \mathsf{hd}(\mathsf{cons}\, r\tilde{t})$ and $t' = r$, ($\triangleright$3) $t = \mathsf{tl}(\mathsf{cons}\, r\tilde{t})$ and $t' = \tilde{t}$,

($\triangleright$4) $t = \mathsf{hd}\,\mathsf{0}^{\mathsf{tree}}$ and $t' = \mathsf{0}^{\mathsf{nat}}$, ($\triangleright$5) $t = \mathsf{tl}\,\mathsf{0}^{\mathsf{tree}} r$ and $t' = \mathsf{0}^{\mathsf{tree}}$.

Inductive Definition of $t \to_\beta^1 t$ for $t, t' \in \mathsf{T}^\tau$.

1. $t \triangleright t' \implies t \to_\beta^1 t'$;
2. $r \to_\beta^1 r' \implies rs \to_\beta^1 r's$; $s \to_\beta^1 s' \implies rs \to_\beta^1 rs'$;
3. $r \to_\beta^1 r' \implies \lambda^\varphi x.r \to_\beta^1 \lambda^\varphi x.r'$.

As usual, $=_\beta$ denotes the reflexive, symmetric and transitive closure of $\to^1$, and t^* is called a *β-normalform of* t if $t^* =_\beta t$ and t^* is in β-normalform (i.e. $\neg\exists t'(t^* \to_\beta^1 t')$).

Theorem.

(a) Every reduction sequence $t_0 \to^1_\beta t_1 \to^1_\beta \ldots$ terminates.

(b) Every $t \in \mathsf{T}^\tau$ has a unique β-normalform $\mathsf{nf}(t) \in \mathsf{T}^\tau$.

Proof: Easy extension of the corresponding proofs for the simply typed λ-calculus.

Definition. $\mathsf{nf}(t) :=$ the β-normalform of t.

Definitions.

$\mathsf{type}_2 := \{\mathsf{tree}\} \cup \{\mathsf{nat}^n {\to} \mathsf{nat} : n \in \mathbb{N}\}$,

$\mathsf{Var}_2 := \{x^\sigma : \sigma \in \mathsf{type}_2\}$

t is Y-*free* iff the constant Y does not occur in t.

$\mathsf{T}_2^\tau := \{t \in \mathsf{T}^\tau : \mathrm{FV}(t) \subseteq \mathsf{Var}_2\}$, $\mathsf{T}_2^{\tau *} := \{t \in \mathsf{T}_2^\tau : t$ Y-free and β-normal$\}$.

Syntactic variables:

x, y: variables of type nat;

$\mathbf{x}^n$: variables and constants of type $\mathsf{nat}^n {\to} \mathsf{nat}$;

X: variables of type tree.

Lemma 3.2.

(a) If $r \in \mathsf{T}_2^{\mathsf{nat}*}$ then r has one of the following shapes:
$\mathbf{x}^n r_1 ... r_n$, $\mathsf{hd}(\mathsf{tl}_{r_1 ... r_n} X)$ with $r_1, ..., r_n \in \mathsf{T}_2^{\mathsf{nat}*}$;

(b) If $t \in \mathsf{T}_2^{\mathsf{tree}*}$ then t has one of the following shapes:
0^{tree}, $\mathsf{cons}\, r\tilde{t}$, $\mathsf{tl}_{r_1 ... r_n} X$ with $r, r_1, ..., r_n \in \mathsf{T}_2^{\mathsf{nat}*}$.

Proof by induction on r, t, resp., simultaneously for (a) and (b):

(a) Obviously $r = \mathbf{x}^n r_1 ... r_n$ or $r = \mathsf{hd}\, t$ with $r_1, ..., r_n \in \mathsf{T}_2^{\mathsf{nat}*}$, $t \in \mathsf{T}_2^{\mathsf{tree}*}$. In addition, $t \neq 0^{\mathsf{tree}}$ and t cannot have the form $\mathsf{cons}\ldots$.. Therefore, by IHb, $t = \mathsf{tl}_{r_1, ..., r_n} X$ with $r_1, ..., r_n \in \mathsf{T}_2^{\mathsf{nat}*}$.

(b) Obviously $t = X$ or $t = \mathsf{tl}\, t_0 r$ or $t = \mathsf{cons}\, r\tilde{t}$ with $r \in \mathsf{T}_2^{\mathsf{nat}*}$, $t_0 \in \mathsf{T}_2^{\mathsf{tree}*}$. In addition, t_0 cannot have the form $\mathsf{cons}\ldots$. Therefore, by IHb, $t_0 = \mathsf{tl}_{r_1, ..., r_n} X$ with $r_1, ..., r_n \in \mathsf{T}_2^{\mathsf{nat}*}$.

Definition.

A Var_2-*assignment* is a mapping $\zeta : \mathsf{Var}_2 {\times} \mathbb{N} \to \mathbb{N}$.

With each Var_2-assignment ζ we associate an assignment $\overline{\zeta}$ in the original sense:

$$\overline{\zeta}(x^\sigma) := \begin{cases} \lambda k_1 ... \lambda k_n . \zeta(x^\sigma, \langle k_1, ..., k_n \rangle) & \text{if } \sigma = \mathsf{nat}^n \to \mathsf{nat} \\ \lambda \nu . \zeta(x^\sigma, \nu) & \text{if } \sigma = \mathsf{tree} \\ [\![0^\sigma]\!] & \text{otherwise} \end{cases} .$$

Then we set $[\![t]\!]_\zeta := [\![t]\!]_{\overline{\zeta}}$ for $t \in \mathsf{T}_2^\tau$.

$\mathsf{T}_2^* := \mathsf{T}_2^{\mathsf{nat}*} \cup \mathsf{T}_2^{\mathsf{tree}*}$.

Definition of $\mathbf{h} : (\mathsf{Var}_2{\times}\mathbb{N} \to \mathbb{N}) \times \mathsf{T}_2^* \to \mathbb{N}$

$$\mathbf{h}_\zeta(\mathbf{x}^n r_1...r_n) := \begin{cases} \zeta(\mathbf{x}^n, \langle \mathbf{h}_\zeta(r_1)...\mathbf{h}_\zeta(r_n)\rangle) & \text{if } \mathbf{x}^n \text{ is a variable} \\ [\![\mathbf{x}^n]\!]\mathbf{h}_\zeta(r_1)\dots\mathbf{h}_\zeta(r_n) & \text{otherwise} \end{cases};$$

$\mathbf{h}_\zeta(\mathsf{hd}(\mathsf{tl}_{r_1...r_n}X)) := \mathbf{h}_\zeta(\mathsf{tl}_{r_1...r_n}X) := \zeta(X, \langle \mathbf{h}_\zeta(r_1), ..., \mathbf{h}_\zeta(r_n)\rangle);$

$\mathbf{h}_\zeta(0^{\mathsf{tree}}) := 0;$

$\mathbf{h}_\zeta(\mathsf{cons}\, r\tilde{t}) := \mathbf{h}_\zeta(r).$

Theorem 3.3.

(a) If $t^\iota \in \mathsf{T}_2^*$ then $\mathbf{h}_\zeta(t^\iota) = \begin{cases} [\![t]\!]_\zeta(0) & \text{if } \iota = \mathsf{tree} \\ [\![t]\!]_\zeta & \text{if } \iota = \mathsf{nat} \end{cases}$.

(b) The functional $\mathbf{h}$ is primitive recursive.

Theorem 3.4.

If $t \in \mathsf{T}_2^{\mathsf{tree}}$ and $\boldsymbol{\varphi}_t \le \psi$, then the functional

$F_t : (\mathsf{Var}_2{\times}\mathbb{N} \to \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$, $F_t(\zeta, \nu) := [\![t]\!]_\zeta(\nu)$ is primitive recursive in ψ.

Proof :

$[\![t]\!]_\zeta(\nu) \overset{\text{L.3.1b}}{=} [\![\mathsf{tl}_\nu\, t^{(m)}]\!]_\zeta(0) = [\![\mathsf{nf}(\mathsf{tl}_\nu\, t^{(m)})]\!]_\zeta(0) \overset{\text{Th.3.3a}}{=} \mathbf{h}_\zeta(\mathsf{nf}(\mathsf{tl}_\nu\, t^{(m)}))$ where $m := \psi(\mathrm{lh}(\nu){+}1)$.

§4 The subsystem $\widehat{\text{TREE}}$; Examples

Definition. A type τ is called *simple* if all its arrows are decorated by the modulus id.

In this section we introduce an especially nice subsystem $\widehat{\text{TREE}} = (\widehat{\mathsf{T}}^\tau)_{\tau\ simple}$ of TREE which has the advantage that one can rather easily decide whether a given term belongs to it or not. On the other side the system is comprehensive enough to cover most of the examples occurring in "practice", as indicated by the examples below.

Definition of $\mathsf{Cr}(t)$

$\mathsf{Cr}(\mathsf{tl}\, t) := \mathrm{FV}(t)$; $\mathsf{Cr}(\mathsf{cons}\, rt) := \mathsf{Cr}(r)$;

$\mathsf{Cr}(rs) := \mathsf{Cr}(r) \cup \mathsf{Cr}(s)$ if $r \ne \mathsf{tl}$ and $r \ne \mathsf{cons}\, r_0$;

$\mathsf{Cr}(\lambda y.r) = \mathsf{Cr}(r) \setminus \{y\}$; $\mathsf{Cr}(y) := \mathsf{Cr}(\mathsf{c}) := \mathsf{Cr}(\mathsf{Y}t) := \emptyset$.

Inductive Definition of $\widehat{\mathsf{T}}^\tau$ **for simple** τ

1.1. If a^τ is a variable or a constant $\ne \mathsf{cons}, \mathsf{tl}$ then $\mathsf{a}^\tau \in \widehat{\mathsf{T}}^\tau$;

1.2. $r \in \widehat{\mathsf{T}}^{\mathsf{nat}}$ & $\tilde{t} \in \widehat{\mathsf{T}}^{\mathsf{nat}\to\mathsf{tree}}$ $\Longrightarrow$ $\mathsf{cons}\, r\tilde{t} \in \widehat{\mathsf{T}}^{\mathsf{tree}}$;

1.3. $t \in \widehat{\mathsf{T}}^{\mathsf{tree}}$ & $\mathsf{Cr}(t) = \emptyset$ $\Longrightarrow$ $\mathsf{tl}\, t \in \widehat{\mathsf{T}}^{\mathsf{nat}\to\mathsf{tree}}$;

2. $r \in \widehat{\mathsf{T}}^{\sigma\to\tau}$ & $s \in \widehat{\mathsf{T}}^\sigma$ $\Longrightarrow$ $rs \in \widehat{\mathsf{T}}^\tau$;

3. $r \in \widehat{\mathsf{T}}^\rho$ & $y \notin \mathsf{Cr}(r)$ & $y : \sigma$ $\Longrightarrow$ $\lambda y.r \in \widehat{\mathsf{T}}^{\sigma\to\rho}$;

4. $t \in \widehat{\mathsf{T}}^\tau$ & $z : \tau$ & $\mathrm{FV}(t) \subseteq \{z\}$ & $t = \lambda\vec{x}.\mathsf{cons}\, r\tilde{t}$ with $z \notin \mathrm{FV}(r) \cup \mathsf{Cr}(\tilde{t})$ $\Longrightarrow$ $\mathsf{Y}\lambda^\ominus z.t \in \widehat{\mathsf{T}}^\tau$.

We prove now that $\widehat{\mathsf{T}}^\tau \subseteq \mathsf{T}^\tau$ (for simple τ), so that for any assignment ξ, each $t \in \widehat{\mathsf{T}}^\tau$ has a well defined value $[\![t]\!]_\xi \in \mathcal{U}^\tau$.

Theorem 4.1. $t \in \widehat{\mathsf{T}}^\tau \implies t \in \mathsf{T}^\tau$ and $\mathsf{m}_x(t) \le \psi[x,t] := \begin{cases} \oplus & \text{if } x \in \mathsf{Cr}(t) \\ \text{id} & \text{otherwise} \end{cases}$.

Proof :

1.1. t a variable or constant: trivial.

1.2. $t = \mathsf{cons}\, r\tilde{t}$: $\mathsf{m}_x(t) = \max\{\mathsf{m}_x(r), \mathsf{m}_x(\tilde{t}) \circ \ominus\} \overset{\text{IH}}{\le} \max\{\psi[x,r], \psi[x,\tilde{t}] \circ \ominus\} \le \max\{\psi[x,t], \text{id}\} = \psi[x,t]$.

1.3. $t = \mathsf{tl}\, s$ with $\mathsf{Cr}(s) = \emptyset$: Then $\mathsf{m}_x(t) = \mathsf{m}_x(s) \circ \oplus$, and since $\mathsf{Cr}(s) = \emptyset$, by IH we have $\mathsf{m}_x(s) \le \text{id}$, and so $\mathsf{m}_x(t) \le \oplus$. If $x \notin \mathsf{Cr}(t)$ then $x \notin \text{FV}(s)$ and so $\mathsf{m}_x(t) = \mathsf{m}_x(s) \circ \oplus = \mathsf{0} \circ \oplus = \mathsf{0}$.

2. $t = rs$ with $r \in \widehat{\mathsf{T}}^{\sigma \to \tau}$: Then $\mathsf{m}_x(t) = \max\{\mathsf{m}_x(r), \mathsf{m}_x(s)\} \overset{\text{IH}}{\le} \max\{\psi[x,r], \psi[x,s]\} = \psi[x,t]$.

3. $t = \lambda y.r$ with $r \in \widehat{\mathsf{T}}^\rho$ and $y \notin \mathsf{Cr}(r)$: By IH we have $r \in \mathsf{T}^\rho$ and $\mathsf{m}_y(r) \le \text{id}$ which yields $t \in \mathsf{T}^\tau$. By IH we also have $\mathsf{m}_x(r) \le \psi[x,r]$, hence (for $x \in \text{FV}(t)$) $\mathsf{m}_x(t) = \mathsf{m}_x(r) \le \psi[x,r] = \psi[x,t]$.

4. $t = \mathsf{Y}\lambda^\ominus z.t_0$ with $\text{FV}(t) = \emptyset$, $t_0 = \lambda \vec{x}.\, \mathsf{cons}\, r\tilde{t} \in \widehat{\mathsf{T}}^\tau$ and $z \notin \text{FV}(r) \cup \mathsf{Cr}(\tilde{t})$:
Then $\mathsf{m}_z(t_0) = \mathsf{m}_z(\mathsf{cons}\, r\tilde{t}) = \max\{\mathsf{m}_z(r), \mathsf{m}_z(\tilde{t}) \circ \ominus\}$. Since $z \notin \text{FV}(r) \cup \mathsf{Cr}(\tilde{t})$, we have $\mathsf{m}_z(r) = \mathsf{0}$ and (by I.H.) $\mathsf{m}_z(\tilde{t}) \le \text{id}$, hence $\mathsf{m}_z(t_0) \le \ominus$. By I.H. we also have $t_0 \in \mathsf{T}^\tau$. Together with $\mathsf{m}_z(t_0) \le \ominus$ this yields $\lambda^\ominus z.t_0 \in \mathsf{T}^{\tau \overset{\ominus}{\to} \tau}$, hence $t = \mathsf{Y}\lambda^\ominus z.t_0 \in \mathsf{T}^\tau$. Further $\mathsf{m}_x(t) = \mathsf{0}$, since $\text{FV}(t) = \emptyset$.

Theorem 4.2. For each $t \in \widehat{\mathsf{T}}_2^{\mathsf{tree}}$ the functional $F_t : (\mathsf{Var}_2 \times \mathbb{N} \to \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$, $F_t(\xi, \nu) := [\![t]\!]_\xi(\nu)$ is primitive recursive.

Proof :

By induction on $t \in \widehat{\mathsf{T}}^\tau$ we prove $\boldsymbol{\varphi}_t \le \text{id}$. Then the claim follows by Theorem 3.4.

The only nontrivial case is $t = \mathsf{Y}\lambda^\ominus z.t_0$ with $t_0 \in \widehat{\mathsf{T}}^\tau$. Then by a subsidiary induction on l we obtain $\boldsymbol{\varphi}_t(l) = l$: $\boldsymbol{\varphi}_t(0) \overset{\text{Def}}{=} 0$, $\boldsymbol{\varphi}_t(l{+}1) \overset{\text{Def}}{=} \max\{\boldsymbol{\varphi}_t(l){+}1, \boldsymbol{\varphi}_{t_0}(l{+}1)\} \overset{\text{SIH}}{=} \max\{l{+}1, \boldsymbol{\varphi}_{t_0}(l{+}1)\} \overset{\text{IH}}{=} l{+}1$.

Examples

In these examples instead of tree we have the ground type str (streams).

The types of the constants $\mathsf{hd}, \mathsf{tl}, \mathsf{cons}$ are now: $\mathsf{hd} : \mathsf{str} \to \mathsf{nat}$, $\mathsf{tl} : \mathsf{str} \overset{\oplus}{\to} \mathsf{str}$, $\mathsf{cons} : \mathsf{nat} \to \mathsf{str} \overset{\ominus}{\to} \mathsf{str}$.

In (1)-(7) one almost immediately sees that the respective term belongs to the subsystem $(\widehat{\mathsf{T}}^\tau)_\tau$. (1)-(6) are choosen in view of [8] where the same examples are treated, but with much more effort than here.

To increase readability we write $\mathsf{F}\vec{x} := r :: t_z(\mathsf{F})$ for $\mathsf{F} := \mathsf{Y}\lambda^\ominus z \lambda \vec{x}.\, \mathsf{cons}\, rt$.

Let $X, X_1, ...$ be variables of type str.

(1) $\mathsf{map}\, f X := f(\mathsf{hd}\, X) :: \mathsf{map}\, f(\mathsf{tl}\, X)$.

(2) $\mathsf{zip}\, f X_1 X_2 := f\,(\mathsf{hd}\, X_1)(\mathsf{hd}\, X_2) :: \mathsf{zip}\, f\,(\mathsf{tl}\, X_1)(\mathsf{tl}\, X_2)$.

(3) $\mathsf{cases}_l\, x X_1 ... X_l := \mathsf{f}_l x(\mathsf{hd}\, X_1)...(\mathsf{hd}\, X_l) :: \mathsf{cases}_l\, x(\mathsf{tl}\, X_1)...(\mathsf{tl}\, X_l)$, where $[\![\mathsf{f}_l]\!] k a_1 ... a_l := \begin{cases} a_{k+1} & \text{if } k < l \\ 0 & \text{otherwise} \end{cases}$.

(4) $\mathsf{merge}\, X_1 X_2 := \mathsf{g}(\mathsf{hd}\, X_1)(\mathsf{hd}\, X_2) :: \mathsf{cases}_3(\mathsf{f}\,(\mathsf{hd}\, X_1)(\mathsf{hd}\, X_2)) \begin{cases} \mathsf{merge}(\mathsf{tl}\, X_1) X_2 \\ \mathsf{merge}(\mathsf{tl}\, X_1)(\mathsf{tl}\, X_2) \\ \mathsf{merge}\, X_1(\mathsf{tl}\, X_2) \end{cases}$

with $[\![\mathsf{g}]\!] a_1 a_2 := \min\{a_1, a_2\}$ and $[\![\mathsf{f}]\!] a_1 a_2 := \begin{cases} 0 & \text{if } a_1 < a_2 \\ 1 & \text{if } a_1 = a_2 \\ 2 & \text{otherwise} \end{cases}$.

(5) $\mathsf{ham} := 1 :: \mathsf{merge}(\mathsf{map}(\times_2)\,\mathsf{ham})(\mathsf{map}(\times_3)\,\mathsf{ham})$ with $[\![\times_k]\!]a := k \cdot a$.

(6) $\mathsf{fib} := 0 :: 1 :: \mathsf{zip}\ \mathsf{add}\ \mathsf{fib}\,(\mathsf{tl}\,\mathsf{fib})$ with $[\![\mathsf{add}]\!]a_1 a_2 := a_1 + a_2$.

(7) $\mathsf{F}_0 X := 1{+}\,\mathsf{hd}\,X :: \mathsf{F}_0(\mathsf{F}_0(\mathsf{F}_0(\mathsf{tl}\,X)))$.

(8) $\mathsf{F}_1 X \overset{\psi}{:=} 1{+}\,\mathsf{hd}\,X :: \mathsf{F}_1(\mathsf{G}X)$, i.e.,
$\mathsf{F}_1 := \mathsf{Y}\lambda^{\ominus} z^{\tau}\lambda^{\psi} X.\mathbf{t}$ with $\mathbf{t} := 1{+}\,\mathsf{hd}\,X :: z^{\tau}(\mathsf{G}X)$, $\mathsf{G} \in \mathsf{T}^{\mathsf{str}\overset{\varphi}{\to}\mathsf{str}}$ and $\tau := \mathsf{str}\overset{\psi}{\to}\mathsf{str}$.

$\mathsf{m}_z(\mathbf{t}) = \mathsf{m}_z(z(\mathsf{G}X)) \circ \ominus = \mathrm{id} \circ \ominus = \ominus$,
$\mathsf{m}_X(\mathbf{t}) = \max\{\mathsf{m}_X(1{+}\,\mathsf{hd}\,X), \mathsf{m}_X(z(\mathsf{G}X)){\circ}\ominus\} = \max\{\mathrm{id}, \varphi{\circ}\psi{\circ}\ominus\}$.

Let us assume that $\mathrm{id} \leq \varphi$. Then for $\psi(l) := \varphi^{(l)}(0)$ we have $\mathsf{m}_X(\mathbf{t}) \leq \psi$ and therefore $\mathsf{F}_1 \in \mathsf{T}^{\tau}$.

(9) $\mathsf{F}_2 X \overset{\psi}{:=} 1{+}\,\mathsf{hd}\,X :: \mathsf{F}_2(\mathsf{F}_2(\mathsf{tl}(\mathsf{tl}\,X)))$, i.e.,
$\mathsf{F}_2 := \mathsf{Y}\lambda^{\ominus} z^{\tau}\lambda^{\psi} X.\mathbf{t}$ with $\mathbf{t} := 1{+}\,\mathsf{hd}\,X :: z^{\tau}(z^{\tau}(\mathsf{tl}(\mathsf{tl}\,X)))$ and $\tau := \mathsf{str}\overset{\psi}{\to}\mathsf{str}$.

By computing the moduli $\mathsf{m}_z(\mathbf{t})$, $\mathsf{m}_X(\mathbf{t})$ we show that the term F_2 does not belong to T^{τ}, independently of how we choose ψ. In addition one can easily see that there is actually no function $F : \mathbb{S} \to \mathbb{S}$ satisfying the equation for F_2.

$\mathsf{m}_z(\mathbf{t}) = \mathsf{m}_z(z(z(\mathsf{tl}(\mathsf{tl}\,X)))){\circ}\ominus = \max\{\mathrm{id}, \mathsf{m}_z(z(\mathsf{tl}(\mathsf{tl}\,X))){\circ}\psi\}{\circ}\ominus =$
$= \max\{\mathrm{id}, \max\{\mathrm{id}, \mathsf{0} \circ \psi\}{\circ}\psi\}{\circ}\ominus = \max\{\mathrm{id}, \psi\}{\circ}\ominus$,

$\mathsf{m}_X(\mathbf{t}) = \max\{\mathrm{id}, \mathsf{m}_X(z(z(\mathsf{tl}(\mathsf{tl}\,X))))\}{\circ}\ominus = \max\{\mathrm{id}, \oplus{\circ}\oplus{\circ}\psi{\circ}\psi\}{\circ}\ominus$.

Let us assume that $\mathsf{m}_z(\mathbf{t}) \leq \ominus$ and $\mathsf{m}_X(\mathbf{t}) \leq \psi$.
Then we have $\psi(2) \leq \mathsf{m}_z(\mathbf{t})(3) \leq 2$ & $1 \leq \mathsf{m}_X(\mathbf{t})(1) \leq \psi(1)$ & $\oplus(\oplus(\psi(\psi(1)))) \leq \mathsf{m}_x(2) \leq \psi(2)$.

From $1 \leq \psi(1)$ it follows that $\oplus(\oplus(\psi(\psi(1)))) = 2 + \psi(\psi(1)) \geq 3$, thence $3 \leq \psi(2) \leq 2$. Contradiction.

So there is no modulus ψ such that $\mathsf{m}_z(\mathbf{t}) \leq \ominus$ and $\mathsf{m}_X(t) \leq \psi$. Therefore F_2 is not a term of the system T.

§5 Cut-elimination for infinitary propositional logic

In this section we apply Theorem 4.2 to define (a version of) Mint's continuous cut-elimination operator R_1 for ω-arithmetic introduced in [5]. Actually our approach is slightly more general insofar as instead of ω-arithmetic we treat infinitary propositional logic where formulas may have transfinite (even non-wellfounded) rank. We will work with a one sided sequent calculus á la Tait. Derivations will be formalized as trees of inferences (strictly speaking, inference symbols), and the relation "d is a derivation of Γ with $\deg(d) \leq \rho$" will be introduced by a co-inductive definition. In order to save some cases in the definition of the cut-elimination operator, we will not eliminate atomic cuts.

We assume the following entities to be given

(1) a set V;

(2) a set $\mathcal{F}$; the elements of $\mathcal{F}$ are called *formulas*;

(2) a linear ordering $(\mathcal{O}, \prec)$;

(3) for each $C \in \mathcal{F}$

- a symbol $\diamond(C) \in \{\mathsf{At}, \bigwedge, \bigvee\}$, we set $\overline{\bigwedge} := \bigvee$, $\overline{\bigvee} := \bigwedge$, $\overline{\mathsf{At}} := \mathsf{At}$,
- an element $\mathrm{rk}(C) \in \mathcal{O}$,
- a subset $|C| \subseteq V$ and formulas $C[\iota]$ $(\iota \in |C|)$ such that $(\diamond(C) = \mathsf{At} \Rightarrow |C| = \emptyset)$ and $\forall \iota \in |C| (\mathrm{rk}(C[\iota]) \prec \mathrm{rk}(C))$;

(4) an operation $\neg : \mathcal{F} \to \mathcal{F}$ such that
$\diamond(\neg C) = \overline{\diamond(C)}$, $|\neg C| = |C|$, $\mathrm{rk}(\neg C) = \mathrm{rk}(C)$, $\forall \iota \in |C| ((\neg C)[\iota] = \neg(C[\iota]))$ and $\neg\neg C = C$.

Definitions.

$\mathcal{F}_{\mathbf{x}} := \{C \in \mathcal{F} : \diamond(C) = \mathbf{x}\}$, for $\mathbf{x} \in \{\mathsf{At}, \bigwedge, \bigvee\}$;

$\mathrm{IS} := \mathcal{F}_{\mathsf{At}} \cup \mathcal{F}_{\bigwedge} \cup \{(\kappa, B) : B \in \mathcal{F}_{\bigvee} \ \&\ \kappa \in |B|\} \cup \{\mathsf{Cut}_C : C \in \mathcal{F}\} \cup \{\mathsf{Rep}\}$ *(inference symbols)*;

$\mathbb{D} := V^{<\omega} \to \mathrm{IS}$ *(derivations)*.

SEQ := set of all *sequents*, i.e., (finite) sets of formulas.

We will use the following *syntactic variables:*

$\iota, \kappa \in V$	$P \in \mathcal{F}_{\mathsf{At}}$
$\nu \in V^{<\omega}$	$A \in \mathcal{F}_{\bigwedge}$
$\rho, \sigma, \tau \in \mathcal{O}$	$B \in \mathcal{F}_{\bigvee}$
$\mathcal{J} \in \mathrm{IS}$	$C \in \mathcal{F}$
$d, e, c \in \mathbb{D}$	$\Gamma \in \mathsf{SEQ}$.

Informally every inference symbol $\mathcal{J} \in \mathrm{IS}$ represents a family of inferences as follows

$$(P)\ \frac{}{\Gamma, \neg P, P} \qquad (A)\ \frac{\ldots \Gamma, A[\iota] \ldots (\iota \in |A|)}{\Gamma, A} \qquad (\kappa, B)\ \frac{\Gamma, B[\kappa]}{\Gamma, B} \qquad (\mathsf{Cut}_C)\ \frac{\Gamma, C \quad \Gamma, \neg C}{\Gamma} \qquad (\mathsf{Rep})\ \frac{\Gamma}{\Gamma}\ .$$

Formally this is rendered by assigning to each inference symbol $\mathcal{J}$ its *arity* $|\mathcal{J}| \subseteq V$ and seqents $\Delta(\mathcal{J})$ *(main part)*, $\Delta_\iota(\mathcal{J})$ $(\iota \in |\mathcal{J}|)$ *(minor part(s))* in such a way that the above inferences get the shape

$$(\mathcal{J})\ \frac{\ldots \Gamma, \Delta_\iota(\mathcal{J}) \ldots (\iota \in |\mathcal{J}|)}{\Gamma, \Delta(\mathcal{J})}\ .$$

The definition of $|\mathcal{J}|$, $\Delta(\mathcal{J})$, $\Delta_\iota(\mathcal{J})$ is shown in the following table

$\mathcal{J}$	P	A	(κ, B)	Cut_C	Rep
$\lvert\mathcal{J}\rvert$	$\emptyset$	$\lvert A\rvert$	$\{0\}$	$\{0, 1\}$	$\{0\}$
$\Delta_\iota(\mathcal{J})$		$\{A[\iota]\}$	$\{B[\kappa]\}$	$\{C\}$, $\{\neg C\}$	$\emptyset$
$\Delta(\mathcal{J})$	$\{\neg P, P\}$	$\{A\}$	$\{B\}$	$\emptyset$	$\emptyset$

Abbreviation.

For $d \in \mathbb{D}$ we set $d_{\langle\rangle} := d(\langle\rangle) \in \mathrm{IS}$, and $d[\iota] := \lambda\nu.d(\langle\iota\rangle{*}\nu) \in \mathbb{D}$.

Definition of functions $\mathcal{I} : \mathcal{F}_\bigwedge \to V \to \mathbb{D} \to \mathbb{D}$, $\mathcal{R} : \mathcal{F}_\bigvee \to \mathbb{D} \to \mathbb{D} \to \mathbb{D}$, $\mathcal{E} : \mathcal{O} \to \mathbb{D} \to \mathbb{D}$

$$(\mathcal{I}A\kappa d)_{\langle\rangle} = \begin{cases} \mathsf{Rep} & \text{if } d_{\langle\rangle} = A \\ d_{\langle\rangle} & \text{otherwise} \end{cases} \qquad (\mathcal{I}A\kappa d)[\iota] = \begin{cases} \mathcal{I}A\kappa\, d[\kappa] & \text{if } d_{\langle\rangle} = A \\ \mathcal{I}A\kappa\, d[\iota] & \text{otherwise} \end{cases}$$

$$(\mathcal{R}Bde)_{\langle\rangle} = \begin{cases} \mathsf{Cut}_{B[\kappa]} & \text{if } d_{\langle\rangle} = (\kappa, B) \\ d_{\langle\rangle} & \text{otherwise} \end{cases} \qquad (\mathcal{R}Bde)[\iota] = \begin{cases} \mathcal{I}\neg B\, \kappa\, e & \text{if } d_{\langle\rangle} = (\kappa, B) \text{ and } \iota = 1 \\ \mathcal{R}Bd[\iota]e & \text{otherwise} \end{cases}$$

$$(\mathcal{E}\sigma d)_{\langle\rangle} = \begin{cases} \mathsf{Rep} & \text{if } d_{\langle\rangle} = \mathsf{Cut}_C \text{ with } \sigma \preceq \mathrm{rk}(C) \text{ and } C \in \mathcal{F}_\bigwedge \cup \mathcal{F}_\bigvee \\ d_{\langle\rangle} & \text{otherwise} \end{cases}$$

$$(\mathcal{E}\sigma d)[\iota] = \begin{cases} \mathcal{E}\sigma(\mathcal{R}B(\mathcal{E}\nu d[0])(\mathcal{E}\nu d[1])) & \text{if } d_{\langle\rangle} = \mathsf{Cut}_B \text{ with } \sigma \preceq \mathrm{rk}(B) =: \nu \\ \mathcal{E}\sigma(\mathcal{R}\neg A(\mathcal{E}\nu d[1])(\mathcal{E}\nu d[0])) & \text{if } d_{\langle\rangle} = \mathsf{Cut}_A \text{ with } \sigma \preceq \mathrm{rk}(A) =: \nu \\ \mathcal{E}\sigma d[\iota] & \text{otherwise} \end{cases}$$

Remark. It follows from Theorem 4.1 (together with Theorem 2.1) that the above equations actually define the desired functions $\mathcal{I}$, $\mathcal{R}$, $\mathcal{E}$. To see this we assume (w.l.o.g.) that $\mathcal{F}_\bigwedge \cup \mathcal{F}_\bigvee \cup \mathcal{O} \cup \mathrm{IS} \subseteq V$ and that $\mathcal{I}, \mathcal{R}, \mathcal{E}$ actually are restrictions of corresponding functions $\mathcal{I} : V \to V \to \mathbb{D} \to \mathbb{D}$, $\mathcal{R} : V \to \mathbb{D} \to \mathbb{D} \to \mathbb{D}$, $\mathcal{E} : V \to \mathbb{D} \to \mathbb{D}$ which are obtained as the interpretations of terms $t_\mathcal{I} \in \widehat{\mathsf{T}}^{\mathsf{nat}^2 \to \mathsf{tree} \to \mathsf{tree}}$, $t_\mathcal{R} \in \widehat{\mathsf{T}}^{\mathsf{nat} \to \mathsf{tree} \to \mathsf{tree} \to \mathsf{tree}}$, $t_\mathcal{E} \in \widehat{\mathsf{T}}^{\mathsf{nat} \to \mathsf{tree} \to \mathsf{tree}}$. In this context the ground types nat, tree are interpreted by $\mathcal{U}^{\mathsf{nat}} := V$, $\iota \approx_l \kappa \;:\Leftrightarrow\; (l \geq 1 \Rightarrow \iota = \kappa)$ and $\mathcal{U}^{\mathsf{tree}} := \mathbb{D}$, $d \approx_l e \;:\Leftrightarrow\; \forall \nu \in V^{<\omega}(\mathrm{lh}(\nu) < l \Rightarrow d(\nu) = e(\nu))$. The constants f_i^n are interpreted by suitable functions $[\![\mathsf{f}_i^n]\!] : V^n \to V$ as needeed in the above definitions.

Definition.
We assume that $-1 \notin \mathcal{O}$, and set $-1 \prec \rho$ for all $\rho \in \mathcal{O}$.
$$\deg(\mathcal{J}) := \begin{cases} \mathrm{rk}(C) & \text{if } \mathcal{J} = \mathsf{Cut}_C \text{ with } C \in \mathcal{F}_\bigwedge \cup \mathcal{F}_\bigvee \\ -1 & \text{otherwise} \end{cases}.$$

Inductive definition of $d \vdash_\rho \Gamma$ *(d is wellfounded derivation of Γ with cut-degree $\preceq \rho$)*
$\deg(d_{\langle\rangle}) \prec \rho \;\&\; \Delta(d_{\langle\rangle}) \subseteq \Gamma \;\&\; \forall \iota \in |d_{\langle\rangle}|(\, d[\iota] \vdash_\rho \Gamma, \Delta_\iota(d_{\langle\rangle})\,) \implies d \vdash_\rho \Gamma$

Coinductive Definition of $d \overset{co}{\underset{\rho}{\vdash}} \Gamma$ *(d is a derivation of Γ with cut-degree $\preceq \rho$)*
$d \overset{co}{\underset{\rho}{\vdash}} \Gamma \implies \deg(d_{\langle\rangle}) \prec \rho \;\&\; \Delta(d_{\langle\rangle}) \subseteq \Gamma \;\&\; \forall \iota \in |d_{\langle\rangle}|(d[\iota] \overset{co}{\underset{\rho}{\vdash}} \Gamma, \Delta_\iota(d_{\langle\rangle}))$.

In other words:

$d \vdash_\rho \Gamma \;:\Leftrightarrow\; \langle d, \Gamma \rangle \in \mu\Phi_\rho$, and $\quad d \overset{co}{\underset{\rho}{\vdash}} \Gamma \;:\Leftrightarrow\; \langle d, \Gamma \rangle \in \nu\Phi_\rho$,

where $\mu\Phi_\rho = \bigcap\{X \subseteq \mathbb{D}\times\mathsf{SEQ} : \Phi_\rho(X) \subseteq X\}$ is the *least*,

and $\nu\Phi_\rho = \bigcup\{X \subseteq \mathbb{D}\times\mathsf{SEQ} : X \subseteq \Phi_\rho(X)\}$ is the *greatest* fixed point

of the monotone operator $\Phi_\rho : \mathcal{P}(\mathbb{D}\times\mathsf{SEQ}) \to \mathcal{P}(\mathbb{D}\times\mathsf{SEQ})$,

$\Phi_\rho(X) := \{\langle d, \Gamma \rangle : \deg(d_{\langle\rangle}) \prec \rho \;\&\; \Delta(d_{\langle\rangle}) \subseteq \Gamma \;\&\; \forall \iota \in |d_{\langle\rangle}|(\, \langle d[\iota]\,;\; \Gamma, \Delta_\iota(d_{\langle\rangle})\rangle \in X\,)\}$.

Theorem 5.1. If $d \overset{co}{\underset{\rho}{\vdash}} \Gamma$ and $\sigma \preceq \rho$ then $\mathcal{E}\sigma d \overset{co}{\underset{\sigma}{\vdash}} \Gamma$.

Proof:
We have to find a set $\mathcal{X}_\sigma \subseteq \mathbb{D}\times\mathsf{SEQ}$ such that $\{\langle \mathcal{E}\sigma d, \Gamma \rangle : \exists \rho(d \overset{co}{\underset{\rho}{\vdash}} \Gamma \;\&\; \sigma \preceq \rho)\} \subseteq \mathcal{X}_\sigma \subseteq \Phi_\sigma(\mathcal{X}_\sigma)$.
Then we obtain $\{\langle \mathcal{E}\sigma d, \Gamma \rangle : \exists \rho(d \overset{co}{\underset{\rho}{\vdash}} \Gamma \;\&\; \sigma \preceq \rho)\} \subseteq \mathcal{X}_\sigma \subseteq \overset{co}{\underset{\sigma}{\vdash}}$, i.e. the claim.

Such a set is $\mathcal{X}_\sigma := \{\langle d, \Gamma \rangle : d \vdash^*_\sigma \Gamma\}$ where $\vdash^*_\sigma$ is defined below. Lemma 5.2 yields $\mathcal{X}_\sigma \subseteq \Phi_\sigma(\mathcal{X}_\sigma)$.

Inductive definition of $d \vdash^*_\sigma \Gamma$

$(\vdash^* 0)$ $d \vdash^{co}_\sigma \Gamma \implies d \vdash^*_\sigma \Gamma$;

$(\vdash^* 1)$ $d \vdash^*_\rho \Gamma \;\&\; \sigma \preceq \rho \implies \mathcal{E}\sigma d \vdash^*_\sigma \Gamma$;

$(\vdash^* 2)$ $\mathrm{rk}(B) \preceq \sigma \;\&\; d \vdash^*_\sigma \Gamma, B \;\&\; e \vdash^*_\sigma \Gamma, \neg B \implies \mathcal{R}Bde \vdash^*_\sigma \Gamma$;

$(\vdash^* 3)$ $d \vdash^*_\sigma \Gamma, A \implies \mathcal{I}A\kappa d \vdash^*_\sigma \Gamma, A[\kappa]$.

Lemma 5.2.

$c \vdash^*_\sigma \Gamma \;\Rightarrow\; \deg(c_{\langle\rangle}) \prec \sigma \;\&\; \Delta(c_{\langle\rangle}) \subseteq \Gamma \;\&\; \forall \iota \in |c_{\langle\rangle}|(c[\iota] \vdash^*_\sigma \Gamma, \Delta_\iota(c_{\langle\rangle}))$.

Proof by induction over $\vdash^*_\sigma$:

0. $c \vdash^{co}_\sigma \Gamma$: Then $\deg(c_{\langle\rangle}) \prec \sigma \;\&\; \Delta(c_{\langle\rangle}) \subseteq \Gamma \;\&\; \forall \iota \in |c_{\langle\rangle}|(c[\iota] \vdash^{co}_\sigma \Gamma, \Delta_\iota(c_{\langle\rangle}))$.
By $(\vdash^* 0)$ we obtain $\forall \iota \in |c_{\langle\rangle}|(c[\iota] \vdash^*_\sigma \Gamma, \Delta_\iota(c_{\langle\rangle}))$.

1. $c = \mathcal{E}\sigma d$ with $d \vdash^*_\rho \Gamma$ and $\sigma \preceq \rho$:
By IH $\deg(d_{\langle\rangle}) \prec \rho \;\&\; \Delta(d_{\langle\rangle}) \subseteq \Gamma \;\&\; \forall \iota \in |d_{\langle\rangle}|(d[\iota] \vdash^*_\rho \Gamma, \Delta_\iota(d_{\langle\rangle}))$.
From the definition of $\mathcal{E}_\sigma d$ it follows that $\deg(c_{\langle\rangle}) \prec \sigma$ and $\Delta(c_{\langle\rangle}) \subseteq \Delta(d_{\langle\rangle}) \subseteq \Gamma$.
1.1. $d_{\langle\rangle} = \mathsf{Cut}_B$ with $\sigma \preceq \mathrm{rk}(B) =: \nu$:
Then $c_{\langle\rangle} = \mathsf{Rep}$, $c[0] = \mathcal{E}\sigma(\mathcal{R}B(\mathcal{E}\nu d[0])(\mathcal{E}\nu d[1]))$ and $\nu = \mathrm{rk}(B) = \deg(d_{\langle\rangle}) \prec \rho$.
$d[0] \vdash^*_\rho \Gamma, B \;\&\; d[1] \vdash^*_\rho \Gamma, \neg B \overset{(\vdash^* 1) + \nu \prec \rho}{\Longrightarrow}$
$\mathcal{E}\nu d[0] \vdash^*_\nu \Gamma, B \;\&\; \mathcal{E}\nu d[1] \vdash^*_\nu \Gamma, \neg B \overset{(\vdash^* 2) + \mathrm{rk}(B) = \nu}{\Longrightarrow}$
$\mathcal{R}B(\mathcal{E}\nu d[0])(\mathcal{E}\nu d[1]) \vdash^*_\nu \Gamma \overset{(\vdash^* 1) + \sigma \preceq \nu}{\Longrightarrow}$
$c[0] \vdash^*_\sigma \Gamma$, i.e. $c[0] \vdash^*_\sigma \Gamma, \Delta_0(c_{\langle\rangle})$, since $\Delta_0(c_{\langle\rangle}) = \Delta_0(\mathsf{Rep}) = \emptyset$.

1.2. $d_{\langle\rangle} = \mathsf{Cut}_{\neg B}$: similar to 1.1.

1.3. otherwise: Then $c[\iota] = \mathcal{E}\sigma d[\iota]$ and from $d[\iota] \vdash^*_\rho \Gamma, \Delta_\iota(d_{\langle\rangle})$ we get $c[\iota] \vdash^*_\sigma \Gamma, \Delta_\iota(d_{\langle\rangle})$.

2. $c = \mathcal{R}Bde$ with $\mathrm{rk}(B) \preceq \sigma \;\&\; d \vdash^*_\sigma \Gamma, B \;\&\; e \vdash^*_\sigma \Gamma, \neg B$:
2.1. $d_{\langle\rangle} = (\kappa, B)$: From the definition of $\mathcal{R}Bde$ it follows that
$c_{\langle\rangle} = \mathsf{Cut}_{B[\kappa]}$, $\Delta(c_{\langle\rangle}) = \emptyset$, $c[0] = \mathcal{R}Bd[0]e$, and $c[1] = \mathcal{I}(\neg B)\kappa e$.
$d \vdash^*_\sigma \Gamma, B \overset{\mathrm{IH}}{\Rightarrow} d[0] \vdash^*_\sigma \Gamma, B, B[\kappa] \overset{(\vdash^* 2)}{\Rightarrow} \mathcal{R}Bd[0]e \vdash^*_\sigma \Gamma, B[\kappa]$, i.e. $c[0] \vdash^*_\sigma \Gamma, \Delta_0(c_{\langle\rangle})$.
$e \vdash^*_\sigma \Gamma, \neg B \overset{(\vdash^* 3)}{\Rightarrow} \mathcal{I}(\neg B)\kappa e \vdash^*_\sigma \Gamma, \neg B[\kappa]$, i.e. $c[1] \vdash^*_\sigma \Gamma, \Delta_1(c_{\langle\rangle})$.

2.2. otherwise: Then $c_{\langle\rangle} = d_{\langle\rangle}$, $c[\iota] = \mathcal{R}Bd[\iota]e$ and $B \notin \Delta(d_{\langle\rangle})$.
Hence $\deg(c_{\langle\rangle}) = \deg(d_{\langle\rangle}) \overset{\mathrm{IH}}{\prec} \sigma$ and $\Delta(c_{\langle\rangle}) = \Delta(d_{\langle\rangle}) \setminus \{B\} \overset{\mathrm{IH}}{\subseteq} \Gamma$.
$d \vdash^*_\sigma \Gamma, B \overset{\mathrm{IH}}{\Rightarrow} d[\iota] \vdash^*_\sigma \Gamma, B, \Delta_\iota(d_{\langle\rangle}) \overset{(\vdash^* 2)}{\Rightarrow} \mathcal{R}Bd[\iota]e \vdash^*_\sigma \Gamma, \Delta_\iota(d_{\langle\rangle})$, i.e. $c[\iota] \vdash^*_\sigma \Gamma, \Delta_\iota(c_{\langle\rangle})$.

3. $c = \mathcal{I}A\kappa d$ and $\Gamma = \Gamma_0, A[\kappa]$ with $d \vdash^*_\sigma \Gamma_0, A$:
3.1. $d_{\langle\rangle} = A$: Then $c_{\langle\rangle} = \mathsf{Rep}$ and $c[0] = \mathcal{I}A\kappa d[\kappa]$. Hence $\deg(c_{\langle\rangle}) = -1$ and $\Delta(c_{\langle\rangle}) = \emptyset$.
IH $\Rightarrow$ $d[\kappa] \vdash^*_\sigma \Gamma_0, A, A[\kappa] \overset{(\vdash^* 3)}{\Rightarrow} \mathcal{I}A\kappa d[\kappa] \vdash^*_\sigma \Gamma_0, A[\kappa]$, i.e. $c[0] \vdash^*_\sigma \Gamma, \Delta_0(c_{\langle\rangle})$.
3.2. otherwise: Then $c_{\langle\rangle} = d_{\langle\rangle}$, $c[\iota] = \mathcal{I}A\kappa d[\iota]$ and $A \notin \Delta(d_{\langle\rangle})$.
Hence $\deg(c_{\langle\rangle}) = \deg(d_{\langle\rangle}) \overset{\mathrm{IH}}{\prec} \sigma$ and $\Delta(c_{\langle\rangle}) = \Delta(d_{\langle\rangle}) \setminus \{A\} \overset{\mathrm{IH}}{\subseteq} \Gamma_0$.
As in 2.2. we obtain $c[\iota] \vdash^*_\sigma \Gamma, \Delta_\iota(c_{\langle\rangle})$.

Remark.

Let $\mathbb{W}$ *(the set of wellfounded derivations)* be inductively defined by: $\forall \iota \in |d_{\langle\rangle}|(\, d[\iota] \in \mathbb{W}\,) \implies d \in \mathbb{W}$.

Then the following holds: $d \vdash_\rho \Gamma \iff d \in \mathbb{W} \;\&\; d \vdash^{co}_\rho \Gamma$.

Proof left to the reader.

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