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Übungen zur Vorlesung "Lineare Algebra II" : Lösungen zu Blatt 6

Aufgabe 21

(i)⇒(iii): Vorlesung. (iii)⇒(i): (iii)⇒ $V \subseteq \text{span}(v_1, ..., v_m) \Rightarrow V$ Basis. (iii)⇒(ii): trivial.

(ii)⇒(iii): $\forall j(\langle v - \sum_i \langle v, v_i \rangle v_i, v_j \rangle = \langle v, v_j \rangle - \sum_i \langle v, v_i \rangle \langle v_i, v_j \rangle = \langle v, v_j \rangle - \langle v, v_j \rangle = 0) \overset{\text{(ii)}}{\Rightarrow} v - \sum_i \langle v, v_i \rangle v_i = 0$.

(iii)⇒(iv): $\langle v, w \rangle = \langle \sum_i \langle v, v_i \rangle v_i, \sum_j \langle w, v_j \rangle v_j \rangle = \sum_{i,j} \langle v, v_i \rangle \overline{\langle w, v_j \rangle} \langle v_i, v_j \rangle =$
$= \sum_{i,j} \langle v, v_i \rangle \langle v_j, w \rangle \delta_{ij} = \sum_i \langle v, v_i \rangle \langle v_i, w \rangle$.

(iv)⇒(v): $\|v\|^2 = \langle v, v \rangle \overset{\text{(iv)}}{=} \sum_i \langle v, v_i \rangle \langle v_i, v \rangle = \sum_i \langle v, v_i \rangle \overline{\langle v, v_i \rangle} = \sum_i |\langle v, v_i \rangle|^2$. (v)⇒(ii): trivial.

Aufgabe 22

(a) $\sigma(A, B) = \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} = \sum_{i,k} a_{ik} b_{ki}$.

σ symmetrisch: $\sum_{i,k} a_{ik} b_{ki} = \sum_{i,k} b_{ki} a_{ik}$.

σ bilinear: $\sigma((A + \lambda A'), B) = \sum\limits_{i,k} (a_{ik} + \lambda a'_{ik}) b_{ki} = \sum\limits_{i,k} a_{ik} b_{ki} + \sum\limits_{i,k} \lambda a'_{ik} b_{ki} = \sigma(A, B) + \lambda \sigma(A', B)$.

σ nicht-ausgeartet: Sei $E_{ij} = (\delta_{ki} \delta_{jl})_{k,l} \in \mathbb{R}^{n \times n}$. Dann $\sigma(A, E_{ij}) = \sum\limits_{k,l} a_{kl} \delta_{li} \delta_{jk} = a_{ji}$.

$\forall B(\sigma(A, B) = 0) \Rightarrow \forall i, j(\sigma(A, E_{i,j}) = 0) \Rightarrow \forall i, j(a_{ji} = 0) \Rightarrow A = 0$.

(b) (i) $A \in U_+ \setminus \{0\} \Rightarrow \sigma(A, A) = \sum_{i,j} a_{ij} a_{ji} = \sum_{i,j} a_{ij}^2 > 0$.
$A \in U_- \setminus \{0\} \Rightarrow \sigma(A, A) = \sum_{i,j} a_{ij} a_{ji} = -\sum_{i,j} a_{ij}^2 < 0$

(ii) 1. $V = U_+ + U_-$: $A = \frac{1}{2}((A + A^{\mathtt{t}}) + (A - A^{\mathtt{t}}))$ und
$(A + A^{\mathtt{t}})^{\mathtt{t}} = A^{\mathtt{t}} + (A^{\mathtt{t}})^{\mathtt{t}} = A^{\mathtt{t}} + A = A + A^{\mathtt{t}}$, $(A - A^{\mathtt{t}})^{\mathtt{t}} = A^{\mathtt{t}} - (A^{\mathtt{t}})^{\mathtt{t}} = A^{\mathtt{t}} - A = -(A - A^{\mathtt{t}})$.

2. $U_+ \cap U_- = \{0\}$: $A \in U_+$ & $A \in U_- \overset{\text{(i)}}{\Rightarrow} A = 0$.

(iii) 1. $U_+ \perp U_-$:
$A^{\mathtt{t}} = A$ & $B^{\mathtt{t}} = -B \Rightarrow \sigma(A, B) = \sum\limits_{i,j} a_{ij} b_{ji} = \sum\limits_{i,j} a_{ji}(-b_{ij}) = -\sum\limits_{i,j} a_{ji} b_{ij} = -\sigma(A, B) \Rightarrow \sigma(A, B) = 0$.

2. Wegen $U_+ \perp U_-$ gilt $U_+ \subseteq U_-^\perp$. Beweis von $U_-^\perp \subseteq U_+$: $A \in U_-^\perp \Rightarrow \forall B \in U_-(\sigma(A, B) = 0) \Rightarrow$
$\forall i, j(\sigma(A, E_{ij} - E_{ji}) = 0) \Rightarrow \forall i, j(\sigma(A, E_{ij}) = \sigma(A, E_{ji})) \Rightarrow \forall i, j(a_{ji} = a_{ij}) \Rightarrow A \in U_+$.

Aufgabe 23

(a) Sei $v_1 := b - a = (1, 2, 2)$ und $v_2 := c - a = (-2, -1, 2)$. Dann $\langle v_1, v_2 \rangle = -2 - 2 + 4 = 0$ und $v_1 \times v_2 = (6, -6, 3)$. Wegen letzterem setzen wir $v_3 := (2, -2, 1)$.

Ecken: $a = (1, 2, 3)$, $a + v_1$, $a + v_2$, $a + v_1 + v_2$, $a + v_3$, $a + v_1 + v_3$, $a + v_2 + v_3$, $a + v_1 + v_2 + v_3$.

(b) (i) $\langle v - w, v + w \rangle = \|v\|^2 - \|w\|^2 \Rightarrow (\|v\| = \|w\| \Leftrightarrow \langle v - w, v + w \rangle = 0)$.

(ii) $\|v - w\|^2 = \|v\|^2 - 2\langle v, w \rangle + \|w\|^2$ & $\|v + w\|^2 = \|v\|^2 + 2\langle v, w \rangle + \|w\|^2 \Longrightarrow$
$\big(\|v - w\| = \|v + w\| \Leftrightarrow -\langle v, w \rangle = \langle v, w \rangle \Leftrightarrow \langle v, w \rangle = 0\big)$.

Aufgabe 24

Bestimmung von Lös$(A;0)$

$A \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$. $x_1 = -x_3 - x_5$ & $x_2 = -x_4 - x_6$

Basis von Lös$(A;0)$: $v_1 := \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 := \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 := \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_4 := \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$,

$w_1 = \frac{1}{\sqrt{2}} v_1$, $w_2 = \frac{1}{\sqrt{2}} v_2$,

$w_3' = v_3 - \langle v_3, w_1 \rangle w_1 - \langle v_3, w_2 \rangle w_2 = v_3 - \frac{1}{2} v_1 - \frac{1}{2} \cdot 0 = v_3 - \frac{1}{2} v_1 = \frac{1}{2}(2v_3 - v_1) = \frac{1}{2}(-1, 0, -1, 0, 2, 0)$

$w_3 = \frac{1}{\sqrt{6}}(-1, 0, -1, 0, 2, 0)$

$w_4' = v_4 - \langle v_4, w_1 \rangle w_1 - \langle v_4, w_2 \rangle w_2 - \langle v_4, w_3 \rangle w_3 = v_4 - 0 \cdot w_1 - \frac{1}{\sqrt{2}} w_2 - 0 \cdot w_3 =$

$(0, -1, 0, 0, 0, 1) - \frac{1}{2}(0, -1, 0, 1, 0, 0) = (0, -\frac{1}{2}, 0, -\frac{1}{2}, 0, 1) = -\frac{1}{2}(0, 1, 0, 1, 0, 2)$

$w_4 = -\frac{1}{\sqrt{6}}(0, 1, 0, 1, 0, 2)$

$(w_1, ..., w_4)$ ist ONB von $\text{span}(v_1, ..., v_4) = \text{Lös}(A; 0)$.

Sei $f(e_i) := w_i$. Dann $\langle f(e_i), f(e_j) \rangle = \langle w_i, w_j \rangle = \delta_{ij} = \langle e_i, e_j \rangle \quad (i, j \in \{1, ..., 4\})$