THE TOPOLOGY OF SYMPLECTIC CIRCLE BUNDLES

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Abstract. We consider circle bundles over compact three-manifolds with symplectic total spaces. We show that the base of such a space must be irreducible or the product of the two-sphere with the circle. We then deduce that such a bundle admits a symplectic form if and only if it admits one that is invariant under the circle action in three special cases: namely if the base is Seifert fibered, has vanishing Thurston norm, or if the total space admits a Lefschetz fibration.

1. Introduction

A conjecture due to Taubes states that if a closed, compact 4-manifold of the form $M \times S^1$ is symplectic, then $M$ must fiber over $S^1$. A natural extension of this conjecture is to the case where $E \to M$ is a possibly nontrivial circle bundle. In [4] it was shown that an $S^1$-bundle admits an $S^1$-invariant symplectic form if and only if its base fibers over $S^1$ and the Euler class $e(E)$ of the total space pairs trivially with the fiber of some fibration. Thus based on the principle that an $S^1$-bundle should admit a symplectic form if and only if it admits an invariant one, one arrives at the following conjecture.

Conjecture 1 (Taubes). If a circle bundle $S^1 \to E \to M$ over a closed, oriented 3-manifold is symplectic, then there is a fibration $\Sigma \to M \to S^1$ such that $e(E)([\Sigma]) = 0$.

If an oriented 3-manifold fibers over $S^1$ with fiber $\Sigma \neq S^2$, then it follows by the long exact homotopy sequence that $M$ is in fact aspherical. So a necessary condition for Conjecture 1 to hold is that any $M$ that is the base of an $S^1$-bundle, whose total space carries a symplectic form, must in fact be aspherical or $S^2 \times S^1$ in the case $\Sigma = S^2$. This observation provides the motivation for the following theorem, which is the main result of the first part of this paper.

Theorem 2. Let $M$ be an oriented, closed 3-manifold, so that some circle bundle $S^1 \to E \to M$ admits a symplectic structure, then, either $M$ is diffeomorphic to $S^2 \times S^1$ and the bundle is trivial, or $M$ is irreducible and aspherical.

A similar statement was proved by McCarthy in [20] for the case $E = M \times S^1$. More precisely, McCarthy showed that if $M \times S^1$ admits a symplectic structure then $M$ decomposes as a connected sum $M = A \# B$ where the first Betti number $b_1(A) \geq 1$ and $B$ has no nontrivial connected covering spaces. This can be refined quite substantially following Perelman’s proof of Thurston’s geometrisation conjecture (see [22], [23] or [21]). For one corollary of geometrisation is that the fundamental group of a closed 3-manifold is residually finite (see [11]), meaning that the $B$ in McCarthy’s theorem must have trivial fundamental group, and hence by the Poincaré Conjecture is diffeomorphic to $S^3$. Thus in fact $M$ must be prime and hence irreducible and aspherical or $S^2 \times S^1$. Theorem 2 is then a generalisation of this more refined statement to the case of nontrivial $S^1$-bundles. Our argument will rely...
on a vanishing result of Kronheimer-Mrowka for the Seiberg-Witten invariants of a manifold that splits into two pieces along a copy of $S^2 \times S^1$, which in itself is of independent interest (cf. Proposition 1). One may also prove Theorem 2 by following the argument of [20], see Remark 1 below.

In the remainder of this paper we will show that Conjecture 1 holds in various special cases. Firstly we will verify the conjecture under certain additional assumptions on the topology of the base manifold $M$. In order to be able to do this we will need to understand when a manifold fibers over $S^1$. One gains significant insight into this problem by considering the Thurston norm $||| \cdot |||_T$ on $H^1(M, \mathbb{R})$, which was introduced by Thurston in [27]. The Thurston norm enables one to see which integral classes $\alpha \in H^1(M, \mathbb{Z})$ can be represented by closed, nonvanishing 1-forms, which in turn induce fibrations of $M$ by compact surfaces.

In [5] Friedl and Vidussi showed that if $E = M \times S^1$ admits a symplectic form and $||| \cdot |||_T \equiv 0$ or $M$ is Seifert fibered, then $M$ must fiber over $S^1$. In Corollary 2 below we will show that in fact Conjecture 1 holds in these two cases. The argument will be based on understanding the Seiberg-Witten invariants of the total space $E$ given that $M$ has vanishing Thurston norm and the Seifert case will be deduced as a corollary of this. Indeed, if $M$ has vanishing Thurston norm and $S^1 \to E \to M$ is symplectic, then the canonical class of $E$ must be trivial. This combined with the restrictions on Seiberg-Witten basic classes of a symplectic manifold as proved by Taubes in [26] means that $K = 0$ is the only Seiberg-Witten basic class and the result then follows by an application of a vanishing result of Lescop (cf. [17] or [28]).

Another special case of the Taubes conjecture is when the total space $E$ admits a Lefschetz fibration, as was considered in [2] and [3] for a trivial bundle. In view of Corollary 2 we will be able to give a comparatively simple proof of the following result.

**Theorem 9.** Let $S^1 \to E \to M$ be a symplectic circle bundle over an irreducible base $M$. If $E$ admits a Lefschetz fibration, then $M$ fibers over $S^1$.

It then follows by considering the Kodaira classification of complex surfaces that Conjecture 1 holds under the assumption that the total space admits a complex structure.

Outline of paper. In Section 2 we will state the relevant vanishing result of Kronheimer-Mrowka in order to prove Theorem 2. In Section 3 we recall the definition of the Thurston norm and quote some well known facts about it. In Section 4 we will use our knowledge of the Thurston norm to verify Conjecture 1 under the assumption that the base is Seifert fibered or has vanishing Thurston norm. Finally in Section 5 we will define Lefschetz fibrations and prove that the conjecture is true when one has a Lefschetz fibration on the total space $E$.

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2. Asphericity of the base $M$

Throughout this article all manifolds will be closed, connected and oriented and $M$ will always denote a manifold of dimension 3. In addition we will make the convention that all (co)homology groups will be taken with integral coefficients unless otherwise stated.

In [20] it was shown that if $M \times S^1$ is symplectic, then $M$ must be irreducible and aspherical or $S^2 \times S^1$. We extend this to the case of a nontrivial $S^1$-bundle. We first collect some relevant lemmas.
**Lemma 1.** Let $M = M_1 \# M_2$ be a nontrivial connect sum decomposition with $b_1(M) \geq 1$, then there is a finite covering $N$ of $M$ that decomposes as a direct sum $N = N_1 \# N_2$ where $b_1(N) \geq k$ for any given $k$.

**Proof.** It follows from Mayer-Vietoris that the Betti numbers are additive for a connect sum, hence by assumption we may assume that $b_1(M_1) \geq 1$. By the proof of geometrisation it follows that the fundamental group of a 3-manifold is residually finite (cf. [11]) and hence $M_2$ has a nontrivial $d$-fold cover $\tilde{M}_2$, with $d \geq 2$. By removing a ball from $M_2$ and its disjoint lifts from $\tilde{M}_2$ and then gluing in $d$ copies of $M_1$ we obtain a cover $\tilde{M}$ of $M = M_1 \# M_2$, and by construction $\tilde{M}$ has a connect sum decomposition as $\tilde{M} = \tilde{M}_1 \# P$, where $b_1(P) \geq 1$. We may now take a $k$-fold cover associated to some surjective homomorphism of $\pi_1(M_1) \to \mathbb{Z}_k$ and glue in copies of $P$ to get a cover of $\tilde{M}$ (and hence of $M$), which decomposes in two pieces one of which has first Betti number at least $k$. One more application of this procedure gives the desired result. \hfill \Box

**Lemma 2.** Let $S^1 \to E \xrightarrow{\pi} M$ be a circle bundle, whose Euler class we denote by $e(E) \in H^2(M)$, then

1. $b_2(E) = \begin{cases} 2b_1(M) - 2, & \text{if } e(E) \text{ is not torsion} \\ 2b_1(M), & \text{if } e(E) \text{ is torsion.} \end{cases}$

2. $b_2^+(E) = b_2^-(E) \geq b_1(M) - 1$.

**Proof.** We consider the Gysin sequence

$$H^0(M) \xrightarrow{\pi^*} H^2(M) \xrightarrow{\pi^*} H^2(E) \xrightarrow{\pi_*} H^1(M) \xrightarrow{\pi^*} H^3(M),$$

where $e \in H^2(M)$ denotes the Euler class of the bundle. By Poincaré duality $H^0(M) = H^3(M) = \mathbb{Z}$ and $b_1(M) = b_2(M)$, so we conclude by exactness that $b_2(E) = 2b_1(M) - 2$ if $e$ is not torsion and $b_2(E) = 2b_1(M)$ if $e$ is torsion. Furthermore since $E$ bounds its associated disc bundle, it has zero signature and hence

$$b_2^+(E) = b_2^-(E) \geq b_1(M) - 1.$$

\hfill \Box

We will need to appeal to a vanishing result for the Seiberg-Witten invariants of manifolds that decompose along $S^2 \times S^1$, which we take from [16]. For this we will need to define a relative notion of $b_2^\pm$ for an oriented 4-manifold $X$ with boundary. This is done by considering the symmetric form induced on rational cohomology that is obtained as the composition

$$H^2(X, \partial X) \times H^2(X, \partial X) \xrightarrow{i^* \times \text{Id}} H^2(X) \times H^2(X, \partial X) \xrightarrow{\cup} \mathbb{Q}.$$  

Here the map $i^*$ is the map coming from the long exact sequence of the pair $(X, \partial X)$ and the second map is nondegenerate by Poincaré duality. This is then a symmetric, possibly degenerate, form on $H^2(X, \partial X)$ and we define $b_2^\pm(X)$ to be the dimension of a maximal positive definite subspace.

**Theorem 1** (Kronheimer-Mrowka, [16]). Let $X = X_1 \cup_{\partial X_1 = \partial X_2} X_2$ where $\partial X_1 = -\partial X_2 = S^2 \times S^1$ and $b_2^\pm(X_1), b_2^\pm(X_2) \geq 1$. Then for all Spin$^c$-structures $\xi$

$$\sum_{\xi^* - \xi \in \text{Tor}} SW(\xi^*) = 0.$$
Although it is not explicitly stated in the book [16], Theorem 1 can be deduced as follows: formula 3.27 (p. 73) allows one to compute the sum of the SW invariants of all $\text{Spin}^c$-structures that differ by torsion as a pairing of certain Floer groups. However these groups are zero for $S^2 \times S^1$ by Proposition 3.10.3 in the case of an untwisted coefficient system and by Proposition 3.10.4 in the twisted case and thus this sum must vanish.

Theorem 1 then implies certain restrictions on the decomposition of symplectic manifolds along a copy of $S^2 \times S^1$.

**Proposition 1.** A symplectic manifold $X$ cannot be decomposed as $X = X_1 \cup_{\partial X_1 = \partial X_2} X_2$, where $\partial X_1 = -\partial X_2 = S^2 \times S^1$ and $b^+_2 (X_1), b^+_2 (X_2) \geq 1$.

**Proof.** By the hypotheses of the proposition, we conclude from Theorem 1 that for every $\text{Spin}^c$-structure $\xi \in \text{Spin}^c(X)$

$$\sum_{\xi^* - \xi \in \text{Tor}} \text{SW} (\xi^*) = 0.$$ 

However as $X$ is symplectic and

$$b^+_2 (X) \geq b^+_2 (X_1) + b^+_2 (X_2) \geq 2$$

the nonvanishing result of Taubes implies $\text{SW} (\xi_{\text{can}}) = \pm 1$, where $\xi_{\text{can}}$ denotes the canonical $\text{Spin}^c$-structure associated to the symplectic structure on $E$ (cf. [25]). Moreover it follows from the constraints on SW basic classes of a symplectic manifold of [26] that if $\xi^*$ is another $\text{Spin}^c$-structure with nontrivial SW invariant and $\xi_{\text{can}} - \xi^* \in \text{Tor}$ then in fact $\xi_{\text{can}} = \xi^*$. Hence

$$\sum_{\xi^* - \xi_{\text{can}} \in \text{Tor}} \text{SW} (\xi^*) = \pm 1$$

which is a contradiction. \qed

**Theorem 2.** Let $M$ be an oriented, closed 3-manifold, so that some circle bundle $S^1 \to E \xrightarrow{\pi} M$ admits a symplectic structure, then $M$ is irreducible and aspherical or $M = S^2 \times S^1$ and the bundle is trivial.

**Proof.** We first show that $M$ must be prime. Since $E$ is symplectic it follows from Lemma 2 that $b_1 (M) \geq 1$. Assume that $M = M_1 \# M_2$ is a nontrivial connected sum, then by taking a suitable covering as in Lemma 1 and pulling back $E$ and its symplectic form we may assume without loss of generality that $b_1 (M_i) \geq 2$. We let $S$ denote the gluing sphere of the connected sum, then as $S$ is nullhomologous the bundle restricted to $S$ is trivial. Thus the connect sum decomposition induces a decomposition $E = E_1 \cup_{S^2 \times S^1} E_2$. Since the bundles $E_i \to M_i \setminus B^3$ are trivial on the boundary we may extend them to bundles $\tilde{E}_i \to M_i$ and as $b_1 (M_i) \geq 2$, Lemma 2 implies that $b^+_2 (\tilde{E}_i) \geq 1$. Further, since $E_i \simeq \tilde{E}_i \setminus (S^1 \times pt)$ we have that

$$b^+_2 (E_i) \geq b^+_2 (\tilde{E}_i) \geq 1,$$

which then contradicts Proposition 1. Hence $M$ is prime, and thus irreducible or $S^2 \times S^1$.

We assume that $M$ is irreducible, then by the sphere theorem $\pi_2 (M) = 0$. Since $b_1 (M) \geq 1$, we have that $\pi_1 (M)$ is infinite so the universal cover $\tilde{M}$ of $M$ is not compact and has $\pi_i (\tilde{M})$ trivial for $i = 1, 2$. The Hurewicz theorem then implies that the first nontrivial $\pi_i (\tilde{M})$ is isomorphic to $H_i (M)$. But since $\tilde{M}$ is not compact $H_3 (M) = 0$ and as $M$ is 3-dimensional
$H_i(\tilde{M}) = 0$ for all $i \geq 4$. Hence $\pi_i(\tilde{M}) = 0$ for all $i \geq 4$ and it follows from Whitehead’s Theorem that $\tilde{M}$ is contractible, that is $M$ is aspherical.

In the case where $M = S^2 \times S^1$ any symplectic bundle must be trivial by Lemma 2. □

**Remark 1.** One may also give a proof of Theorem 2 that uses the covering construction of [20]. In order to do this one first takes finite coverings on each of the two pieces in the connect sum decomposition. Then one glues these together to find a covering $\tilde{M}$ where the sphere of the connect sum lifts to a sphere that is nontrivial in real cohomology. This sphere then lifts to the total space of the pullback bundle $\tilde{E}$ over $\tilde{M}$. One may also assume by Lemma 1 that $b_1(\tilde{M})$ is large and hence $b_1^2(\tilde{E})$ is large. Then a standard vanishing theorem for the SW invariants (cf. [13]) implies that all invariants are zero, which then contradicts Taubes’ result if $E$ and hence $\tilde{E}$ is symplectic.

By considering the long exact homotopy sequence we have the following corollary that was first proved by Kotschick in [14].

**Corollary 1.** Let $S^1 \to E \to M$ be a symplectic circle bundle over an oriented 3-manifold $M$. Then the map $\pi_1(S^1) \to \pi_1(E)$ induced by the inclusion of the fiber is injective. In particular a fixed point free circle action on a symplectic 4-manifold can never have contractible orbits.

3. **The Thurston norm**

In this section we will define and collect several relevant facts about the Thurston norm. We first define the negative Euler characteristic or *complexity* of a possibly disconnected, orientable surface $\Sigma = \bigsqcup_i \Sigma_i$ to be

$$\chi_-(\Sigma) = \sum_{\chi^2(\Sigma_i) \leq 0} -\chi(\Sigma_i)$$

where $\chi$ denotes the Euler characteristic of the surface.

Next we define the Thurston norm $\| \|_T$ as a map on $H_1(M)$ by

$$\|\sigma\|_T = \min \{ \chi_-(\Sigma) \mid PD(\Sigma) = \sigma \}.$$  

It is a basic fact that this map extends uniquely to a (semi)norm on $H_1(M, \mathbb{R})$, which we will denote again by $\| \|_T$. One particularly important property of the Thurston norm is that its unit ball, which we denote by $B_T$, is a (possibly noncompact) convex polytope with finitely many faces. If $B^*_T$ denotes the unit ball in the dual space we have the following characterisation of $B_T$.

**Theorem 3** ([27], p. 106). The unit ball $B^*_T$ is a polyhedron whose vertices are integral lattice points, $\pm \beta_1, ..., \pm \beta_k$ and the unit ball $B_T$ is defined by the following inequalities

$$B_T = \{ \alpha \mid \|\beta_i(\alpha)\| \leq 1, \quad 1 \leq i \leq k \}.$$  

We are interested in understanding how a manifold fibers over $S^1$ and the following theorem says that the Thurston norm determines precisely which cohomology classes can be represented by fibrations.

**Theorem 4** ([27], p. 120). Let $M$ be a compact, oriented 3-manifold. The set $F$ of cohomology classes in $H^1(M, \mathbb{R})$ representable by nonsingular closed 1-forms is the union of the open cones on certain top-dimensional open faces of $B_T$, minus the origin. The set of elements in $H^1(M, \mathbb{Z})$ whose Poincaré dual is represented by the fiber of some fibration consists of the set of lattice points in $F$. 

We call a top-dimensional face of the unit ball $B_T$ fibered, if some integral class, and hence all, in the cone over its interior can be represented by a fibration. One also understands how the Thurston norm behaves under finite covers by the following result of Gabai.

**Theorem 5** ([7], Cor. 6.13). Let $\tilde{M} \to M$ be a finite connected $d$-sheeted covering then for $\sigma \in H^1(M, \mathbb{R})$ we have

$||\sigma||_T = \frac{1}{d} ||p^*\sigma||_T$.

These facts then allow us to completely characterise the Thurston norm of an irreducible Seifert fibered manifold.

**Proposition 2.** If $M$ is irreducible and Seifert fibered, then either the Thurston norm of $M$ vanishes identically or $M$ fibers over $S^1$ and

$||\sigma||_T = m.||\sigma(\gamma)||$

for some class $\gamma \in H_1(M)$.

**Proof.** Since $M$ is irreducible and Seifert fibered either $M$ has a horizontal surface, i.e. a closed surface transverse to all fibers, or every surface is isotopic to a vertical surface, i.e. a surface that is a union of fibers (cf. [9] Prop 1.11) and is hence a union of tori so the Thurston norm is identically zero. If $M$ has a horizontal surface $F$, which we may assume to be connected, then $M$ is a mapping torus with monodromy $\phi \in Diff^+(F)$ so that $\phi^n = Id$ for some $n$. This means that $M$ is covered by $\tilde{M} = F \times S^1$. If $\tilde{\gamma} = pt \times S^1$, then the Thurston norm of $\tilde{M}$ is given by

$||\sigma||_T = \chi_-(F)||\sigma(\tilde{\gamma})||$.

We let $\gamma = p_*(\tilde{\gamma})$, then by Theorem 5 the norm on $M$ is given by

$||\sigma||_T = \frac{1}{n}||p^*\sigma||_T = \frac{\chi_-(F)}{n} ||p^*\sigma(\tilde{\gamma})|| = \frac{\chi_-(F)}{n} ||\sigma(\gamma)|| = m.||\sigma(\gamma)||$.

□

**Example 1** (Seifert fibered spaces with horizontal surfaces). We note that in the second case of Proposition 2 the Thurston ball $B_T$ consists of two (noncompact) faces that are both fibered and that the Thurston norm is identically zero on a codimension one subspace $K$. Thus by [4] any bundle over such an $M$ will admit an $S^1$-invariant symplectic form except possibly in the case where the Euler class $e(E)$ pairs trivially with all elements in $K$, that is $e(E)$ pairs trivially with all tori in $M$. By taking the pullback bundle of the cover $\tilde{M} = F \times S^1 \to M$ we may assume that we have a bundle $E$ over $F \times S^1$ that is symplectic and has Euler class that again pairs trivially with embedded tori in $\tilde{M}$ and is thus a nonzero multiple of $PD(\tilde{\gamma})$. This in turn has a covering $\tilde{E}$ that is an $S^1$-bundle with Euler class equal to $PD(\tilde{\gamma})$. Now if we let $T = \tilde{\gamma} \times S^1$ and $X = \tilde{M} \times S^1$ then the SW polynomial of $X$ can be computed to be

$SW_X = (t_T - t_T^{-1})^{2g-2}$

where $g$ is the genus of $F$. Then by the formula of Baldridge in [1] it follows that all the SW invariants of $\tilde{E}$ are zero, contradicting Taubes’ nonvanishing result for the SW invariants of a symplectic manifold. So in fact Conjecture 1 holds for Seifert fibered spaces that have horizontal surfaces.
4. The case of vanishing Thurston norm

In [5] Friedl and Vidussi showed that if $E = M \times S^1$ admits a symplectic form and $|||_T \equiv 0$ or $M$ is Seifert fibered, then $M$ must fiber over $S^1$. In this section we shall extend this to the case of a nontrivial $S^1$-bundle and then show that Conjecture 1 holds in both of these cases. From now on we shall assume that $M$ is irreducible, which in view of Theorem 2 only excludes the case where $M = S^2 \times S^1$ and the bundle is trivial. Our argument will be based on that of [5] and we begin with the following lemma.

**Lemma 3.** If $S^1 \to E \xrightarrow{\pi} M$ is a bundle over an $M$ that has vanishing Thurston norm, then

$$H^2(E)/\text{Tor} = V \oplus W$$

where $V, W$ are isotropic subspaces that admit a basis of embedded tori.

**Proof.** We consider the Gysin sequence

$$\mathbb{Z} \to H^2(M) \xrightarrow{\pi_*} H^2(E) \xrightarrow{\pi_*} H^1(M) \to \mathbb{Z}.$$ 

Here $s$ is a section defined on the image of $\pi_*$ as follows: we represent an element of $\sigma \in H^1(M)$ by an embedded surface $\Sigma$. By exactness, $\sigma$ will be in $\text{Im}(\pi_*)$ precisely when the bundle is trivial on $\Sigma$ and in this case we may lift $\Sigma$ to some $\tilde{\Sigma}$ in $E$. As $H^1(M)$ is free, we define $s$ on a $\mathbb{Z}$-basis $\{\sigma_i\}$ by $s(\sigma_i) = \tilde{\Sigma}_i$. We set $V = \pi^*(H^2(M))$ and $W = s(H^1(M))$, then $V$ is clearly spanned by embedded tori and the statement for $W$ is precisely the assumption on the Thurston norm. \qed

**Proposition 3.** Let $S^1 \to E \xrightarrow{\pi} M$ be an $S^1$-bundle with torsion Euler class $e(E)$, then there is a finite cover $\tilde{M} \xrightarrow{\tilde{\pi}} M$ such that the pullback bundle $p^*E \to \tilde{M}$ is trivial.

**Proof.** We choose a splitting of $H_1(M) = F \oplus T$ where $T$ is the torsion subgroup and $F$ is any free complement. We take the cover $\tilde{M} \xrightarrow{\tilde{\pi}} M$ associated to the kernel of the composition

$$\pi_1(M) \to H_1(M) \xrightarrow{\phi} T,$$

where $\phi$ is the projection with kernel $F$. Note that the composition $H_1(\tilde{M}) \xrightarrow{p_*} H_1(M) \xrightarrow{\phi} T$ is zero. Then by the Universal Coefficient Theorem we have the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & \text{Ext}(H_1(\tilde{M}), \mathbb{Z}) & \to & H^2(\tilde{M}) & \to & \text{Hom}(H_2(\tilde{M}), \mathbb{Z}) & \to & 0 \\
\uparrow{(p_*)^*} & & \uparrow{p^*} & & \uparrow{(p_*)^*} & & \uparrow{(p_*)^*} & & \\
0 & \to & \text{Ext}(H_1(M), \mathbb{Z}) & \to & H^2(M) & \to & \text{Hom}(H_2(M), \mathbb{Z}) & \to & 0. \\
\end{array}
\]

This implies that $p^*$ is zero on torsion in $H^2(M)$ so the pullback bundle is indeed trivial. \qed

**Theorem 6.** If $S^1 \to E \xrightarrow{\pi} M$ is a symplectic circle bundle over an irreducible manifold for which $|||_T$ is identically zero, then $M$ fibers over $S^1$.

After this paper had been submitted I was informed that Friedl and Vidussi have independently proved this result (cf. [6]).
Proof. Since $E$ is symplectic it has an associated canonical $\text{Spin}^c$-structure $\xi_{\text{can}}$ and canonical class that we denote by $K$. We claim that our assumption on the Thurston norm of the base implies that $K$ must be torsion. For by Taubes’ nonvanishing result $\xi_{\text{can}}$ has nontrivial SW invariant. If $\alpha \in H^2(E)$, the adjunction inequality (see [15]) and Lemma 3 imply that

$$|\alpha.K| = 0.$$  

This also holds in the case $b_2^+(E) = 1$ (cf. [18] Theorem E, [19] Theorem B). As $M$ is irreducible and $b_2(M) \geq 1$ the assumption on the vanishing of the Thurston norm implies that $M$ contains an embedded, incompressible torus $T \hookrightarrow M$. Then by Proposition 7 of [12] either $T$ is the fiber of some fibration or there is a finite cover $\bar{M} \rightarrow M$ with large $b_1$, say $b_1(\bar{M}) \geq 4$. We assume that the latter holds. Then the pullback $\bar{E} = p^*E$ will be symplectic with canonical class $\bar{K} = p^*K$, symplectic form $\bar{\omega} = p^*\omega$ and $b_2^+(\bar{E}) \geq 2$. Then for any $\text{Spin}^c$-structure $\xi_{\text{can}} \otimes F$ that has nontrivial SW invariant we have by [26]

$$0 \leq F.\bar{\omega} \leq \bar{K}.\bar{\omega}.\]$$

Moreover, since $\bar{K}$ is torsion and equality on the left implies $F = 0$, we conclude that in fact $\bar{K} = 0$. Thus $\bar{K} = 0$, so $\xi_{\text{can}}$ is trivial and this is the only $\text{Spin}^c$-structure with nonzero SW invariant. We now need to consider two cases. We first assume that $e(E)$ and hence $e(\bar{E})$ is nontorsion. In this case we compute

$$\pm 1 = \sum_{\xi_\ast \in \text{Spin}^c(E)} SW_\bar{E}^4(\xi_\ast) = \sum_{\xi_\ast \in \text{Spin}^c(E)} \sum_{\xi \equiv \xi_\ast \mod \bar{e}} SW_\bar{M}^3(\xi) = \sum_{\xi \in \text{Spin}^c(E)} SW_\bar{M}^3(\xi),$$

where the second inequality follows from Theorem 1 in [1]. However as $b_1(\bar{M}) \geq 4$ this sum is zero (cf. [28] p.114) a contradiction. If the Euler class is torsion we may assume by Proposition 3 that it is indeed zero and the above calculation reduces to

$$\pm 1 = \sum_{\xi \in \text{Spin}^c(E)} SW_\bar{E}^4(\xi) = \sum_{\xi \in \text{Spin}^c(E)} SW_\bar{M}^3(\xi) = 0.$$

In either case we obtain a contradiction and hence $M$ must fiber over $S^1$. □

As a consequence of this theorem we conclude that Conjecture 1 holds if $M$ has vanishing Thurston norm or is Seifert fibered.

Corollary 2. Conjecture 1 holds if $M$ is Seifert fibered or $|||T \equiv 0$.

Proof. If $M$ has vanishing Thurston norm, then by Theorem 4 we conclude that if one class in $H^1(M)$ can be represented by a fibration then so can all classes and by the construction of [4] every bundle over $M$ admits an $S^1$-invariant symplectic form. If $M$ is Seifert fibered it either has vanishing Thurston norm by Proposition 2 and we proceed as in the previous case or $M$ has a horizontal surface and the claim follows by Example 1 above. □

5. The case where $E$ admits a Lefschetz fibration

In [2] Chen and Matveyev showed that if $S^1 \times M$ admits a symplectic Lefschetz fibration then $M$ fibers over $S^1$. This was extended by Etgü in [3] to the case where the fibration may or may not be symplectic. In this section we shall show that the same statement holds for arbitrary $S^1$-bundles. Let us begin with some definitions and basic facts concerning Lefschetz fibrations.
Definition 7. Let $E$ be a compact, connected, oriented smooth 4-manifold, a Lefschetz fibration is a map $E \xrightarrow{p} B$ to an orientable surface so that any critical point has an oriented chart on which $p(z_1, z_2) = z_1^2 + z_2^2$.

We list some basic properties of Lefschetz fibrations (for proofs see [8]).

1. There are finitely many critical points, so the generic preimage of a point will be a surface and we may assume that this is connected. To each critical point one associates a vanishing cycle in the fiber.
2. A Lefschetz fibration admits a symplectic form so that the fiber is a symplectic submanifold if the class $[F]$ of the fiber is nontorsion in $H_2(E)$. Moreover this is always true if $\chi(F) \neq 0$.
3. We have a formula for the Euler characteristic given by
   
   $$\chi(E) = \chi(B) \cdot \chi(F) + \#\{\text{critical points}\}.$$  

We will first show that for a symplectic circle bundle any Lefschetz fibration will actually be a proper fibration, i.e. cannot have any critical points. The following lemma is essentially Lemma 3.4 of [2].

Lemma 4. Let $S^1 \to E \xrightarrow{\pi} M$ be a circle bundle that admits a Lefschetz fibration $E \xrightarrow{p} B$, then $p$ has no critical points.

Proof. We first consider the case where $F = S^2$. Since $E$ is spin, it has an even intersection form and thus all vanishing cycles are nonseparating in the fiber $F = S^2$. However this means there cannot be any since $S^2$ is simply connected and hence $p$ has no critical points.

If $F = T^2$, the equation

$$0 = \chi(E) = \chi(B) \cdot \chi(F) + \#\{\text{critical points}\}.$$  

implies that $E$ has no critical points.

We now consider the case when $F$ has genus greater than 1. We know that $E$ admits a symplectic Lefschetz fibration by (2) above. Thus by the adjunction formula for symplectic surfaces we see that

$$K.F = \chi_-(F) \neq 0$$  

where $K$ is the canonical class on $E$. If $b_2^+ > 1$ then it follows from Taubes’ result that $K$ is a basic class and thus the adjunction inequality holds. In the case where $b_2^+(E) = 1$ we may apply the adjunction inequality exactly as in the case of $b_2^+ > 1$ by ([18] Theorem E). Now we assume that our fibration has a critical point and hence a vanishing cycle $\gamma$, then we know that this is nonseparating so the fiber $F$ is homologous to a surface obtained by collapsing $\gamma$ to a point and this can in turn be thought of as the image of a map $F' \xrightarrow{\pi} E$ where $\chi_-(F') < \chi_-(F)$. Hence the image $\pi_*[F]$ may be represented by a surface of complexity at most $\chi_-(F')$ (see [7]). We know that any basic class of a circle bundle is a pullback of a class on the base (see [1]) thus by the adjunction inequality (which still holds for $b_2^+ = 1$) and equation (1)

$$\chi_-(F) = |K |F | = |K \cdot \pi_*F | \leq \|\pi_*F ||_T \leq \chi_-(F') < \chi_-(F)$$  

which is a contradiction.  

Our proof of Theorem 9 below, which differs from those of [2] and [3], will rely on a theorem of Stallings that characterises fibered 3-manifolds in terms of their fundamental group.
Theorem 9. Let $G$ where

\[ 1 \to G \to \pi_1(M) \to \mathbb{Z} \to 1 \]

is finitely generated and $G \neq \mathbb{Z}_2$, then $M$ fibers over $S^1$.

We now come to the main result of this section.

Theorem 9. Let $S^1 \to E \xrightarrow{p} M$ be a symplectic circle bundle over an irreducible base $M$. If $E$ admits a Lefschetz fibration, then $M$ fibers over $S^1$.

Proof. First of all by Lemma 4 we have that $E$ actually admits a fibration $F \to E \xrightarrow{p} B$. In addition we note that the fiber $\gamma$ of any oriented circle bundle lies in the centre of the fundamental group of the total space. We shall have to consider two distinct cases according to whether $\gamma$ is in the kernel of $p_*$ or not.

**Case 1:** $p_*(\gamma) \neq 1$.

Since $\gamma$ was central in the fundamental group of $E$ the fact that $p_*(\gamma)$ is nontrivial in $\pi_1(B)$ means that $B$ must be a torus. Hence the long exact homotopy sequence of the fibration gives the following short exact sequence

\[ 1 \to \pi_1(F) \to \pi_1(E) \xrightarrow{p_*} \pi_1(T^2) = \mathbb{Z}^2 \to 1. \tag{2} \]

Since $M$ is assumed to be irreducible and hence aspherical we also have the following exact sequence from the homotopy exact sequence of the fibration $S^1 \to E \xrightarrow{p} M$:

\[ 1 \to \pi_1(S^1) = \langle \gamma \rangle \to \pi_1(E) \xrightarrow{p_*} \pi_1(M) \to 1. \tag{3} \]

Because $\gamma$ is central in $\pi_1(E)$, the sequence (2) gives the following exact sequence

\[ 1 \to \pi_1(F) \xrightarrow{i_*} \pi_1(E)/\langle \gamma \rangle \xrightarrow{p_*} \mathbb{Z}^2/\langle p_*(\gamma) \rangle \to 1. \]

Moreover since $p_*(\gamma) \neq 1$ we have that $\mathbb{Z}^2/\langle p_*(\gamma) \rangle = \mathbb{Z} \oplus \mathbb{Z}_k$ for some $k$. If we let $H = p_*^{-1}(\mathbb{Z}_k)$ we see that $H$ has $\pi_1(F)$ as a finite index subgroup and is thus also finitely generated. Then by taking the projection to $\mathbb{Z}$ in the above sequence we obtain

\[ 1 \to H \to \pi_1(E)/\langle \gamma \rangle = \pi_1(M) \xrightarrow{p_*} \mathbb{Z} \to 1. \]

This is exact and $H \neq \mathbb{Z}_2$ since it contains $\langle \gamma \rangle$. As $M$ is irreducible, the hypotheses of Theorem 8 are satisfied and we conclude that $M$ fibers over $S^1$.

**Case 2:** $p_*(\gamma) = 1$.

In this case $\langle \gamma \rangle \subset \pi_1(F)$ and hence $F = T^2$. Thus sequence (2) above yields the following

\[ 1 \to \mathbb{Z} \to \pi_1(E) \xrightarrow{p_*} \pi_1(B) \to 1 \]

and $\langle \gamma \rangle \subset \mathbb{Z}^2$. Again by taking the quotient by $\langle \gamma \rangle$ we obtain the following short exact sequence

\[ 1 \to \mathbb{Z} \oplus \mathbb{Z}_k = \mathbb{Z}^2/\langle \gamma \rangle \to \pi_1(E)/\langle \gamma \rangle = \pi_1(M) \xrightarrow{p_*} \pi_1(B) \to 1. \]

However since $M$ is irreducible and hence prime and $\pi_1(M)$ is infinite it follows from ([10], Corollary 9.9) that $\pi_1(M)$ is torsion free. Hence $k = 0$ and $\pi_1(M)$ contains an infinite cyclic
normal subgroup, thus by ([10], Corollary 12.8) it is in fact Seifert fibered and the result follows from Corollary 2 above.

Theorem 9 then allows us to prove Conjecture 1 under the assumption that the total space is a complex manifold.

Corollary 3. Conjecture 1 holds in the case that $E$ is a complex manifold.

Proof. By considering the Kodaira classification and noting that $E$ is spin, symplectic and has $\chi(E) = 0$ one concludes that one of the following must hold (cf. [3] Theorem 5.1)

(1) $E = S^2 \times T^2$
(2) $E$ is a $T^2$-bundle over $T^2$
(3) $E$ is a Seifert fibration over a hyperbolic orbifold.

If $E = S^2 \times T^2$ then $M = S^2 \times S^1$ and one clearly has an $S^1$-invariant symplectic form. In the second case it follows from the argument above that $M$ is a $T^2$-bundle over $S^1$ and hence has vanishing Thurston norm. In the final case $M$ must be Seifert fibered as in Case 2 in the proof of Theorem 9 and hence the claim holds in the latter two cases by Corollary 2. 

References


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