Comment on "Time-like flows of energy-momentum and particle trajectories for the Klein–Gordon equation"

Roderich Tumulka*

April 30, 2002

Abstract

Horton, Dewdney, and Nesteruk [1] have proposed Bohm-type particle trajectories accompanying a Klein–Gordon wave function ψ on Minkowski space. From two vector fields on space-time, W^+ and W^-, defined in terms of ψ, they intend to construct a timelike vector field W, the integral curves of which are the possible trajectories, by the following rule: at every space-time point, take either $W = W^+$ or $W = W^-$ depending on which is timelike.

This procedure, however, is ill-defined as soon as both are timelike, or both spacelike. Indeed, they cannot both be timelike, but they can well both be spacelike, contrary to the central claim of [1]. We point out the gap in their proof, provide a counterexample, and argue that, even for a rather arbitrary wave function, the points where both W^+ and W^- are spacelike can form a set of positive measure.

Let $\psi = \mathrm{e}^{P+\mathrm{i}S}$ (where P and S are real) solve the Klein-Gordon equation, $-\Box\psi = m^2\psi$. Set $P_\mu = \partial_\mu P$, $S_\mu = \partial_\mu S$, and

$$\theta = \sinh^{-1} \frac{P^\mu P_\mu - S^\mu S_\mu}{2P^\mu S_\mu}\,.$$

That P_μ and S_μ are orthogonal is an exceptional case that we neglect, like the authors of [1]. For W_μ one is supposed to take either $W^+_\mu = \mathrm{e}^\theta P_\mu + S_\mu$

*address: Mathematisches Institut der Universität München, Theresienstr. 39, D-80333 München, Germany. E-mail: tumulka@mathematik.uni-muenchen.de

or $W_\mu^- = -\mathrm{e}^{-\theta} P_\mu + S_\mu$, depending on which is timelike; they cannot both be timelike since they are orthogonal. The question is, could they both be spacelike?

The authors of [1] declare that W_μ^+ and W_μ^- cannot both be spacelike and argue like this: otherwise there exists a Lorentz frame such that $W_0^+ = 0$ and $W_0^- = 0$, thus $\mathrm{e}^\theta P_0 = -\mathrm{e}^{-\theta} P_0$, from which they conclude $\mathrm{e}^\theta = -\mathrm{e}^{-\theta}$, which is impossible.

It is correct that any two orthogonal spacelike vectors span a spacelike 2-plane (corresponding to $x^0 = 0$ in the appropriate Lorentz frame), but no contradiction arises since in this case P_0 would be 0 (in this frame). This is the mistake in the proof.

Together with $W_0^+ = 0$ (or $W_0^- = 0$), $P_0 = 0$ implies $S_0 = 0$. Hence, for W_μ^+ and W_μ^- to be spacelike it is necessary and sufficient that P_μ and S_μ span a spacelike 2-plane.

Can this case occur? Clearly: since the Klein–Gordon equation is of second order, one may choose ψ and $\partial_0 \psi$ *ad libitum* on the $x^0 = 0$ hyperplane. Can it also occur for the first-order Klein–Gordon equation $-\mathrm{i}\partial_0 \psi = \sqrt{m^2 - \Delta}\, \psi$, or, equivalently, for functions from the positive-energy subspace? Here is an example: let ψ be a superposition of three[1] plane waves

$$\psi(x) = \sum_{i=1}^{3} c_i \, \mathrm{e}^{\mathrm{i} k_\mu^{(i)} x^\mu}$$

with wave vectors $k_\mu^{(1)} = \Big(m, 0, 0, 0\Big)$, $k_\mu^{(2)} = \Big(\sqrt{27}m, \sqrt{26}m, 0, 0\Big)$, $k_\mu^{(3)} = \Big(\sqrt{27}m, 0, \sqrt{26}m, 0\Big)$, and $c_1 = 3$, $c_2 = -1/\sqrt{3} - \mathrm{i}$, $c_3 = \mathrm{i}$. Then, at the coordinate origin, we find $P_\mu = (0, \alpha, -\alpha, 0)$ and $S_\mu = (0, -\beta, 0, 0)$ with $\alpha = \sqrt{26}m/\gamma$, $\beta = \alpha/\sqrt{3}$ and $\gamma = 3 - 1/\sqrt{3}$. This example could also be made square-integrable by replacing the plane waves $\exp(\mathrm{i} k_\mu^{(i)} x^\mu)$ by positive-energy L^2 Klein–Gordon functions $\varphi^{(i)}(x)$ with the properties $\varphi^{(i)}(0) = 1$ and $\partial_\mu \varphi^{(i)}(0) = \mathrm{i} k_\mu^{(i)}$.

One may suspect, however, that perhaps this particular wave function ψ is very exceptional, and perhaps even that for this special wave function the coordinate origin is a rather atypical point, so that the sort of situation just described can be ignored. After all, we would be willing to ignore the

[1] Two will not suffice for an example since P_μ and S_μ are linear combinations of the k_μ vectors.

case where P_μ and S_μ are orthogonal because in the 8-dimensional space of all possible pairs of vectors P_μ, S_μ it corresponds to a subset of dimension 7, and therefore one would expect that the space-time points where this happens form a set of measure zero.

But since $W^+_\mu W^{+\mu}$ and $W^-_\mu W^{-\mu}$ are continuous functions of P_μ and S_μ, the set of pairs P_μ, S_μ where both W^+ and W^- are spacelike is open (and nonempty) and thus has positive measure in 8 dimensions. I know of nothing precluding any pairs P_μ, S_μ from arising from a Klein–Gordon wave function, so it seems reasonable to expect that the space-time points with spacelike W^+ and W^- form a set of positive measure for many wave functions, perhaps for most.

Acknowledgements. I wish to thank Sheldon Goldstein for improvements and simplifications of the arguments.

References

[1] G. Horton, C. Dewdney, A. Nesteruk: "Timelike Flows of Energy-Momentum and Particle Trajectories for the Klein–Gordon Equation", J. Phys. A: Math. Gen. **33** (2000) 7337. arXiv: quant-ph/0103114