Global Existence of Bell's Time-Inhomogeneous Jump Process for Lattice Quantum Field Theory

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Abstract

We consider the time-inhomogeneous Markovian jump process introduced by John S. Bell [4] for a lattice quantum field theory, which runs on the associated configuration space. Its jump rates, tailored to give the process the quantum distribution $|\Psi_t|^2$ at all times t, typically exhibit singularities. We establish the existence of a unique such process for all times, under suitable assumptions on the Hamiltonian or the initial state vector Ψ_0. The proof of non-explosion takes advantage of the special role of the $|\Psi_t|^2$ distribution.

Key words. Markov jump processes, non-explosion, time-dependent jump rates, equivariant distributions, Bell's process, lattice quantum field theory.
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1 Introduction

This paper deals with the existence of Markov jump processes on countable sets having time-inhomogeneous transition rates of a particular form proposed by John S. Bell [4] in his observer-independent formulation of lattice quantum field theories. Bell's setting is as follows.

Bell's model. Let E be the configuration space for a variable but finite number of particles on a countable lattice $\Lambda \subseteq \mathbb{R}^3$. A configuration $x \in E$ is mathematically represented by a function $x : \Lambda \to \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ indicating the number of particles $x(\boldsymbol{r})$ at a site $\boldsymbol{r} \in \Lambda$. Thus

$$E = \Big\{ x \in \mathbb{Z}_+^\Lambda : \sum_{\boldsymbol{r}\in\Lambda} x(\boldsymbol{r}) < \infty \Big\}.$$

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Hence E is countably infinite. (In Bell's proposal, $x(\boldsymbol{r})$ is the number of *fermions* at site $\boldsymbol{r}$, but this is of no relevance here.)

Next, Bell considers the Hilbert space $\mathscr{H}$ and the Hamiltonian H of a lattice quantum field theory. This means that H is a self-adjoint operator on $\mathscr{H}$ determining the quantum state at time t via

$$\Psi_t = \mathrm{e}^{-\mathrm{i}Ht/\hbar}\,\Psi_0 \tag{1}$$

for some initial state vector Ψ_0. These quantities are related to the configuration space by a projection-valued measure (PVM) P on E acting on $\mathscr{H}$. That is, for every $x \in E$ there exists an associated projection $P(x)$ such that $\sum_{x\in E} P(x) = I$, where I is the identity operator, and $P(x)P(y) = 0$ when $x \neq y$. Specifically, $P(x)$ is the projection to the joint eigenspace of the (commuting) fermion number operators $N(\boldsymbol{r})$ associated with the eigenvalues $x(\boldsymbol{r})$. In particular, $\langle \Psi_t | P(x) \Psi_t \rangle$ is the quantum probability of a configuration x at time t. Bell then introduces the transition rate

$$\sigma_t(y|x) = \frac{[(2/\hbar)\,\mathrm{Im}\,\langle \Psi_t | P(y) H P(x) \Psi_t \rangle]^+}{\langle \Psi_t | P(x) \Psi_t \rangle} \tag{2}$$

for a jump from x to $y \in E$, where $a^+ = \max(a, 0)$ denotes the positive part of a. Note that $\sigma_t(x|x) = 0$ because $\langle \Psi_t | P(x) H P(x) \Psi_t \rangle = \langle P(x)\Psi_t | H P(x) \Psi_t \rangle$ is real. A formal calculation yields that this choice of the jump rates is compatible with the process having distribution $\langle \Psi_t | P(\,\cdot\,) \Psi_t \rangle$ at each time t. See [17] for an extensive discussion of this jump rate formula. In this paper we will choose the time unit such that $\hbar = 2$.

Probabilistic questions. One of the main features of the transition rates (2) is that they become singular at times t when x becomes a "node" of Ψ_t, i.e., when the denominator $\langle \Psi_t | P(x) \Psi_t \rangle$ in (2) vanishes. So, at such times the process would not know how to proceed. Fortunately, it turns out that the increase of the rates close to such singularities has the positive effect of forcing the process to jump away before the singularity time is reached.

A more serious problem is the possibility of explosion in finite time; that is, the jump times T_n could accumulate so that $\zeta = \sup_n T_n < \infty$ with positive probability. The standard criteria for non-explosion of pure jump processes are confined to transition rates that are homogeneous in time, relying heavily on the fact that the holding times are then exponentially distributed and independent; see, e.g., Section 2.7 of [21] or Proposition 10.21 of [19]. This independence, however, fails to hold in the case of time-dependent jump rates, and the singularities of Bell's transition rates do not allow any simple bounds excluding explosion. The only thing one knows is that the process is designed to have the prescribed quantum distribution at fixed (deterministic) times, and it is this fact we will exploit.

Our proof will not make any use of the particular construction or meaning of E and P. We will merely assume that E is a countable set and P a PVM on E acting on $\mathscr{H}$. Actually we only need that P is a positive-operator-valued measure; see Section 2 below. Steps towards an existence proof for Bell's process have already been made by Bacciagaluppi [1, 2]; his approach is, however, very different from ours.

Physical Perspective. Bell's observer-independent formulation of lattice quantum field theories has attracted increasing attention recently [11, 13, 14, 15, 16, 17]. Apart from its relevance to the foundations of quantum theory, it has proven useful for numerical simulations [13], and has been found distinguished among all $|\Psi|^2$ distributed processes as the minimal one [17, 24], involving the least amount of stochasticity.

There are close connections between Bell's model and two well-known $|\Psi|^2$ distributed processes associated with nonrelativistic quantum mechanics in $\mathbb{R}^3$: E. Nelson's *stochastic mechanics* [20, 18, 8, 9] and *Bohmian mechanics* [7, 3, 5, 6]. These processes are similar in spirit to Bell's process, and can be combined with Bell's stochastic jumps to include particle creation and annihilation [15, 16, 17]. Bell's process has also been utilized for modal interpretations of quantum theory [2]. Bohmian mechanics arises as the continuum limit of Bell's process for a suitable choice of H and E [26, 27], and in general the continuum limit presumably resembles the combined Bell–Bohm model of [16]. A generalization of Bell's jump rate (2) to continuum spaces E is given in [17]. The global existence problem of stochastic mechanics has been solved in [8] (see also [9, 20]) and the one of Bohmian mechanics in [6], whereas for combined models with jumps, such as the ones considered in [16, 17], it is still open. The existence problems of stochastic mechanics and Bohmian mechanics have two aspects in common with that of Bell's process: First, since the law of motion (as defined by the drift in stochastic mechanics, the velocity in Bohmian mechanics, and the jump rate in Bell's model) is ill-defined at the nodes of the wave function, one needs to show that the process never reaches a node. Second, while in stochastic mechanics and Bohmian mechanics there are no jumps that could accumulate, one needs to exclude (and has excluded) the analogous possibility that the process could escape to infinity in finite time.

2 The Result

The basic ingredients of the model are:

- a complex Hilbert space $\mathscr{H}$ with inner product $\langle \cdot | \cdot \rangle$, the space of quantum states,
- a self-adjoint operator H acting on $\mathscr{H}$, the Hamiltonian,
- an initial state vector $\Psi_0 \in \mathscr{H}$ with $\|\Psi_0\| = 1$,
- a countable set E, physically thought of as configuration space and serving as state space of the jump process to be constructed, and
- a positive-operator-valued measure (POVM) $P(\,\cdot\,)$ on E acting on $\mathscr{H}$.

Here, a POVM is a family $(P(x))_{x\in E}$ of positive bounded self-adjoint operators on $\mathscr{H}$ such that, for each $F \subseteq E$, the sum $P(F) := \sum_{x\in F} P(x)$ exists in the sense of the weak operator topology, and $P(E) = I$. In fact, the countable additivity then also holds in the strong topology [12]. In particular,

$$\forall\, \Phi \in \mathscr{H} : \sum_{x\in E} P(x)\Phi \text{ converges in the } L^2 \text{ sense to } \Phi. \tag{3}$$

Every PVM is a POVM but not vice versa. As has already been pointed out in [17], the jump rate formula (2) still makes sense if $P(\,\cdot\,)$ is a POVM rather than a PVM.

In quantum field theory, the "configuration observable" $P(\,\cdot\,)$ is often a POVM; a typical situation is that $\mathscr{H}$ is a subspace (e.g., the positive spectral subspace of the free Hamiltonian) of a larger Hilbert space $\mathscr{H}_0$ containing also unphysical states, and $P(\,\cdot\,) = P'P_0(\,\cdot\,)I'$ where P' is the projection $\mathscr{H}_0 \to \mathscr{H}$, I' is the embedding $\mathscr{H} \hookrightarrow \mathscr{H}_0$, and $P_0(\,\cdot\,)$ is a PVM (the configuration observable) acting on $\mathscr{H}_0$.

To establish the existence of a Markovian jump process with rates (2) we need the following joint assumption on H, P, and the initial state vector Ψ_0.

Assumption A *The Hamiltonian H, the POVM P and the state vector $\Psi_0 \in \mathscr{H}$ satisfy the conditions*

(A1) *For all $t \in \mathbb{R}$ and $x \in E$, Ψ_t and $P(x)\Psi_t$ belong to the domain of H.*

(A2) *For all $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$,*

$$\int_{t_0}^{t_1} dt \sum_{x,y \in E} \big|\langle \Psi_t | P(y) H P(x) \Psi_t \rangle\big| < \infty\,.$$

For given H and P, Assumption A can also be understood as an assumption on Ψ_0, thus defining a set $\mathscr{D} \equiv \mathscr{D}_{H,P} \subseteq \mathscr{H}$ of "good" state vectors for which the process is well-defined. This $\mathscr{D}$ is invariant under the time evolution but not necessarily a subspace of $\mathscr{H}$ because assumption (A2) is not linear in Ψ_t. The following proposition provides conditions on H under which Assumption A holds for all $\Psi_0 \in \mathscr{H}$, so that $\mathscr{D} = \mathscr{H}$. (For general H we do not know how large $\mathscr{D}$ is, and whether it is dense, as would be physically desirable. We will not presuppose this but instead construct the process solely for initial state vectors $\Psi_0 \in \mathscr{D}$.)

Proposition 1 *Assumption A holds for all $\Psi_0 \in \mathscr{H}$ when either*

(a) *H is bounded and E is finite, or*

(b) *H is a Hilbert–Schmidt operator, i.e., $\operatorname{tr} H^2 < \infty$.*

The proof is postponed until Section 5. Assumption (A1) implies that $P(y)HP(x)\Psi_t$ exists, and thus that

$$\sigma_t(y|x) \text{ is well defined whenever } \langle \Psi_t | P(x)\Psi_t \rangle \neq 0.$$

When $\langle \Psi_t | P(x)\Psi_t \rangle = 0$, we set $\sigma_t(y|x) := \infty$ for all y; thus, $\sigma_t(y|x)$ is always defined as a $[0,\infty]$-valued function. (When suitably reinterpreted, the numerator of (2) still exists if $P(x)\Psi_t$ and $P(y)\Psi_t$ merely lie in the form domain, rather than the domain, of H. We will not pursue here this kind of greater generality.)

As was pointed out in the introduction, the rates $\sigma_t(y|x)$ are constructed in such a way that the corresponding Markov process X_t should have the quantum distribution

$$\pi_t(x) := \langle \Psi_t | P(x) \Psi_t \rangle \,, \quad x \in E, \tag{4}$$

at any time $t \in \mathbb{R}$. In other words, the family $(\pi_t)_{t\in\mathbb{R}}$ should be *equivariant*, or an *entrance law*, for the process. Here is our main result stating that such a process does exist.

Theorem 1 *Suppose Assumption A holds. Then there exists a right-continuous (time-inhomogenous) Markovian pure jump process $(X_t)_{t\in\mathbb{R}}$ in E with transition rates* (2) *and such that, for each t, X_t has distribution π_t. The process is unique in distribution.*

3 The construction

We fix some starting time $t_0 \in \mathbb{R}$ and construct the process first on the time interval $[t_0, \infty[$. We also introduce an auxiliary "cemetery" configuration $\triangle$ in order to deal with the possibility that the process explodes or runs into a node. In the next section we will show that this does in fact not occur. We write

$$\mathcal{N} := \{(t,x) \in \mathbb{R} \times E : \pi_t(x) = 0\} \tag{5}$$

for the node-set of all exceptional times and positions for which the transition rates (2) are infinite. Likewise,

$$E_t := \{x \in E : \pi_t(x) > 0\} \tag{6}$$

is the set of all admissible positions at time $t \in \mathbb{R}$. Finally, for $(t,x) \notin \mathcal{N}$ we let

$$\theta_{t,x} := \inf\{s > t : (s,x) \in \mathcal{N}\} \tag{7}$$

be the first time instant after t at which x becomes a node; here we set $\inf \emptyset := \infty$. Let us start with a technical lemma; its proof follows later in this section. Formula (8) relies on our convention $\hbar = 2$.

Lemma 1 *For every $x \in E$, the mapping $t \mapsto \pi_t(x)$ is differentiable with locally integrable derivative*

$$\dot{\pi}_t(x) = \operatorname{Im} \langle \Psi_t | P(x) H \Psi_t \rangle. \tag{8}$$

In particular, the function $\pi_\cdot(x)$ is locally absolutely continuous. Also, the jump rates $\sigma_t(y|x)$ depend measurably on t with values in $[0,\infty]$, and the total jump rate

$$\gamma_x(t) := \sum_{y \in E} \sigma_t(y|x) \tag{9}$$

is finite whenever $(t,x) \notin \mathcal{N}$.

The process $(X_t)_{t\geq t_0}$ will be constructed on the enlarged position space $E \cup \{\triangle\}$ by means of a suitable sequence of random jump times T_n and jump destinations Z_n. To achieve this we need two key quantities: the distribution $\mu_{t,x}$ of the holding time in $x \in E$ (i.e., the random waiting time before the next jump) after a given time t, and the distribution $p_{t,x}$ of the the jump destination at the jump time. Our assumption that the process $(X_t)_{t\geq t_0}$ should have the transition rates (2) simply means that $\mu_{t,x}$ should be the distribution with "failure rate function" (or "hazard rate function") γ_x; cf. e.g. [23, pp. 276 ff., 577]. That is, for any $(t,x) \notin \mathcal{N}$ we let $\mu_{t,x}$ be the unique probability measure on $]t,\infty]$ with "survival probabilities"

$$\mu_{t,x}([u,\infty]) = \mathrm{e}^{-\Gamma_{t,x}(u)} \text{ for all } u > t, \tag{10}$$

where

$$\Gamma_{t,x}(u) = \int_t^u \gamma_x(s)\, ds\,; \tag{11}$$

in (10) and below we set $\mathrm{e}^{-\infty} = 0$. (Note that $\Gamma_{t,x}$ is left-continuous by the monotone convergence theorem, so that there exists indeed a unique probability measure $\mu_{t,x}$ having $\mathrm{e}^{-\Gamma_{t,x}}$ as right-sided distribution function.) In particular, $\mu_{t,x}(\{\infty\}) > 0$ if and only if $\Gamma_{t,x}(\infty) < \infty$; thus, in this case there is a non-zero probability for the process to be frozen in x. If $t = \infty$ or $x =\triangle$ we let $\mu_{t,x} = \delta_\infty$ be the Dirac measure at $+\infty$. The following lemma collects the essential properties of $\mu_{t,x}$.

Lemma 2 *Suppose $(t,x) \notin \mathcal{N}$. Then the following statements hold:*

(a) *On $]t,\theta_{t,x}[$, $\mu_{t,x}$ has the density function $\gamma_x\, \mathrm{e}^{-\Gamma_{t,x}}$.*

(b) *If $\theta_{t,x} < \infty$ then $\mu_{t,x}([\theta_{t,x},\infty]) = 0$.*

(c) $\mu_{t,x}(0 < \gamma_x < \infty) = 1$.

(d) *For any $u \in]t,\theta_{t,x}[$, $\mu_{t,x}(]u,\infty]) > 0$ and $\mu_{u,x} = \mu_{t,x}(\,\cdot\,|\,]u,\infty])$.*

The proof will be given later in this section. It follows readily from (a) that if we take $\mu_{t,x}$ as the the distribution of the holding time at x then the jump rate is indeed γ_x, as intended. Assertion (b) states that the process cannot run into a node by sitting on an x until it becomes a node: right before x becomes a node, the rate γ_x grows so fast that the process has probability 1 to jump away. In particular, the unboundedness of the jump rates (even for bounded H, even for Hilbert–Schmidt H) *favours* the global existence rather than preventing it. Statement (d) expresses a loss-of-memory property which is responsible for the Markov property of the process.

As for the distribution of the jump destinations, we obviously have to define

$$p_{t,x}(y) = \sigma_t(y|x)/\gamma_x(t) \qquad \text{if } 0 < \gamma_x(t) < \infty\,. \tag{12}$$

Otherwise we set $p_{t,x} = \delta_\triangle$, the Dirac measure at $\triangle$. The next lemma states that $p_{t,x}$ is supported on the set E_t of non-nodes defined in (6).

Lemma 3 $p_{t,x}(E_t) = 1$ *whenever* $0 < \gamma_x(t) < \infty$.

With these ingredients we are now ready to construct the process $(X_t)_{t\geq t_0}$ on $E\cup\{\Delta\}$. Let $(T_n, Z_n)_{n\geq 0}$ be a sequence of random variables with the following properties. Let $T_0 := t_0$ and $Z_0 \in E_{t_0}$ be a random variable with distribution π_{t_0}. Then, for any $n \geq 0$, let (T_{n+1}, Z_{n+1}) have the conditional distribution

$$\mathbb{P}\big(T_{n+1} \in \, dt, Z_{n+1} = y \,|T_0, Z_0, \ldots, T_n, Z_n\big) = \mu_{T_n,Z_n}(dt)\, p_{t,Z_n}(y)\,. \tag{13}$$

The existence of such a sequence $(T_n, Z_n)_{n\geq 0}$ on a suitable probability space $(\Omega_{t_0}, \mathcal{F}_{t_0}, \mathbb{P}_{t_0})$ follows from the Ionescu–Tulcea theorem [19, Theorem 5.17]. Moreover, Lemmas 2 and 3 imply that $T_{n+1} < \theta_{T_n,Z_n}$ and $Z_n \in E_{T_n}$ almost surely for all n. We also have $T_n < T_{n+1}$ as long as $T_n < \infty$. So we define

$$X_t = Z_n \text{ when } T_n \leq t < T_{n+1}, \text{ and } X_t = \Delta \text{ for } t \geq \zeta = \sup_n T_n\,. \tag{14}$$

It is then clear that $(X_t)_{t\geq t_0}$ is right-continuous, and $(t, X_t) \notin \mathcal{N}$ for all $t \in [t_0, \zeta[$ with probability 1. In the next section we will show that in fact $\zeta = \infty$ almost surely.

We now turn to the proofs of the lemmas above. Recall our convention that $\hbar = 2$.

Proof of Lemma 1: Since Ψ_t belongs to the domain of H by Assumption (A1), the $\mathscr{H}$-valued mapping $t \mapsto \Psi_t$ is differentiable with derivative $d\Psi_t/dt = -\frac{\mathrm{i}}{2}H\Psi_t$ [22, p. 265]. Hence

$$\big(\pi_{t+s}(x) - \pi_t(x)\big)/s = \langle\Psi_{t+s}|P(x)(\Psi_{t+s} - \Psi_t)/s\rangle + \langle(\Psi_{t+s} - \Psi_t)/s|P(x)\Psi_t\rangle$$

converges, as $s \to 0$, to

$$\dot\pi_t(x) := -\frac{\mathrm{i}}{2}\langle\Psi_t|P(x)H\Psi_t\rangle + \frac{\mathrm{i}}{2}\langle H\Psi_t|P(x)\Psi_t\rangle = \mathrm{Im}\,\langle\Psi_t|P(x)H\Psi_t\rangle\,.$$

As a limit of the continuous difference ratios, $t \mapsto \dot\pi_t(x)$ is measurable. Moreover, using (3) we can write

$$\begin{aligned}\big|\dot\pi_t(x)\big| &\leq \big|\langle\Psi_t|P(x)H\Psi_t\rangle\big| = \big|\langle HP(x)\Psi_t|\Psi_t\rangle\big| \\ &= \Big|\sum_{y\in E}\langle HP(x)\Psi_t|P(y)\Psi_t\rangle\Big| \leq \sum_{y\in E}\big|\langle\Psi_t|P(x)HP(y)\Psi_t\rangle\big|\,.\end{aligned}$$

Together with Assumption (A2), it follows that $\dot\pi_t(x)$ is locally integrable. In particular, $t \mapsto \pi_t(x)$ is locally absolutely continuous and an integral function of $t \mapsto \dot\pi_t(x)$; see [10, Theorem 6.3.10] or [25, Theorems 8.21 and 8.17].

Concerning the measurability of the jump rates $\sigma_t(y|x)$, it is sufficient to show that $\langle\Psi_t|P(y)HP(x)\Psi_t\rangle$ depends measurably on t. (This is because $\pi_\cdot(x)$ is continuous and the ratio of nonnegative measurable functions is measurable.) To this end, we introduce the cutoff function $f_n(a) := (a \wedge n) \vee (-n)$ and observe that, for every t,

$$\langle\Psi_t|P(y)f_n(H)P(x)\Psi_t\rangle \to \langle\Psi_t|P(y)HP(x)\Psi_t\rangle$$

as $n \to \infty$. Hence $t \mapsto \langle \Psi_t | P(y) H P(x) \Psi_t \rangle$ is a pointwise limit of continuous functions and thereby measurable. In particular, the total jump rate γ_x is measurable. For $(t,x) \notin \mathcal{N}$ we have

$$\gamma_x(t) \leq \sum_{y \in E} \big| \langle \Psi_t | P(y) H P(x) \Psi_t \rangle \big| / \pi_t(x) \,.$$

The last sum is finite because, due to (3), the series $\sum_y \langle \Psi_t | P(y) H P(x) \Psi_t \rangle$ converges to $\langle \Psi_t | H P(x) \Psi_t \rangle$ in every ordering, and is therefore absolutely convergent. □

Before proving Lemma 2 we establish the following result, a key fact for showing that the process never runs into a node. Recall the definitions (7) and (11).

Lemma 4 *Suppose* $(t,x) \notin \mathcal{N}$. *Then* $\Gamma_{t,x}(u) < \infty$ *if* $t < u < \theta_{t,x}$, *while* $\Gamma_{t,x}(\theta_{t,x}) = \infty$ *if* $\theta_{t,x} < \infty$.

Proof: Consider first the case $t < u < \theta_{t,x}$. Since $\pi_\cdot(x)$ is continuous and positive on $[t,u]$, it stays bounded away from zero on this interval. On the other hand, we have

$$\int_t^u ds \sum_{y \in E} \big[\mathrm{Im}\, \langle \Psi_s | P(y) H P(x) \Psi_s \rangle \big]^+ \leq \int_t^u ds \sum_{y \in E} \big| \langle \Psi_s | P(y) H P(x) \Psi_s \rangle \big| \,,$$

and the last integral is finite due to Assumption (A2). This proves the first assertion.

Consider now the case $\theta_{t,x} < \infty$. Since $\sum_i a_i^+ \geq \big[\sum_i a_i \big]^+$ in general, we have for all $t < s < \theta_{t,x}$

$$\gamma_x(s) \geq \Big[\mathrm{Im} \sum_y \langle \Psi_s | P(y) H P(x) \Psi_s \rangle \Big]^+ / \pi_s(x) \,.$$

In view of (3) and Lemma 1, the last expression is equal to

$$\big[\mathrm{Im}\, \langle \Psi_s | H P(x) \Psi_s \rangle \big]^+ / \pi_s(x) = \big[-\dot{\pi}_s(x) \big]^+ / \pi_s(x) \,.$$

Since always $a^+ \geq a$, we arrive at the key inequality

$$\gamma_x(s) \geq -\frac{d}{ds} \log \pi_s(x) \,.$$

The last derivative is integrable over any interval $[t,u]$ with $t < u < \theta_{t,x}$ because $\pi_s(x)$ is bounded away from zero on such an interval and $\dot{\pi}_s(x)$ is locally integrable by Lemma 1. By the general fundamental theorem of calculus as in [10, Theorem 6.3.10] or [25, Theorem 8.21], it follows that

$$\int_t^u \gamma_x(s)\, ds \geq -\int_t^u \frac{d}{ds} \log \pi_s(x)\, ds = \log \pi_t(x) - \log \pi_u(x).$$

Letting $u \uparrow \theta_{t,x}$ and using the continuity of $\pi_u(x)$ we arrive at the second statement of the lemma. □

We are now ready for the proof of Lemma 2.

Proof of Lemma 2: (a) Let $t < u < \theta_{t,x}$. Instead of using the fundamental theorem of calculus (which would be possible), we prefer to give here a direct argument which is based on Fubini's theorem. In view of Lemma 4, $\Gamma_{t,x}$ is finite on $[t, u]$. Thus we can write, omitting the indices t, x,

$$\int_t^u \gamma(s)\, \mathrm{e}^{-\Gamma(s)} ds = \int_t^u ds\, \gamma(s) \int_0^\infty dr\, \mathrm{e}^{-r}\, 1_{\{\Gamma(s) \le r\}}$$
$$= \int_0^\infty dr\, \mathrm{e}^{-r} \int_t^u ds\, \gamma(s)\, 1_{\{\Gamma(s)\le r\}} = \int_0^\infty dr\, \mathrm{e}^{-r}\, (r \wedge \Gamma(u))\,.$$

The last equality uses the fact that Γ is continuous and increasing. Since $r \wedge \Gamma(u) = r - [r - \Gamma(u)]^+$, the last integral coincides with $1 - \mathrm{e}^{-\Gamma(u)} = \mu_{t,x}(]t, u])$, thus proving assertion (a).

(b) This is immediate from (10) and Lemma 4.

(c) This comes from statements (a) and (b) together with Lemma 1.

(d) Let $t < u < \theta_{t,x}$. Since $\Gamma_{t,x}(u) < \infty$ by Lemma 4, Equation (10) shows that $\mu_{t,x}(\,]u, \infty]) > 0$. Moreover, for $v > u$ we have

$$\mu_{t,x}\big(\,]v,\infty]\,\big|\,]u,\infty]\big) = \mathrm{e}^{-\Gamma_{t,x}(v)+\Gamma_{t,x}(u)} = \mathrm{e}^{-\Gamma_{u,x}(v)} = \mu_{u,x}\big(\,]v,\infty]\,\big)$$

by Equation (11). This proves the final statement. □

We conclude this section with the proof of Lemma 3.

Proof of Lemma 3: We only have to to show that $\sigma_t(y|x) = 0$ whenever $\pi_t(y) = 0$. But since $\|P(y)^{1/2}\Psi_t\|^2 = \pi_t(y)$, we then have $P(y)^{1/2}\Psi_t = 0$. Hence $P(y)\Psi_t = 0$ and therefore $\langle \Psi_t | P(y) H P(x) \Psi_t\rangle = \langle P(y)\Psi_t | H P(x)\Psi_t\rangle = 0$, which gives the result. □

4 Non-explosion

In the last section we have constructed a process $(X_t)_{t\ge t_0}$ that stays in the configuration space E until some possibly finite explosion time $\zeta = \sup_n T_n$, at which it jumps into the cemetery $\triangle$. We will now show that ζ is in fact almost surely infinite. To this end we consider the random number

$$S(t) := \#\{n \ge 1 : t_0 < T_n \le t\} \in \mathbb{Z}_+ \cup \{\infty\} \tag{15}$$

of jumps during the time interval $]t_0, t]$ for any $t > t_0$. We want to show that $S(t)$ has finite expectation. To this end we start from the following formula.

Lemma 5 *For all $t > t_0$,*

$$\mathbb{E}_{t_0}\, S(t) = \int_{t_0}^t ds \sum_{x,y\in E} \mathbb{P}_{t_0}(X_s = x)\, \sigma_s(y|x)\,.$$

To estimate the last expression we will show:

Lemma 6 $\mathbb{P}_{t_0}(X_t = x) \leq \pi_t(x)$ *for all* $x \in E$ *and* $t > t_0$.

In other words, though the rates are constructed in such a way that the process should follow the equivariant distribution π_t, we cannot exclude *a priori* that some mass is lost at the cemetery $\triangle$. Combining these two lemmas we obtain

$$\begin{aligned} \mathbb{E}_{t_0}\, S(t) &\leq \int_{t_0}^{t} ds \sum_{x,y\in E} \pi_s(x)\,\sigma_s(y|x) \\ &= \int_{t_0}^{t} ds \sum_{x,y\in E} \big[\mathrm{Im}\,\langle\Psi_s|P(y)HP(x)\Psi_s\rangle\big]^+ \\ &\leq \int_{t_0}^{t} ds \sum_{x,y\in E} \big|\langle\Psi_s|P(y)HP(x)\Psi_s\rangle\big|\,, \end{aligned}$$

and the last expression is finite by Assumption (A2). Hence $S(t) < \infty$ almost surely, and thereby $\zeta > t$ almost surely. As t was arbitrary, we conclude that $\zeta = \infty$ almost surely, as we wanted to show. We now turn to the proofs of the two lemmas above.

Proof of Lemma 5: Using Equation (13) and Lemma 2 we can write

$$\begin{aligned} \mathbb{E}_{t_0}\, S(t) &= \sum_{n\geq 0} \mathbb{P}_{t_0}(t_0 \leq T_{n+1} \leq t) = \sum_{n\geq 0} \mathbb{E}\Big(\mathbb{P}_{t_0}\big(t_0 \leq T_{n+1} \leq t \,|\, T_k, Z_k : k \leq n\big)\Big) \\ &= \sum_{n\geq 0} \mathbb{E} \int_{t_0}^{t} ds\, 1_{\{T_n < s < \theta_{T_n,Z_n}\}}\, \gamma_{Z_n}(s)\, \mathrm{e}^{-\Gamma_{T_n,Z_n}(s)} \\ &= \sum_{n\geq 0} \int_{t_0}^{t} ds\, \mathbb{E}\Big(1_{\{T_n<s\}}\, \gamma_{Z_n}(s)\, \mathbb{P}_{t_0}\big(T_{n+1} > s \,|\, T_k, Z_k : k \leq n\big)\Big) \\ &= \int_{t_0}^{t} ds \sum_{x\in E} \gamma_x(s)\, \mathbb{E}\Big(\sum_{n\geq 0} 1_{\{T_n<t<T_{n+1},\, Z_n = x\}}\Big) \\ &= \int_{t_0}^{t} ds \sum_{x\in E} \gamma_x(s)\, \mathbb{P}_{t_0}(X_s = x)\,. \end{aligned}$$

Together with (9) the lemma follows. □

For the proof of Lemma 6 we consider the integral equation

$$\rho_t(x) = \pi_{t_0}(x)\, \mathrm{e}^{-\Gamma_{t_0,x}(t)} + \sum_{y\in E} \int_{t_0}^{t} ds\; \rho_s(y)\, \sigma_s(x|y)\, \mathrm{e}^{-\Gamma_{s,x}(t)}\,, \tag{16}$$

$t \geq t_0$, $x \in E$, for a time-dependent subprobability measure ρ_t on E. Lemma 6 follows directly from the next two results.

Lemma 7 *The mapping* $(t,x) \mapsto \mathbb{P}_{t_0}(X_t = x)$ *is the minimal solution of* (16)*.*

Lemma 8 *The mapping* $(t,x) \mapsto \pi_t(x)$ *is a solution of* (16) *for arbitrary* t_0*.*

Proof of Lemma 7: For any $x \in E$ and $t > t_0$ we can write

$$\mathbb{P}_{t_0}(X_t = x) = \sum_{n \geq 0} A_n(t,x) \quad \text{with} \quad A_n(t,x) := \mathbb{P}_{t_0}\big(T_n \leq t < T_{n+1},\, Z_n = x\big).$$

It follows from (10) that

$$A_0(t,x) = \pi_{t_0}(x)\, \mathrm{e}^{-\Gamma_{t_0,x}(t)} \tag{17}$$

and, for $n \geq 1$,

$$\begin{aligned} A_n(t,x) = & \sum_{x_0,\ldots,x_{n-1} \in E} \quad \int\cdots\int_{t_0 < t_1 < \cdots < t_n \leq t} \\ & \times \mathbb{P}_{t_0}\Big(T_1 \in dt_1, \ldots, T_n \in dt_n, T_{n+1} > t, Z_0 = x_0, \ldots, Z_n = x_n\Big) \\ = & \sum_{x_0,\ldots,x_{n-1} \in E} \quad \int\cdots\int_{t_0 < t_1 < \cdots < t_n \leq t} dt_1 \cdots dt_n \\ & \times \pi_{t_0}(x_0) \left(\prod_{i=1}^{n} \mathrm{e}^{-\Gamma_{t_{i-1},x_{i-1}}(t_i)} \sigma_{t_i}(x_i | x_{i-1}) \right) \mathrm{e}^{-\Gamma_{t_n,x}(t)}\,, \end{aligned}$$

where $x_n := x$. In particular, separating the summation over x_{n-1} and the integration over t_n we find that

$$A_n(t,x) = \sum_{y \in E} \int_{t_0}^{t} ds\ A_{n-1}(s,y)\, \sigma_s(x|y)\, \mathrm{e}^{-\Gamma_{s,x}(t)}\,. \tag{18}$$

This shows that $\mathbb{P}_{t_0}(X_t = x)$ satisfies (16).

Now let $\rho_t(x)$ be an arbitrary (nonnegative) solution of (16). An $(N-1)$-fold iteration of (16) then leads to the equation

$$\rho_t(x) = \sum_{n=0}^{N-1} A_n(t,x) + R_N(t,x),$$

with $A_n(t,x)$ defined by (17) and (18), and the remainder term

$$\begin{aligned} R_N(t,x) = & \sum_{x_0,\ldots,x_{N-1} \in E} \quad \int\cdots\int_{t_0 < t_1 < \cdots < t_N \leq t} dt_1 \cdots dt_N \\ & \times \rho_{t_1}(x_0) \left(\prod_{i=1}^{N-1} \sigma_{t_i}(x_i | x_{i-1})\, \mathrm{e}^{-\Gamma_{t_i,x_i}(t_{i+1})} \right) \sigma_{t_N}(x | x_{N-1})\, \mathrm{e}^{-\Gamma_{t_N,x}(t)}\,. \end{aligned}$$

(Compared with $A_N(t,x)$, $R_N(t,x)$ involves ρ_{t_1} rather than $\rho_{t_0} = \pi_{t_0}$, and the σ's and $e^{-\Gamma}$'s run in a different order.) Since $R_N(t,x) \geq 0$, we see that $\rho_t(x)$ exceeds each partial sum of the infinite series constituting $\mathbb{P}_{t_0}(X_t = x)$. This proves Lemma 7. □

Proof of Lemma 8: We start from the observation that, by (8), (3), (2) and the self-adjointness of H and $P(x)$,

$$\begin{aligned}\dot{\pi}_t(x) &= \operatorname{Im}\langle HP(x)\Psi_t|\Psi_t\rangle = \sum_{y\in E} \operatorname{Im}\langle HP(x)\Psi_t|P(y)\Psi_t\rangle \\ &= \sum_{y\in E}\Big(\pi_t(y)\,\sigma_t(x|y) - \pi_t(x)\,\sigma_t(y|x)\Big) \\ &= \sum_{y\in E}\pi_t(y)\,\sigma_t(x|y) - \pi_t(x)\,\gamma_x(t)\end{aligned}$$

This means that the integral equation (16) for $\pi_t(x)$ takes the form

$$\pi_t(x) - \pi_{t_0}(x)\,e^{-\Gamma_{t_0,x}(t)} = \int_{t_0}^{t} ds\,\big(\dot{\pi}_s(x) + \pi_s(x)\,\gamma_x(s)\big)e^{-\Gamma_{s,x}(t)}. \tag{19}$$

To establish this equation we write for brevity $f(s) = \pi_s(x)$ and $g(s) = e^{-\Gamma_{s,x}(t)}$ and distinguish two cases.

Case 1: $f > 0$ on $[t_0, t]$; that is, x is never a node on this interval. Then, by Lemma 4, γ_x is integrable over $[t_0, t]$, whence $s \mapsto \Gamma_{s,x}(t)$ is absolutely continuous with derivative $-\gamma_x$. Since the exponential function is Lipschitz on $]-\infty, 0]$, it follows that g is absolutely continuous with derivative $\dot{g} = \gamma_x\, g$ Lebesgue-almost-everywhere; see [10, Corollary 6.3.7] or [25, Theorem 8.17]. Equation (19) is thus equivalent to the partial integration formula

$$f(t)g(t) - f(t_0)g(t_0) = \int_{t_0}^{t}\big(\dot{f}(s)g(s) + f(s)\dot{g}(s)\big)ds$$

which holds according to Corollary 6.3.8 of [10].

Case 2: $f(s) = 0$ for some $s \in [t_0, t]$; that is, x is a node at some time s. By the continuity of f, there exists then a largest such s in $[t_0, t]$, say θ. Suppose first that $\theta = t_0$. We can then apply Case 1 to each subinterval $[t_*, t]$ of $[t_0, t]$, which yields (19) with t_* in place of t_0. Let $I(t_*)$ be the corresponding integral on the right-hand side. Since the integrand is nonnegative, we can use the monotone convergence theorem to conclude that $I(t_*) \uparrow I(t_0)$ as $t_* \downarrow t_0$. On the other hand, $f(t_*)g(t_*) \to 0 = f(t_0)g(t_0)$ as $t_* \downarrow t_0$ because f is continuous and $0 \leq g \leq 1$. This proves (19) in the case $\theta = t_0$.

If $\theta > t_0$, we observe that $\Gamma_{s,x}(t) = \infty$ for all $s < \theta$. Indeed, we even have $\Gamma_{s,x}(\theta) = \infty$. This is evident when $]s, \theta[$ consists only of node-times because then γ_x is infinite on this interval; otherwise it follows from the second statement of Lemma 4 applied to the segment from a non-node-time between t and θ to the next node-time. Consequently, the left-hand side of (19) is equal to $f(t)$, while the integrand on the right-hand side vanishes on $[t_0, \theta[$. This means that we have to establish (19) with t_0 replaced by θ. But this is trivial when $\theta = t$ because then both sides vanish, and otherwise follows from the previous paragraph. $\square$

It is now easy to complete the proof of the theorem.

Proof of Theorem 1: As we have shown above, for any $t_0 \in \mathbb{R}$ there exists a right-continuous pure jump process $(X_t)_{t \geq t_0}$ on a suitable probability space $(\Omega_{t_0}, \mathcal{F}_{t_0}, \mathbb{P}_{t_0})$. Since $\zeta = \infty$ almost surely, this process avoids the cemetery $\triangle$ and thus takes values in E. Hence $\sum_{x \in E} \mathbb{P}_{t_0}(X_t = x) = 1$ for all $t \geq t_0$. Lemma 6 therefore implies that $\mathbb{P}_{t_0}(X_t = x) = \pi_t(x)$ for all $x \in E$ and $t > t_0$. In particular, if E_t is given by (6) then $X_t \in E_t$ for all $t \geq t_0$ with probability 1.

We also note that $(X_t)_{t \geq t_0}$ is Markovian; its transition matrix from time s to time t given by

$$P_{s,t}(x, \cdot) = \mathbb{P}_s(X_t = \cdot \mid X_s = x)$$

when $x \in E_s$, and arbitrary otherwise. This follows directly from the construction together with Lemma 2(d). In particular, the distribution P_{t_0} of $(X_t)_{t \geq t_0}$ on the Skorohod space $D([t_0, \infty[, E)$ of all càdlàg[1] paths from $[t_0, \infty[$ to E is uniquely determined, and the family $(\mathrm{P}_{t_0})_{t_0 \in \mathbb{R}}$ is consistent. Kolmogorov's extension theorem [19, Theorem 5.16] therefore provides us with a probability measure P on $E^{\mathbb{R}}$ which extends all distributions P_{t_0} and is therefore concentrated on $D(\mathbb{R}, E)$, the space of all càdlàg paths on $\mathbb{R}$. Under P, the canonical coordinate process constitutes the global Markov jump process with the desired properties. $\square$

5 Proof of Proposition 1

First we consider case (a). Since H is bounded, assumption (A1) holds trivially. The boundedness of H also implies that the expression $\langle \Psi_t | P(x) H P(y) \Psi_t \rangle$ is (well defined and) a continuous function of t for every x and y. As E is finite, the integrand in Assumption (A2) is continuous and therefore locally integrable.

Turning to case (b), we observe first that assumption (A1) is again trivially satisfied because Hilbert–Schmidt operators are bounded. Assumption (A2) will follow from the inequality

$$\sum_{x,y \in E} \big| \langle \Psi | P(x) H P(y) \Psi \rangle \big| \leq \|\Psi\|^2 \sqrt{\operatorname{tr} H^2} \quad \forall\, \Psi \in \mathscr{H} \tag{20}$$

which we prove now.

We start with a general remark. Let $\mathscr{I}$ be a countable index set and A_i and B_i, $i \in \mathscr{I}$, any Hilbert–Schmidt operators with (possibly different) adjoints A_i^* resp. B_i^*; i.e., we have $\operatorname{tr} A_i^* A_i < \infty$ and similarly for B_i. The Cauchy–Schwarz inequality then asserts that

$$\sum_{i \in \mathscr{I}} |\operatorname{tr} A_i^* B_i| \leq \Big(\sum_{i \in \mathscr{I}} \operatorname{tr} A_i^* A_i \Big)^{1/2} \Big(\sum_{i \in \mathscr{I}} \operatorname{tr} B_i^* B_i \Big)^{1/2} \tag{21}$$

[1] continues à droite avec des limites à gauche = right-continuous with left limits.

whenever both terms on the right hand side are finite. (Note that we can put the modulus sign inside of the sum because we can replace A_i by $z_i A_i$ with $z_i = (\operatorname{tr} A_i^* B_i)/|\operatorname{tr} A_i^* B_i|$ whenever $\operatorname{tr} A_i^* B_i \neq 0$.)

To obtain (20) from (21), we set $\mathscr{I} = E \times E$, $A_{x,y} = P(x)^{1/2} P_\Psi P(y)^{1/2}$ with $P_\Psi = |\Psi\rangle\langle\Psi|$ the projection to $\mathbb{C}\Psi$, and $B_{x,y} = P(x)^{1/2} H P(y)^{1/2}$. Then $\operatorname{tr} A_{x,y}^* B_{x,y} = \langle\Psi|P(x)HP(y)\Psi\rangle$. To see that $A_{x,y}$ is a Hilbert–Schmidt operator, we note that

$$\operatorname{tr} A_{x,y}^* A_{x,y} = \operatorname{tr}\Big(P(y)^{1/2} P_\Psi P(x) P_\Psi P(y)^{1/2}\Big) = \langle\Psi|P(x)\Psi\rangle\langle\Psi|P(y)\Psi\rangle < \infty.$$

It follows further from (3) that

$$\sum_{x,y} \operatorname{tr} A_{x,y}^* A_{x,y} = \sum_{x,y} \langle\Psi|P(x)\Psi\rangle\langle\Psi|P(y)\Psi\rangle = \|\Psi\|^4.$$

Next we show that $B_{x,y}$ is a Hilbert–Schmidt operator. Note that $0 \leq P(x) \leq I$ since $I - P(x) = P(E \setminus \{x\}) \geq 0$. This implies that $0 \leq \langle\Phi|P(x)\Phi\rangle \leq \langle\Phi|\Phi\rangle$ for all $\Phi \in \mathscr{H}$. Setting $\Phi := C\phi_n$ for any Hilbert–Schmidt operator C and an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ of $\mathscr{H}$ we find

$$\sum_{n\in\mathbb{N}} \langle\phi_n|C^* P(x) C\phi_n\rangle \leq \sum_{n\in\mathbb{N}} \langle\phi_n|C^* C\phi_n\rangle$$

and thus

$$\operatorname{tr} C^* P(x) C \leq \operatorname{tr} C^* C. \tag{22}$$

That is, if C is a Hilbert–Schmidt operator then so is $P(x)^{1/2}C$; and so is $CP(x)^{1/2} = (P(x)^{1/2}C^*)^*$. As a consequence, $B_{x,y}$ is a Hilbert–Schmidt operator.

Finally, we need to show that

$$\sum_{x,y\in E} \operatorname{tr} B_{x,y}^* B_{x,y} \leq \operatorname{tr} H^2.$$

For every finite subset $F \subseteq E$ we have, using the linearity of the trace and its invariance under cyclic permutations,

$$\begin{aligned}\sum_{x,y\in F} \operatorname{tr} B_{x,y}^* B_{x,y} &= \sum_{x,y\in F} \operatorname{tr} HP(x)HP(y) = \operatorname{tr} HP(F)HP(F)\\ &= \operatorname{tr}\big(P(F)^{1/2}HP(F)^{1/2}\big)^*\big(P(F)^{1/2}HP(F)^{1/2}\big) \leq \operatorname{tr} H^2\,.\end{aligned}$$

The last inequality comes from the fact that, according to (22), the Hilbert–Schmidt norm $\|C\|_{HS} = \sqrt{\operatorname{tr} C^*C}$ of an operator C can only decrease when C is multiplied, from the left or from the right, by $P^{1/2}$ where $0 \leq P \leq I$. Taking the supremum over all finite subsets F and combining all inequalities above we arrive at (20).

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