Quantum Hamiltonians and Stochastic Jumps

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Abstract

With many Hamiltonians one can naturally associate a $|\Psi|^2$-distributed Markov process. For nonrelativistic quantum mechanics, this process is in fact deterministic, and is known as Bohmian mechanics. For the Hamiltonian of a quantum field theory, it is typically a jump process on the configuration space of a variable number of particles. We define these processes for regularized quantum field theories, thereby generalizing previous work of John S. Bell [3] and of ourselves [11]. We introduce a formula expressing the jump rates in terms of the interaction Hamiltonian, and establish a condition for finiteness of the rates.

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Contents

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1 Introduction

The central formula of this paper is

$$\sigma(dq|q') = \frac{[(2/\hbar)\,\mathrm{Im}\,\langle\Psi|P(dq)H\,P(dq')|\Psi\rangle]^{+}}{\langle\Psi|P(dq')|\Psi\rangle}\,. \tag{1}$$

It plays a role similar to that of Bohm's equation of motion (2). Together, these two equations make possible a formulation of quantum field theory (QFT) that makes no

reference to observers or measurements, while implying that observers, when making measurements, will arrive at precisely the results that QFT is known to predict. Special cases of formula (1) have been utilized before [3, 11, 31]. Part of what we explain in this paper is what this formula means, how to arrive at it, when it can be applied, and what its consequences are. Such a formulation of QFT takes up ideas from the seminal paper of John S. Bell [3], and we will often refer to theories similar to the model suggested by Bell in [3] as "Bell-type QFTs". (What similar means here will be fleshed out in the course of this paper.)

The aim of this paper is to define a canonical Bell-type model for more or less any regularized QFT. We assume a well-defined Hamiltonian as given; to achieve this, it is often necessary to introduce cut-offs. We shall assume this has been done where needed. In cases in which one has to choose between several possible position observables, for example because of issues related to the Newton–Wigner operator [26, 19], we shall also assume that a choice has been made.

The primary variables of Bell-type QFTs are the positions of the particles. Bell suggested a dynamical law, governing the motion of the particles, in which the Hamiltonian H and the state vector Ψ determine the jump rates σ. We point out how Bell's rates fit naturally into a more general scheme summarized by (1). Since these rates are in a sense the smallest choice possible (as explained in Section 5), we call them the *minimal jump rates*. By construction, they preserve the $|\Psi|^2$ distribution. Most of this paper concerns the properties and mathematical foundations of minimal jump rates. In Bell-type QFTs, which can be regarded as extensions of Bohmian mechanics, the stochastic jumps often correspond to the creation and annihilation of particles. We will discuss further aspects of Bell-type QFTs and their construction in our forthcoming work [12].

The paper is organized as follows. In Section 2 we introduce all the main ideas and reasonings; a superficial reading should focus on this section. Some examples of processes defined by minimal jump rates are presented in Section 3. In Section 4 we provide conditions for the rigorous existence and finiteness of the minimal jump rates. In Section 5 we explain in what sense the rates (1) are minimal. Section 6 concerns further properties of processes defined by minimal jump rates. In Section 7 we conclude.

2 The Jump Rate Formula

2.1 Review of Bohmian Mechanics and Equivariance

Bohmian mechanics [4, 14, 16] is a non-relativistic theory about N point particles moving in 3-space, according to which the configuration $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N)$ evolves according to[1]

$$\frac{dQ}{dt} = v(Q), \qquad v = \hbar \operatorname{Im} \frac{\Psi^* \nabla \Psi}{\Psi^* \Psi}. \tag{2}$$

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[1]The masses m_k of the particles have been absorbed in the Riemann metric $g_{\mu\nu}$ on configuration space $\mathbb{R}^{3N}$, $g_{ia,jb} = m_i \, \delta_{ij} \, \delta_{ab}$, $i, j = 1 \ldots N$, $a, b = 1, 2, 3$, and ∇ is the gradient associated with $g_{\mu\nu}$, i.e., $\nabla = (m_1^{-1} \nabla_{\boldsymbol{q}_1}, \ldots, m_N^{-1} \nabla_{\boldsymbol{q}_N})$.

$\Psi = \Psi_t(q)$ is the wave function, which evolves according to the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \,, \tag{3}$$

with

$$H = -\frac{\hbar^2}{2}\Delta + V \tag{4}$$

for spinless particles, with $\Delta = \operatorname{div} \nabla$. For particles with spin, Ψ takes values in the appropriate spin space $\mathbb{C}^k$, V may be matrix valued, and numerator and denominator of (2) have to be understood as involving inner products in spin space. The secret of the success of Bohmian mechanics in yielding the predictions of standard quantum mechanics is the fact that the configuration Q_t is $|\Psi_t|^2$-distributed in configuration space at all times t, provided that the initial configuration Q_0 (part of the Cauchy data of the theory) is so distributed. This property, called *equivariance* in [14], suffices for empirical agreement between *any* quantum theory (such as a QFT) and *any* version thereof with additional (often called "hidden") variables Q, provided the outcomes of all experiments are registered or recorded in these variables. That is why equivariance will be our guide for obtaining the dynamics of the particles.

The equivariance of Bohmian mechanics follows immediately from comparing the continuity equation for a probability distribution ρ associated with (2),

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho v) \,, \tag{5}$$

with the equation satisfied by $|\Psi|^2$ which follows from (3),

$$\frac{\partial |\Psi|^2}{\partial t}(q,t) = \frac{2}{\hbar}\operatorname{Im}\left[\Psi^*(q,t)\,(H\Psi)(q,t)\right]. \tag{6}$$

In fact, it follows from (4) that

$$\frac{2}{\hbar}\operatorname{Im}\left[\Psi^*(q,t)\,(H\Psi)(q,t)\right] = -\operatorname{div}\left[\hbar\operatorname{Im}\Psi^*(q,t)\nabla\Psi(q,t)\right] \tag{7}$$

so, recalling (2), one obtains that

$$\frac{\partial |\Psi|^2}{\partial t} = -\operatorname{div}\left(|\Psi|^2 v\right), \tag{8}$$

and hence that if $\rho_t = |\Psi_t|^2$ at some time t then $\rho_t = |\Psi_t|^2$ for *all* times. Equivariance is an expression of the compatibility between the Schrödinger evolution for the wave function and the law, such as (2), governing the motion of the actual configuration. In [14], in which we were concerned only with the Bohmian dynamics (2), we spoke of the distribution $|\Psi|^2$ as being equivariant. Here we wish to find processes for which we have equivariance, and we shall therefore speak of equivariant processes and motions.

2.2 Equivariant Markov Processes

The study of example QFTs like that of [11] has lead us to the consideration of Markov processes as candidates for the equivariant motion of the configuration Q for Hamiltonians H more general than those of the form (4).

Consider a Markov process Q_t on configuration space. The transition probabilities are characterized by the *backward generator* L_t, a (time-dependent) linear operator acting on functions f on configuration space:

$$L_t f(q) = \frac{d}{ds} \mathbb{E}(f(Q_{t+s})|Q_t = q) \tag{9}$$

where d/ds means the right derivative at $s = 0$ and $\mathbb{E}(\,\cdot\,|\,\cdot\,)$ denotes the conditional expectation. Equivalently, the transition probabilities are characterized by the *forward generator* $\mathscr{L}_t$ (or, as we shall simply say, *generator*), which is also a linear operator but acts on (signed) measures on the configuration space. Its defining property is that for every process Q_t with the given transition probabilities, the distribution ρ_t of Q_t evolves according to

$$\frac{\partial \rho_t}{\partial t} = \mathscr{L}_t \rho_t \,. \tag{10}$$

$\mathscr{L}_t$ is the dual of L_t in the sense that

$$\int f(q)\, \mathscr{L}_t \rho(dq) = \int L_t f(q)\, \rho(dq) \,. \tag{11}$$

We will use both L_t and $\mathscr{L}_t$, whichever is more convenient. We will encounter several examples of generators in the subsequent sections.

We can easily extend the notion of equivariance from deterministic to Markov processes. Given the Markov transition probabilities, we say that *the $|\Psi|^2$ distribution is equivariant* if and only if for all times t and t' with $t < t'$, a configuration Q_t with distribution $|\Psi_t|^2$ evolves, according to the transition probabilities, into a configuration $Q_{t'}$ with distribution $|\Psi_{t'}|^2$. In this case, we also simply say that the transition probabilities are *equivariant*, without explicitly mentioning $|\Psi|^2$. Equivariance is equivalent to

$$\mathscr{L}_t |\Psi_t|^2 = \frac{\partial |\Psi_t|^2}{\partial t} \tag{12}$$

for all t. When (12) holds (for a fixed t) we also say that $\mathscr{L}_t$ is an *equivariant generator* (with respect to Ψ_t and H). Note that this definition of equivariance agrees with the previous meaning for deterministic processes.

We call a Markov process Q *equivariant* if and only if for every t the distribution ρ_t of Q_t equals $|\Psi_t|^2$. For this to be the case, equivariant transition probabilities are necessary but not sufficient. (While for a Markov process Q to have equivariant transition probabilities amounts to the property that if $\rho_t = |\Psi_t|^2$ for one time t, where ρ_t denotes the distribution of Q_t, then $\rho_{t'} = |\Psi_{t'}|^2$ for every $t' > t$, according to our definition of an equivariant Markov process, in fact $\rho_t = |\Psi_t|^2$ for all t.) However, for equivariant transition probabilities there exists a unique equivariant Markov process.

The crucial idea for our construction of an equivariant Markov process is to note that (6) is completely general, and to find a generator $\mathscr{L}_t$ such that the right hand side of (6) can be read as the action of $\mathscr{L}$ on $\rho = |\Psi|^2$,

$$\frac{2}{\hbar} \operatorname{Im} \Psi^* H \Psi = \mathscr{L} |\Psi|^2 \,. \tag{13}$$

We shall implement this idea beginning in Section 2.4, after a review of jump processes and some general considerations. But first we shall illustrate the idea with the familiar case of Bohmian mechanics.

For H of the form (4), we have (7) and hence that

$$\frac{2}{\hbar} \operatorname{Im} \Psi^* H \Psi = - \operatorname{div} \left(\hbar \operatorname{Im} \Psi^* \nabla \Psi \right) = - \operatorname{div} \left(|\Psi|^2 \hbar \operatorname{Im} \frac{\Psi^* \nabla \Psi}{|\Psi|^2} \right) \,. \tag{14}$$

Since the generator of the (deterministic) Markov process corresponding to the dynamical system $dQ/dt = v(Q)$ given by a velocity vector field v is

$$\mathscr{L} \rho = - \operatorname{div} \left(\rho v \right) \,, \tag{15}$$

we may recognize the last term of (14) as $\mathscr{L} |\Psi|^2$ with $\mathscr{L}$ the generator of the deterministic process defined by (2). Thus, as is well known, Bohmian mechanics arises as the natural equivariant process on configuration space associated with H and Ψ.

To be sure, Bohmian mechanics is not the only solution of (13) for H given by (4). Among the alternatives are Nelson's stochastic mechanics [25] and other velocity formulas [8]. However, Bohmian mechanics is the most natural choice, the one most likely to be relevant to physics. (It is, in fact, the canonical choice, in the sense of minimal process which we shall explain in [12, Sec. 5.2].)

An important class of equivariant Markov processes are equivariant jump processes, which we discuss in the next three sections. They arise naturally in QFT, as we shall explain in Section 2.6.

2.3 Equivariant Jump Processes

Let $\mathcal{Q}$ denote the configuration space of the process, whatever sort of space that may be (vector space, lattice, manifold, etc.); mathematically speaking, we need that $\mathcal{Q}$ be a measurable space. A (pure) jump process is a Markov process on $\mathcal{Q}$ for which the only motion that occurs is via jumps. Given that $Q_t = q$, the probability for a jump to q', i.e., into the infinitesimal volume dq' about q', by time $t + dt$ is $\sigma_t(dq'|q) \, dt$, where σ is called the *jump rate*. In this notation, σ is a finite measure in the first variable; $\sigma(B|q)$ is the rate (the probability per unit time) of jumping to somewhere in the set $B \subseteq \mathcal{Q}$, given that the present location is q. The overall jump rate is $\sigma(\mathcal{Q}|q)$.

It is often the case that $\mathcal{Q}$ is equipped with a distinguished measure, which we shall denote by dq or dq', slightly abusing notation. For example, if $\mathcal{Q} = \mathbb{R}^d$, dq may be the Lebesgue measure, or if $\mathcal{Q}$ is a Riemannian manifold, dq may be the Riemannian volume element. When $\sigma(\,\cdot\,|q)$ is absolutely continuous relative to the distinguished measure,

we also write $\sigma(q'|q)\,dq'$ instead of $\sigma(dq'|q)$. Similarly, we sometimes use the letter ρ for denoting a measure and sometimes the density of a measure, $\rho(dq) = \rho(q)\,dq$.

A jump first occurs when a random waiting time T has elapsed, after the time t_0 at which the process was started or at which the most recent previous jump has occurred. For purposes of simulating or constructing the process, the destination q' can be chosen at the time of jumping, $t_0 + T$, with probability distribution $\sigma_{t_0+T}(\mathcal{Q}|q)^{-1}\,\sigma_{t_0+T}(\,\cdot\,|q)$. In case the overall jump rate is time-independent, T is exponentially distributed with mean $\sigma(\mathcal{Q}|q)^{-1}$. When the rates are time-dependent—as they will typically be in what follows—the waiting time remains such that

$$\int_{t_0}^{t_0+T} \sigma_t(\mathcal{Q}|q)\,dt$$

is exponentially distributed with mean 1, i.e., T becomes exponential after a suitable (time-dependent) rescaling of time. For more details about jump processes, see [6].

The generator of a pure jump process can be expressed in terms of the rates:

$$\mathscr{L}_\sigma \rho(dq) = \int_{q'\in\mathcal{Q}} \Big(\sigma(dq|q')\rho(dq') - \sigma(dq'|q)\rho(dq) \Big)\,, \tag{16}$$

a "balance" or "master" equation expressing $\partial\rho/\partial t$ as the gain due to jumps to dq minus the loss due to jumps away from q.

We shall say that jump rates σ are *equivariant* if $\mathscr{L}_\sigma$ is an equivariant generator. It is one of our goals in this paper to describe a general scheme for obtaining equivariant jump rates. In Sections 2.4 and 2.5 we will explain how this leads us to formula (1).

2.4   Integral Operators Correspond to Jump Processes

What characterizes jump processes versus continuous processes is that some amount of probability that vanishes at $q \in \mathcal{Q}$ can reappear in an entirely different region of configuration space, say at $q' \in \mathcal{Q}$. This is manifest in the equation for $\partial\rho/\partial t$, (16): the first term in the integrand is the probability increase due to arriving jumps, the second the decrease due to departing jumps, and the integration over q' reflects that q' can be anywhere in $\mathcal{Q}$. This suggests that Hamiltonians for which the expression (6) for $\partial|\Psi|^2/\partial t$ is naturally an integral over dq' correspond to pure jump processes. So when is the left hand side of (13) an integral over dq'? When H is an integral operator, i.e., when $\langle q|H|q'\rangle$ is not merely a formal symbol, but represents an integral kernel that exists as a function or a measure and satisfies

$$(H\Psi)(q) = \int dq'\,\langle q|H|q'\rangle\,\Psi(q')\,. \tag{17}$$

(For the time being, think of $\mathcal{Q}$ as $\mathbb{R}^d$ and of wave functions as complex valued.) In this case, we should choose the jump rates in such a way that, when $\rho = |\Psi|^2$,

$$\sigma(q|q')\,\rho(q') - \sigma(q'|q)\,\rho(q) = \frac{2}{\hbar}\,\mathrm{Im}\,\Psi^*(q)\,\langle q|H|q'\rangle\,\Psi(q')\,, \tag{18}$$

7

and this suggests, since jump rates must be nonnegative (and the right hand side of (18) is anti-symmetric), that

$$\sigma(q|q')\,\rho(q') = \left[\frac{2}{\hbar}\,\mathrm{Im}\,\Psi^*(q)\,\langle q|H|q'\rangle\,\Psi(q')\right]^+$$

(where x^+ denotes the positive part of $x \in \mathbb{R}$, that is, x^+ is equal to x for $x > 0$ and is zero otherwise), or

$$\sigma(q|q') = \frac{\left[(2/\hbar)\,\mathrm{Im}\,\Psi^*(q)\,\langle q|H|q'\rangle\,\Psi(q')\right]^+}{\Psi^*(q')\,\Psi(q')}. \qquad (19)$$

These rates are an instance of what we call the *minimal jump rates* associated with H (and Ψ). They are also an instance of formula (1), as will become clear in the following section. The name comes from the fact that they are actually the minimal possible values given (18), as is expressed by the inequality (96) and will be explained in detail in Section 5. Minimality entails that at any time t, one of the transitions $q_1 \to q_2$ or $q_2 \to q_1$ is forbidden. We will call the process defined by the minimal jump rates the *minimal jump process* (associated with H).

In contrast to jump processes, continuous motion, as in Bohmian mechanics, corresponds to such Hamiltonians that the formal matrix elements $\langle q|H|q'\rangle$ are nonzero only infinitesimally close to the diagonal, and in particular to differential operators like the Schrödinger Hamiltonian (4), which has matrix elements of the type $\delta''(q - q') + V(q)\,\delta(q - q')$.

The minimal jump rates as given by (19) have some nice features. The possible jumps for this process correspond to the nonvanishing matrix elements of H (though, depending on the state Ψ, even some of the jump rates corresponding to nonvanishing matrix elements of H might happen to vanish). Moreover, in their dependence on the state Ψ, the jump rates σ depend only "locally" upon Ψ: the jump rate for a given jump $q' \to q$ depends only on the values $\Psi(q')$ and $\Psi(q)$ corresponding to the configurations linked by that jump. Discretizing $\mathbb{R}^3$ to a lattice $\varepsilon\mathbb{Z}^3$, one can obtain Bohmian mechanics as a limit $\varepsilon \to 0$ of minimal jump processes [31, 32], whereas greater-than-minimal jump rates lead to Nelson's stochastic mechanics [25] and similar diffusions; see [32, 17]. If the Schrödinger operator (4) is approximated in other ways by operators corresponding to jump processes, e.g., by $H_\varepsilon = e^{-\varepsilon H}He^{-\varepsilon H}$, the minimal jump processes presumably also converge to Bohmian mechanics.

We have reason to believe that there are lots of self-adjoint operators which do not correspond to any stochastic process that can be regarded as defined, in any reasonable sense, by (19).[2] But such operators seem never to occur in QFT. (The Klein–Gordon operator $\sqrt{m^2c^4 - \hbar^2c^2\Delta}$ does seem to have a process, but it requires a more detailed discussion which will be provided in a forthcoming work [13].)

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[2]Consider, for example, $H = p\cos p$ where p is the one-dimensional momentum operator $-i\hbar\partial/\partial q$. Its formal kernel $\langle q|H|q'\rangle$ is the distribution $-\frac{i}{2}\delta'(q - q' - 1) - \frac{i}{2}\delta'(q - q' + 1)$, for which (19) would not have a meaning. From a sequence of smooth functions converging to this distribution, one can obtain a sequence of jump processes with rates (19): the jumps occur very frequently, and are by amounts of approximately ± 1. A limiting process, however, does not exist.

2.5   Minimal Jump Rates

The reasoning of the previous section applies to a far more general setting than just considered: to arbitrary configuration spaces $\mathcal{Q}$ and "generalized observables"—POVMs—defining, for our purposes, what the "position representation" is. We now present this more general reasoning, which leads to formula (1).

The process we construct relies on the following ingredients from QFT:

1. A Hilbert space $\mathscr{H}$ with scalar product $\langle \Psi | \Phi \rangle$.

2. A unitary one-parameter group U_t in $\mathscr{H}$ with Hamiltonian H,

$$U_t = e^{-\frac{i}{\hbar} tH} \, ,$$

so that in the Schrödinger picture the state Ψ evolves according to

$$i\hbar \frac{d\Psi_t}{dt} = H\Psi_t \, . \tag{20}$$

U_t could be part of a representation of the Poincaré group.

3. A positive-operator-valued measure (POVM) $P(dq)$ on $\mathcal{Q}$ acting on $\mathscr{H}$, so that the probability that the system in the state Ψ is localized in dq at time t is

$$\mathbb{P}_t(dq) = \langle \Psi_t | P(dq) | \Psi_t \rangle \, . \tag{21}$$

Mathematically, a POVM P on $\mathcal{Q}$ is a countably additive set function ("measure"), defined on measurable subsets of $\mathcal{Q}$, with values in the positive (bounded self-adjoint) operators on (a Hilbert space) $\mathscr{H}$, such that $P(\mathcal{Q})$ is the identity operator.[3] Physically, for our purposes, $P(\,\cdot\,)$ represents the (generalized) position observable, with values in $\mathcal{Q}$. The notion of POVM generalizes the more familiar situation of observables given by a set of commuting self-adjoint operators, corresponding, by means of the spectral theorem, to a projection-valued measure (PVM): the case where the positive operators are projection operators. A typical example is the single Dirac particle: the position operators on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ induce there a natural PVM $P_0(\,\cdot\,)$: for any Borel set $B \subseteq \mathbb{R}^3$, $P_0(B)$ is the projection to the subspace of functions that vanish outside B, or, equivalently, $P_0(B)\Psi(q) = \mathbf{1}_B(q)\,\Psi(q)$ with $\mathbf{1}_B$ the indicator function of the set B. Thus, $\langle \Psi | P_0(dq) | \Psi \rangle = |\Psi(q)|^2 dq$. When one considers as Hilbert space $\mathscr{H}$ only the subspace of positive energy states, however, the localization probability is given by $P(\,\cdot\,) = P_+ P_0(\,\cdot\,)I$ with $P_+ : L^2(\mathbb{R}^3, \mathbb{C}^4) \to \mathscr{H}$ the projection and $I : \mathscr{H} \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ the inclusion mapping. Since P_+ does not commute with most of the operators $P_0(B)$, $P(\,\cdot\,)$ is no longer a PVM but a genuine POVM[4] and consequently does not correspond to

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[3]The countable additivity is to be understood as in the sense of the weak operator topology. This in fact implies that countable additivity also holds in the strong topology.

[4]This situation is indeed more general than it may seem. By a theorem of Naimark [7, p. 142], every POVM $P(\,\cdot\,)$ acting on $\mathscr{H}$ is of the form $P(\,\cdot\,) = P_+ P_0(\,\cdot\,) P_+$ where P_0 is a PVM on a larger Hilbert space, and P_+ the projection to $\mathscr{H}$.

any position operator—although it remains true (for Ψ in the positive energy subspace) that $\langle\Psi|P(dq)|\Psi\rangle = |\Psi(q)|^2 dq$. That is why in QFT, the position observable is indeed more often a POVM than a PVM. POVMs are also relevant to photons [1, 22]. In one approach, the photon wave function $\Psi : \mathbb{R}^3 \to \mathbb{C}^3$ is subject to the constraint condition $\nabla\cdot\Psi = \partial_1\Psi_1 + \partial_2\Psi_2 + \partial_3\Psi_3 = 0$. Thus, the physical Hilbert space $\mathscr{H}$ is the (closure of the) subspace of $L^2(\mathbb{R}^3, \mathbb{C}^3)$ defined by this constraint, and the natural PVM on $L^2(\mathbb{R}^3, \mathbb{C}^3)$ gives rise, by projection, to a POVM on $\mathscr{H}$. So much for POVMs. Let us get back to the construction of a jump process.

The goal is to specify equivariant jump rates $\sigma = \sigma^{\Psi, H, P}$, i.e., such rates that

$$\mathscr{L}_\sigma \mathbb{P} = \frac{d\mathbb{P}}{dt}\,. \tag{22}$$

To this end, one may take the following steps:

1. Note that
$$\frac{d\mathbb{P}_t(dq)}{dt} = \frac{2}{\hbar}\operatorname{Im}\langle\Psi_t|P(dq)H|\Psi_t\rangle\,. \tag{23}$$

2. Insert the resolution of the identity $I = \int\limits_{q'\in\mathcal{Q}} P(dq')$ and obtain

$$\frac{d\mathbb{P}_t(dq)}{dt} = \int\limits_{q'\in\mathcal{Q}} \mathbb{J}_t(dq, dq')\,, \tag{24}$$

where

$$\mathbb{J}_t(dq, dq') = \frac{2}{\hbar}\operatorname{Im}\langle\Psi_t|P(dq)HP(dq')|\Psi_t\rangle\,. \tag{25}$$

3. Observe that $\mathbb{J}$ is anti-symmetric, $\mathbb{J}(dq', dq) = -\mathbb{J}(dq, dq')$. Thus, since $x = x^+ - (-x)^+$,

$$\mathbb{J}(dq, dq') = [(2/\hbar)\operatorname{Im}\langle\Psi|P(dq)HP(dq')|\Psi\rangle]^+ - [(2/\hbar)\operatorname{Im}\langle\Psi|P(dq')HP(dq)|\Psi\rangle]^+\,.$$

4. Multiply and divide both terms by $\mathbb{P}(\,\cdot\,)$, obtaining that

$$\int\limits_{q'\in\mathcal{Q}} \mathbb{J}(dq, dq') = \int\limits_{q'\in\mathcal{Q}} \Bigg(\frac{[(2/\hbar)\operatorname{Im}\langle\Psi|P(dq)HP(dq')|\Psi\rangle]^+}{\langle\Psi|P(dq')|\Psi\rangle}\mathbb{P}(dq') -$$
$$- \frac{[(2/\hbar)\operatorname{Im}\langle\Psi|P(dq')HP(dq)|\Psi\rangle]^+}{\langle\Psi|P(dq)|\Psi\rangle}\mathbb{P}(dq) \Bigg)\,.$$

5. By comparison with (16), recognize the right hand side of the above equation as $\mathscr{L}_\sigma\mathbb{P}$, with $\mathscr{L}_\sigma$ the generator of a Markov jump process with jump rates (1), which we call the *minimal jump rates*. We repeat the formula for convenience:

$$\sigma(dq|q') = \frac{[(2/\hbar)\operatorname{Im}\langle\Psi|P(dq)HP(dq')|\Psi\rangle]^+}{\langle\Psi|P(dq')|\Psi\rangle}\,.$$

Mathematically, the right hand side of this formula as a function of q' must be understood as a density (Radon–Nikodým derivative) of one measure relative to another. The plus symbol denotes the positive part of a signed measure; it can also be understood as applying the plus function, $x^+ = \max(x, 0)$, to the density, if it exists, of the numerator.

To sum up, we have argued that with H and Ψ is naturally associated a Markov jump process Q_t whose marginal distributions coincide at all times by construction with the quantum probability measure, $\rho_t(\,\cdot\,) = \mathbb{P}_t(\,\cdot\,)$, so that Q_t is an equivariant Markov process.

In Section 4, we establish precise conditions on H, P, and Ψ under which the jump rates (1) are well-defined and finite $\mathbb{P}$-almost everywhere, and prove that in this case the rates are equivariant, as suggested by the steps 1-5 above. It is perhaps worth remarking at this point that any H can be approximated by Hamiltonians H_n (namely Hilbert–Schmidt operators) for which the rates (1) are always (for all Ψ) well-defined and equivariant, as we shall prove in Section 4.2.1.

2.6 Bell-Type QFT

A Bell-type QFT is about particles moving in physical 3-space; their number and positions are represented by a point Q_t in configuration space $\mathcal{Q}$, with $\mathcal{Q}$ defined as follows. Let $\Gamma\mathbb{R}^3$ denote the configuration space of a variable (but finite) number of identical particles in $\mathbb{R}^3$, i.e., the union of $(\mathbb{R}^3)^n$ modulo permutations,

$$\Gamma\mathbb{R}^3 = \bigcup_{n=0}^{\infty} (\mathbb{R}^3)^n / S_n \,. \tag{26}$$

$\mathcal{Q}$ is the Cartesian product of several copies of $\Gamma\mathbb{R}^3$, one for each species of particles. For a discussion of the space $\Gamma\mathbb{R}^3$, and indeed of ΓS for any other measurable space S playing the role of physical space, see [12, Sec. 2.8].

A related space, for which we write $\Gamma_{\neq}\mathbb{R}^3$, is the space of all finite subsets of $\mathbb{R}^3$; it is contained in $\Gamma\mathbb{R}^3$, after obvious identifications. In fact, $\Gamma_{\neq}\mathbb{R}^3 = \Gamma\mathbb{R}^3 \setminus \Delta$, where Δ is the set of coincidence configurations, i.e., those having two or more particles at the same position. $\Gamma_{\neq}\mathbb{R}^3$ is the union of the spaces $\mathcal{Q}_{\neq}^{(n)}$ for $n = 0, 1, 2, \ldots$, where $\mathcal{Q}_{\neq}^{(n)}$ is the space of subsets of $\mathbb{R}^3$ with n elements, a manifold of dimension $3n$ (see [10] for a discussion of Bohmian mechanics on this manifold). The set Δ of coincidence configurations has codimension 3 and thus can usually be ignored. We can thus replace $\Gamma\mathbb{R}^3$ by the somewhat simpler space $\Gamma_{\neq}\mathbb{R}^3$.

Q_t follows a Markov process in $\mathcal{Q}$, which is governed by a state vector Ψ in a suitable Hilbert space $\mathscr{H}$. $\mathscr{H}$ is related to $\mathcal{Q}$ by means of a PVM or POVM P.

The Hamiltonian of a QFT usually comes as a sum, such as

$$H = H_0 + H_I \tag{27}$$

with H_0 the free Hamiltonian and H_I the interaction Hamiltonian. If several particle species are involved, H_0 is itself a sum containing one free Hamiltonian for each species.

The left hand side of (13), which should govern our choice of the generator, is then also a sum,

$$\frac{2}{\hbar} \operatorname{Im} \Psi^* H_0 \Psi + \frac{2}{\hbar} \operatorname{Im} \Psi^* H_I \Psi = \mathscr{L} |\Psi|^2 \,. \tag{28}$$

This opens the possibility of finding a generator $\mathscr{L}$ by setting $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I$, provided we have generators $\mathscr{L}_0$ and $\mathscr{L}_I$ corresponding to H_0 and H_I in the sense that

$$\frac{2}{\hbar} \operatorname{Im} \Psi^* H_0 \Psi = \mathscr{L}_0 |\Psi|^2 \tag{29a}$$

$$\frac{2}{\hbar} \operatorname{Im} \Psi^* H_I \Psi = \mathscr{L}_I |\Psi|^2 \,. \tag{29b}$$

This feature of (13) we call *process additivity*; it is based on the fact that the left hand side of (13) is linear in H.

In a Bell-type QFT, the generator $\mathscr{L}$ is of the form $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I$, where $\mathscr{L}_0$ is usually the generator of a deterministic process, usually defined by the Bohmian or Bohm–Dirac law of motion, see below, and $\mathscr{L}_I$ is the generator of a pure jump process, which is our main focus in this paper. The process generated by $\mathscr{L}$ is then given by deterministic motion determined by $\mathscr{L}_0$, randomly interrupted by jumps at a rate determined by $\mathscr{L}_I$.

We thus need to define two equivariant processes, one (the "free process") associated with H_0 and the other (the "interaction process") with H_I. The interaction process is the pure jump process with rates given by (1) with H_I in place of H. We now give a description of the free process for the two most relevant free Hamiltonians: the second-quantized Schrödinger operator and the second-quantized Dirac operator. We give a more general and more detailed discussion of free processes in [12]; there we provide a formula, roughly analogous to (1), for $\mathscr{L}_0$ in terms of H_0, and an algorithm for obtaining the free process from a one-particle process that is roughly analogous to the "second quantization" procedure for obtaining H_0 from a one-particle Hamiltonian.

The free process associated with a second-quantized Schrödinger operator arises from Bohmian mechanics. Fock space $\mathscr{H} = \mathscr{F}$ is a direct sum

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}^{(n)}, \tag{30}$$

where $\mathscr{F}^{(n)}$ is the n-particle Hilbert space. $\mathscr{F}^{(n)}$ is the subspace of symmetric (for bosons) or anti-symmetric (for fermions) functions in $L^2(\mathbb{R}^{3n}, (\mathbb{C}^{2s+1})^{\otimes n})$ for spin-s particles. Thus, $\Psi \in \mathscr{F}$ can be decomposed into a sequence $\Psi = (\Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(n)}, \ldots)$, the n-th member $\Psi^{(n)}$ being an n-particle wave function, the wave function representing the n-particle sector of the quantum state vector. The obvious way to obtain a process on $\mathcal{Q} = \Gamma \mathbb{R}^3$ is to let the configuration $Q(t)$, containing $N = \#Q(t)$ particles, move according to the N-particle version of Bohm's law (2), guided by $\Psi^{(N)}$.[5] This is

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[5] As defined, configurations are unordered, whereas we have written Bohm's law (2) for ordered configurations. Thanks to the (anti-)symmetry of the wave function, however, all orderings will lead to the same particle motion. For more about such considerations, see our forthcoming work [10].

indeed an equivariant process since H_0 has a block diagonal form with respect to the decomposition (30),

$$H_0 = \bigoplus_{n=0}^{\infty} H_0^{(n)} \, ,$$

and $H_0^{(n)}$ is just a Schrödinger operator for n noninteracting particles, for which, as we already know, Bohmian mechanics is equivariant. We used a very similar process in [11] (the only difference being that particles were numbered in [11]).

Similarly, if H_0 is the second quantized Dirac operator, we let a configuration Q with N particles move according to the usual N-particle Bohm–Dirac law [5, p. 274]

$$\frac{dQ}{dt} = c \frac{\Psi^*(Q) \, \alpha_N \, \Psi(Q)}{\Psi^*(Q) \, \Psi(Q)} \tag{31}$$

where c denotes the speed of light and $\alpha_N = (\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(N)})$ with $\boldsymbol{\alpha}^{(k)}$ acting on the spin index of the k-th particle.

This completes the construction of the Bell-type QFT. An explicit example of a Bell-type process for a simple QFT is described in [11], which we take up again in Section 3.12 below to point out how its jump rates fit into the scheme (1). Another such example, concerning electron–positron pair creation in an external electromagnetic field, is described in [12, Sec. 3.3.].

3 Examples

In this section, we present various special cases of the jump rate formula (1) and examples of its application. We also point out how the jump rates of the models in [11] and [3] are contained in (1).

3.1 A First Example

To begin with, we consider $\mathcal{Q} = \mathbb{R}^d$, $\mathscr{H} = L^2(\mathbb{R}^d, \mathbb{C})$, and P the natural PVM, which may be written $P(dq) = |q\rangle\langle q| \, dq$. Then, $\mathbb{P}(dq) = \langle \Psi | P(dq) | \Psi \rangle = |\Psi(q)|^2 dq$, and the jump rate formula (1) reads

$$\sigma(q|q') = \frac{[(2/\hbar) \, \mathrm{Im} \, \Psi^*(q) \, \langle q|H|q'\rangle \, \Psi(q')]^+}{\Psi^*(q') \, \Psi(q')} \tag{32a}$$

$$= \left[\frac{2}{\hbar} \, \mathrm{Im} \, \frac{\Psi^*(q) \, \langle q|H|q'\rangle}{\Psi^*(q')} \right]^+ . \tag{32b}$$

Note that (32a) is the same expression as (19). As a simple example of an operator H_I with a kernel, consider a convolution operator, $H_I = V\star$, where V may be complex-valued and $V(-q) = V^*(q)$,

$$(H_I \Psi)(q) = \int V(q - q') \, \Psi(q') \, dq' \, .$$

13

The kernel is $\langle q|H_I|q'\rangle = V(q-q')$. Together with $H_0 = -\frac{\hbar^2}{2}\Delta$, we obtain a baby example of a Hamiltonian $H = H_0 + H_I$ that goes beyond the form (4) of Schrödinger operators, in particular in that it is no longer local in configuration space. Recall that H_0 is associated with the Bohmian motion (2). Combining the two generators on the basis of process additivity, we obtain a process that is piecewise deterministic, with jump rates (19) and Bohmian trajectories between successive jumps.

3.2 Wave Functions with Spin

Let us next become a bit more general and consider wave functions with spin, i.e., with values in $\mathbb{C}^k$. We have $\mathcal{Q} = \mathbb{R}^d$, $\mathscr{H} = L^2(\mathbb{R}^d, \mathbb{C}^k)$ and P the natural PVM, which may be written $P(dq) = \sum_{i=1}^k |q,i\rangle\langle q,i|\, dq$, where i indexes the standard basis of $\mathbb{C}^k$. Another way of viewing P is to understand $\mathscr{H}$ as the tensor product $L^2(\mathbb{R}^d, \mathbb{C}) \otimes \mathbb{C}^k$, and $P(dq) = P_0(dq) \otimes I_{\mathbb{C}^k}$ with P_0 the natural PVM on $L^2(\mathbb{R}^d, \mathbb{C})$ and $I_{\mathbb{C}^k}$ the identity operator on $\mathbb{C}^k$. Using the notation $\langle\!\langle \Phi(q)|\Psi(q)\rangle\!\rangle$ for the scalar product in $\mathbb{C}^k$, we can write $\mathbb{P}(dq) = \langle\Psi|P(dq)|\Psi\rangle = \langle\!\langle \Psi(q)|\Psi(q)\rangle\!\rangle\, dq$, and the jump rate formula (1) reads

$$\sigma(q|q') = \frac{[(2/\hbar)\,\mathrm{Im}\,\langle\!\langle \Psi(q)|K(q,q')|\Psi(q')\rangle\!\rangle]^+}{\langle\!\langle \Psi(q')|\Psi(q')\rangle\!\rangle} \tag{33}$$

with $K(q,q')$, the kernel of H, a $k \times k$ matrix. If we write $\Phi^*(q)\,\Psi(q)$ for $\langle\!\langle \Phi(q)|\Psi(q)\rangle\!\rangle$, as we did in (2) and (31), and $\langle q|H|q'\rangle$ for $K(q,q')$, (33) reads

$$\sigma(q|q') = \frac{[(2/\hbar)\,\mathrm{Im}\,\Psi^*(q)\,\langle q|H|q'\rangle\,\Psi(q')]^+}{\Psi^*(q')\,\Psi(q')}\,,$$

which is (19) again, interpreted in a different way.

3.3 Vector Bundles

Next consider, instead of the fixed value space $\mathbb{C}^k$, a vector bundle E over a Riemannian manifold $\mathcal{Q}$, and cross-sections of E as wave functions. In order to have a scalar product of wave functions, we need that every bundle fiber E_q be equipped with a Hermitian inner product $\langle\!\langle \cdot | \cdot \rangle\!\rangle_q$. We consider $\mathscr{H} = L^2(E)$ (the space of square-integrable cross-sections) and P the natural PVM. For any q and q', $K(q,q')$ then has to be a $\mathbb{C}$-linear mapping $E_{q'} \to E_q$, so that the kernel of H is a cross-section of the bundle $\bigcup_{q,q'} E_q \otimes E_{q'}^*$ over $\mathcal{Q} \times \mathcal{Q}$. (1) then reads

$$\sigma(q|q') = \frac{[(2/\hbar)\,\mathrm{Im}\,\langle\!\langle \Psi(q)|K(q,q')\,\Psi(q')\rangle\!\rangle_q]^+}{\langle\!\langle \Psi(q')|\Psi(q')\rangle\!\rangle_{q'}}\,. \tag{34}$$

In the following we will use the notation $\Phi^*(q)\,\Psi(q)$ for $\langle\!\langle \Phi(q)|\Psi(q)\rangle\!\rangle_q$ and $\langle q|H|q'\rangle$ for $K(q,q')$, so that

$$\sigma(q|q') = \frac{[(2/\hbar)\,\mathrm{Im}\,\Psi^*(q)\,\langle q|H|q'\rangle\,\Psi(q')]^+}{\Psi^*(q')\,\Psi(q')}\,,$$

which looks like (19) again.

3.4 Kernels of the Measure Type

The kernel $\langle q|H|q'\rangle$ can be less regular than a function. Since the numerator of (1) is a measure in q and q', the formula still makes sense (for P the natural PVM) when the kernel $\langle q|H|q'\rangle$ is a complex measure in q and q'. The mathematical details will be discussed in Section 4.2. For instance, the kernel can have singularities like a Dirac δ, but it cannot have singularities worse than δ, such as derivatives of δ (as would arise from an operator whose position representation is a differential operator). It can happen that the kernel is not a function but a measure even for a very well-behaved (even bounded) operator. For example, this is the case for H a multiplication operator (i.e., a function $V(\hat{q})$ of the position operator), $\langle q|H|q'\rangle = V(q)\,\delta(q-q')$. Note, though, that multiplication operators correspond to zero jump rates.

A nontrivial example of an operator with δ singularities in the kernel is $H = 1 - \cos(p/p_0)$ where $p = -i\hbar\partial/\partial q$ is the momentum operator in one dimension, $\mathscr{H} = L^2(\mathbb{R}, \mathbb{C})$, and p_0 is a constant. The dispersion relation $E = 1 - \cos(p/p_0)$ begins at $p = 0$ like $\frac{1}{2}(p/p_0)^2$ but deviates from the parabola for large p. In the position representation, H is the convolution with ($(2\pi)^{-1/2}$ times) the inverse Fourier transform of the function $1 - \cos(\hbar k/p_0)$, and thus $\langle q|H|q'\rangle = \delta(q-q') - \frac{1}{2}\delta(q-q'+\frac{\hbar}{p_0}) - \frac{1}{2}\delta(q-q'-\frac{\hbar}{p_0})$. In this case, (1) leads to

$$\sigma(q|q') = \frac{[(-1/\hbar)\,\mathrm{Im}\,\Psi^*(q)\,\Psi(q')]^+}{\Psi^*(q')\,\Psi(q')}\left(\delta(q-q'+\tfrac{\hbar}{p_0}) + \delta(q-q'-\tfrac{\hbar}{p_0})\right). \qquad (35)$$

(Note that nonnegative factors can be drawn out of the plus function.) This formula may be viewed as contained in (19) as well, in a formal sense. As a consequence of (35), only jumps by an amount of $\pm\frac{\hbar}{p_0}$ can occur in this case.

3.5 Infinite Rates

There also exist Markov processes that perform *infinitely many jumps* in every finite time interval (e.g., Glauber dynamics for infinitely many spins). These processes, which we do not count among the jump processes, may appear pathological, and we will not investigate them in this paper, but we note that some Hamiltonians may correspond to such processes. They could arise from jump rates $\sigma(\,\cdot\,|q')$ given by (1) that form not a finite but merely a σ-finite measure, so that $\sigma(\mathcal{Q}|q') = \infty$. Here is an (artificial) example of σ-finite (but not finite) rates, arising from an operator H that is even bounded.

Let $\mathcal{Q} = \mathbb{R}$, $\mathscr{H} = L^2(\mathbb{R})$ with $P(\,\cdot\,)$ the position PVM, and let H, in Fourier representation, be multiplication by $f(k) = \sqrt{\pi/2}\,\mathrm{sign}(k)$. H is bounded since f is. f is the Fourier transform of i/x, understood as the distribution defined by the principal value integral. As a consequence, H has, in position representation, the kernel $\langle q|H|q'\rangle = i/(q-q')$. From (19) we obtain the jump rates

$$\sigma(q|q') = \frac{2}{\hbar}\frac{1}{\Psi^*(q')\,\Psi(q')}\left[\frac{\mathrm{Re}\,\Psi^*(q)\,\Psi(q')}{q-q'}\right]^+, \qquad (36)$$

15

which entails that $\sigma(\mathbb{R}|q') = \int \sigma(q|q')\,dq = \infty$ at least whenever Ψ is continuous (and nonvanishing) at q'. Nonetheless, since the rate for jumping anywhere outside the interval $[q'-\varepsilon, q'+\varepsilon]$ is finite for every $\varepsilon > 0$ and since $\int_{q'-\varepsilon}^{q'+\varepsilon} |q - q'|\sigma(q|q')\,dq < \infty$, a process with these rates should exist: among the jumps that the process would have to make per unit time, the large ones would be few and the frequent ones would be tiny—too tiny to significantly contribute.

3.6 Discrete Configuration Space

Now consider a discrete configuration space $\mathcal{Q}$. Mathematically, this means $\mathcal{Q}$ is a countable set. In this case, measures are determined by their values on singletons $\{q\}$, and we can specify all jump rates by specifying the rate $\sigma(q|q')$ for each transition $q' \to q$. (1) then reads

$$\sigma(q|q') = \frac{[(2/\hbar)\operatorname{Im}\langle\Psi|P\{q\}HP\{q'\}|\Psi\rangle]^+}{\langle\Psi|P\{q'\}|\Psi\rangle}. \tag{37}$$

We begin with the particularly simple case that there is an orthonormal basis of $\mathscr{H}$ labeled by $\mathcal{Q}$, $\{|q\rangle : q \in \mathcal{Q}\}$, and P is the PVM corresponding to this basis, $P\{q\} = |q\rangle\langle q|$. In this case, the notation $\langle q|H|q'\rangle$ and the name "matrix element" can be taken literally. The rates (1) then simplify to

$$\sigma(q|q') = \frac{[(2/\hbar)\operatorname{Im}\langle\Psi|q\rangle\langle q|H|q'\rangle\langle q'|\Psi\rangle]^+}{\langle\Psi|q'\rangle\langle q'|\Psi\rangle} \tag{38a}$$

$$= \left[\frac{2}{\hbar}\operatorname{Im}\frac{\langle\Psi|q\rangle\langle q|H|q'\rangle}{\langle\Psi|q'\rangle}\right]^+. \tag{38b}$$

Note that (38a) is the obvious discrete analogue of (19); in fact, one can regard (19) as another way of writing (38a) in this case.

Consider now the more general case that a basis of Hilbert space is indexed by two "quantum numbers," the configuration q and another index i. Then the POVM is given by the PVM $P\{q\} = \sum_i |q,i\rangle\langle q,i|$, the projection onto the subspace associated with q (whose dimension might depend on q); such a PVM may be called "degenerate." We have $\mathbb{P}(q) = \langle\Psi|P\{q\}|\Psi\rangle = \sum_i \langle\Psi|q,i\rangle\langle q,i|\Psi\rangle$, and (1) becomes

$$\sigma(q|q') = \frac{\left[\frac{2}{\hbar}\operatorname{Im}\sum_{i,i'}\langle\Psi|q,i\rangle\langle q,i|H|q',i'\rangle\langle q',i'|\Psi\rangle\right]^+}{\sum_{i'}\langle\Psi|q',i'\rangle\langle q',i'|\Psi\rangle}. \tag{39}$$

We may also write (39) as (38a), understanding $\langle\Psi|q\rangle$ and $\langle q'|\Psi\rangle$ as multi-component, $\langle q|H|q'\rangle$ as a matrix, and products as inner products. In case that the dimension of the subspace associated with q is always k, independent of q, (39) is a discrete analogue of the rate formula (33) for spinor-valued wave functions.

Apart from serving as mathematical examples, discrete configuration spaces are relevant for several reasons: First, they provide particularly simple cases of jump processes

with minimal rates that are easy to study. Second, any *numerical* computation is discrete by nature. Third, one may consider approximating or replacing the $\mathbb{R}^3$ that is supposed to model physical space by a lattice $\mathbb{Z}^3$; after all, lattice approaches have often been employed in QFT, for various reasons. Moreover, Bell-type QFTs will usually have as configurations the positions of a variable number of particles; so the configuration has a certain continuous aspect, the positions, and a certain discrete aspect, the number of particles. Sometimes one wishes to study simplified models, and in this vein it may be interesting to have only the particle number as a state variable, and thus the set of nonnegative integers as configuration space.

3.7 Bell's Process

The model Bell specified in [3] is a case of a minimal jump process on a discrete set. "For simplicity," Bell considers a lattice Λ instead of continuous 3-space, and a Hamiltonian of a lattice QFT. As a consequence, the configuration space $\mathcal{Q} = \Gamma(\Lambda)$ is countable. (Bell even makes $\mathcal{Q}$ finite, but this is not relevant here. We also remark that according to Bell's formulation, even distinguishable particles have configuration space $\Gamma(\Lambda)$.)

Bell chooses as the configuration the number of *fermions* at every lattice site, rather than the total particle number (i.e., in our terminology he takes $P\{q\}$ to be the projection to the joint eigenspace of the fermion number operators for all lattice sites with eigenvalues the occupation numbers corresponding to $q \in \Gamma(\Lambda)$). He thus gives the fermionic degrees of freedom a status different from the bosonic ones. That is to say, boson particles do not exist in Bell's model, despite the fact that $\mathscr{H} = \mathscr{H}_{\text{fermions}} \otimes \mathscr{H}_{\text{bosons}}$ and the presence of bosonic terms in the Hamiltonian.

Thus the PVM $P\{q\} = P_{\text{fermions}}\{q\} \otimes \mathbf{1}_{\text{bosons}}$ is "doubly" degenerate: the fermionic occupation number operators do not form a complete set of commuting operators, because of both the spin and the bosonic degrees of freedom. Different spin states and different quantum states of the bosonic fields are compatible with the same fermion occupation numbers. So a further index i is necessary to label a basis $\{|q, i\rangle\}$ of $\mathscr{H}$. The jump rates Bell prescribes are then (39), and are thus a special case of (1). We emphasize that here the index i does not merely label different spin states, but states of the quantized radiation as well.

3.8 A Case of POVM

Consider for $\mathscr{H}$ the space of Dirac wave functions of positive energy. The POVM $P(\,\cdot\,)$ we defined on it in Section 2.5 is, as we have already remarked, not a PVM but a genuine POVM and arises from the natural PVM $P_0(\,\cdot\,)$ on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by $P(\,\cdot\,) = P_+ P_0(\,\cdot\,) I$ with $P_+ : L^2(\mathbb{R}^3, \mathbb{C}^4) \to \mathscr{H}$ the projection and $I : \mathscr{H} \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ the inclusion mapping. We can extend any given interaction Hamiltonian H on $\mathscr{H}$ to an operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$, $H_{\text{ext}} = IHP_+$. If H_{ext} possesses a kernel $\langle q|H_{\text{ext}}|q'\rangle$, then H corresponds to a jump process, and the rates (1) can be expressed in terms of this kernel, since for $\Psi \in \mathscr{H}$, $\langle \Psi|P(dq)HP(dq')|\Psi\rangle = \langle \Psi|P_+ P_0(dq)IHP_+ P_0(dq')I|\Psi\rangle = \langle \Psi|P_0(dq)H_{\text{ext}}P_0(dq')|\Psi\rangle =$

$\Psi^*(q) \langle q|H_{\mathrm{ext}}|q'\rangle \Psi(q') \, dq \, dq'$. We thus obtain

$$\sigma(q|q') = \frac{\left[(2/\hbar) \operatorname{Im} \Psi^*(q) \langle q|H_{\mathrm{ext}}|q'\rangle \Psi(q') \right]^+}{\Psi^*(q') \, \Psi(q')}. \tag{40}$$

This POVM is used in the pair creation model of [12, Sec. 3.3].

3.9   Another Case of POVM

Let $\mathscr{H} = L^2(\mathbb{R}^d)$ and let $P_0(\,\cdot\,)$ be the natural PVM. We obtain a POVM P by smearing out P_0 with a profile function $\varphi : \mathbb{R}^d \to [0, \infty)$ with $\int \varphi(q) \, dq = 1$ and $\varphi(-q) = \varphi(q)$, e.g., a Gaussian:

$$P(B) = \int\limits_{q \in B} dq \int\limits_{q' \in \mathbb{R}^d} \varphi(q' - q) \, P_0(dq'). \tag{41}$$

Whereas $P_0(B)$ is multiplication by $\mathbf{1}_B$, $P(B)$ is multiplication by $\varphi \star \mathbf{1}_B$. It leads to $\mathbb{P}(dq) = (\varphi \star |\Psi|^2)(q) \, dq$.

The jump rate formula (1) then yields

$$\sigma(q|q') = \frac{\left[(2/\hbar) \operatorname{Im} \int dq'' \int dq''' \, \varphi(q'' - q) \Psi^*(q'') \langle q''|H|q'''\rangle \Psi(q''') \, \varphi(q''' - q') \right]^+}{\int dq'' \, \varphi(q'' - q') \, \Psi^*(q'') \, \Psi(q'')},$$

i.e., the denominator gets smeared out with φ, and the square bracket in the numerator gets smeared out with φ in each variable.

3.10   Identical Particles

The n-particle sector of the configuration space (without coincidence configurations) of identical particles $\Gamma_{\neq}(\mathbb{R}^3)$ is the manifold of n-point subsets of $\mathbb{R}^3$; let $\mathcal{Q}$ be this manifold. The most common way of describing the quantum state of n fermions is by an anti-symmetric (square-integrable) wave function Ψ on $\hat{\mathcal{Q}} := \mathbb{R}^{3n}$; let $\mathscr{H}$ be the space of such functions. Whereas for bosons Ψ could be viewed as a function on $\mathcal{Q}$, for fermions Ψ is not a function on $\mathcal{Q}$.

Nonetheless, the configuration observable still corresponds to a PVM P on $\mathcal{Q}$: for $B \subseteq \mathcal{Q}$, we set $P(B)\Psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) = \Psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ if $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\} \in B$ and zero otherwise. In other words, $P(B)$ is multiplication by the indicator function of $\pi^{-1}(B)$ where π is the obvious projection mapping $\hat{\mathcal{Q}} \setminus \Delta \to \mathcal{Q}$, with Δ the set of coincidence configurations.

To obtain other useful expressions for this PVM, we introduce the formal kets $|\hat{q}\rangle$ for $\hat{q} \in \hat{\mathcal{Q}}$ (to be treated like elements of $L^2(\hat{\mathcal{Q}})$), the anti-symmetrization operator S (i.e., the projection $L^2(\hat{\mathcal{Q}}) \to \mathscr{H}$), and the normalized anti-symmetrizer[6] $s = \sqrt{n!}\, S$, and

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[6]The name means this: since S is a projection, $S\Psi$ is usually not a unit vector when Ψ is. Whenever $\Psi \in L^2(\hat{\mathcal{Q}})$ is supported by a fundamental domain of the permutation group, i.e., by a set $\Omega \subseteq \hat{\mathcal{Q}}$ on which (the restriction of) π is a bijection to $\mathcal{Q}$, the norm of $S\Psi$ is $1/\sqrt{n!}$, so that $s\Psi$ is again a unit vector.

the formal kets $|s\hat{q}\rangle := s|\hat{q}\rangle$ (to be treated like elements of $\mathscr{H}$). The $|\hat{q}\rangle$ and $|s\hat{q}\rangle$ are normalized in the sense that

$$\langle\hat{q}|\hat{q}'\rangle = \delta(\hat{q} - \hat{q}') \text{ and } \langle s\hat{q}|s\hat{q}'\rangle = (-1)^{\varrho(\hat{q},\hat{q}')}\,\delta(q - q'),$$

where $q = \pi(\hat{q})$, $q' = \pi(\hat{q}')$, $\varrho(\hat{q}, \hat{q}')$ is the permutation that carries $\hat{q}$ into $\hat{q}'$ given that $q = q'$, and $(-1)^\varrho$ is the sign of the permutation ϱ. Now we can write

$$P(dq) = \sum_{\hat{q}\in\pi^{-1}(q)} |\hat{q}\rangle\langle\hat{q}|\,dq = n!\,S|\hat{q}\rangle\langle\hat{q}|\,dq = |s\hat{q}\rangle\langle s\hat{q}|\,dq, \tag{42}$$

where the sum is over the $n!$ ways of numbering the n points in q; the last two terms actually do not depend on the choice of $\hat{q}\in\pi^{-1}(q)$, the numbering of q.

The probability distribution arising from this PVM is

$$\mathbb{P}(dq) = \sum_{\hat{q}\in\pi^{-1}(q)} |\Psi(\hat{q})|^2\,dq = n!\,|\Psi(\hat{q})|^2\,dq = |\langle s\hat{q}|\Psi\rangle|^2\,dq \tag{43}$$

with arbitrary $\hat{q}\in\pi^{-1}(q)$.

If an operator $\hat{H}$ on $L^2(\hat{\mathcal{Q}})$ is permutation invariant,

$$U_\varrho^{-1}\hat{H}U_\varrho = \hat{H} \text{ for every permutation } \varrho, \tag{44}$$

where U_ϱ is the unitary operator on $L^2(\hat{\mathcal{Q}})$ performing the permutation ϱ, then $\hat{H}$ maps anti-symmetric functions to anti-symmetric functions, and thus defines an operator H on $\mathscr{H}$. If $\hat{H}$ has a kernel $\langle\hat{q}|\hat{H}|\hat{q}'\rangle$ then the kernel is permutation invariant in the sense that

$$\langle\varrho(\hat{q})|\hat{H}|\varrho(\hat{q}')\rangle = \langle\hat{q}|\hat{H}|\hat{q}'\rangle \quad \forall\varrho, \tag{45}$$

where $\varrho(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) := (\boldsymbol{q}_{\varrho(1)}, \ldots, \boldsymbol{q}_{\varrho(n)})$, and H also possesses a kernel,

$$\langle s\hat{q}|H|s\hat{q}'\rangle = n!\,\langle\hat{q}|S\hat{H}S|\hat{q}'\rangle = \frac{1}{n!}\sum_{\varrho,\varrho'}\langle\varrho(\hat{q})|\hat{H}|\varrho'(\hat{q}')\rangle.$$

In this case (1) yields

$$\sigma(q|q') = \frac{\left[\frac{2}{\hbar}\operatorname{Im}\sum_{\hat{q},\hat{q}'}\Psi^*(\hat{q})\,\langle\hat{q}|\hat{H}|\hat{q}'\rangle\,\Psi(\hat{q}')\right]^+}{\sum_{\hat{q}'}\Psi^*(\hat{q}')\,\Psi(\hat{q}')} \tag{46a}$$

$$= \frac{\left[\frac{2}{\hbar}\operatorname{Im}\langle\Psi|s\hat{q}\rangle\langle s\hat{q}|H|s\hat{q}'\rangle\langle s\hat{q}'|\Psi\rangle\right]^+}{\langle\Psi|s\hat{q}'\rangle\langle s\hat{q}'|\Psi\rangle} \tag{46b}$$

where $\hat{q}\in\pi^{-1}(q)$ and $\hat{q}'\in\pi^{-1}(q')$, as running variables in (46a) and as arbitrary but fixed in (46b).

3.11 Another View of Fermions

There is a way of viewing fermion wave functions as being defined on $\mathcal{Q}$, rather than $\mathbb{R}^{3n}$, by regarding them as cross-sections of a particular 1-dimensional vector bundle over $\mathcal{Q}$. To this end, define an $n!$-dimensional vector bundle E by

$$E_q := \bigoplus_{\hat{q} \in \pi^{-1}(q)} \mathbb{C} \,. \tag{47}$$

Every function $\Psi : \mathbb{R}^{3n} \to \mathbb{C}$ naturally gives rise to a cross-section Φ of E, defined by

$$\Phi(q) := \bigoplus_{\hat{q} \in \pi^{-1}(q)} \Psi(\hat{q}) \,. \tag{48}$$

The anti-symmetric functions form a 1-dimensional subbundle of E (see also [10] for a discussion of this bundle). The jump rate formula for vector bundles (34) can be applied to either the subbundle or E, depending on the way in which the kernel of H is given. The kernel $\langle \hat{q} | \hat{H} | \hat{q}' \rangle$ above translates directly into a kernel on $\mathcal{Q} \times \mathcal{Q}$ with values in $E_q \otimes E_{q'}^*$, for which the rate formula for bundles (34) is the same as the rate formula for identical particles (46a) derived in the previous section.

Another alternative view of a fermion wave function is to regard it as a complex differential form of full rank, a $3n$-form, on $\mathcal{Q}$. (See, e.g., [10]. This would not work if the dimension of physical space were even.) Of course, the complex $3n$-forms are nothing but the sections of a certain 1-dimensional bundle, usually denoted $\mathbb{C} \otimes \Lambda^{3n} \mathcal{Q}$, which is equivalent to the subbundle of E considered in the previous paragraph, and which is contained in the bundle $\mathbb{C} \otimes \Lambda \mathcal{Q}$ of Grassmann numbers over $\mathcal{Q}$.

3.12 A Simple QFT

We presented a simple example of a Bell-type QFT in [11], and we will now briefly point to the aspects of this model that are relevant here. The model is based on one of the simplest possible QFTs [30, p. 339].

The relevant configuration space $\mathcal{Q}$ for a QFT (with a single particle species) is the configuration space of a variable number of identical particles in $\mathbb{R}^3$, which is the set $\Gamma(\mathbb{R}^3)$, or, ignoring the coincidence configurations (as they are exceptions), the set $\Gamma_{\neq}(\mathbb{R}^3)$ of all finite subsets of $\mathbb{R}^3$. The n-particle sector of this is a manifold of dimension $3n$; this configuration space is thus a union of (disjoint) manifolds of different dimensions. The relevant configuration space for a theory with several particle species is the Cartesian product of several copies of $\Gamma_{\neq}(\mathbb{R}^3)$. In the model of [11], there are two particle species, a fermion and a boson, and thus the configuration space is

$$\mathcal{Q} = \Gamma_{\neq}(\mathbb{R}^3) \times \Gamma_{\neq}(\mathbb{R}^3) \,. \tag{49}$$

We will denote configurations by $q = (x, y)$ with x the configuration of the fermions and y the configuration of the bosons.

For simplicity, we replaced in [11] the sectors of $\Gamma_{\neq}(\mathbb{R}^3) \times \Gamma_{\neq}(\mathbb{R}^3)$, which are manifolds, by vector spaces of the same dimension (by artificially numbering the particles), and obtained the union

$$\hat{\mathcal{Q}} = \bigcup_{n=0}^{\infty} (\mathbb{R}^3)^n \times \bigcup_{m=0}^{\infty} (\mathbb{R}^3)^m \,, \tag{50}$$

with n the number of fermions and m the number of bosons. Here, however, we will use (49) as the configuration space. In comparison with (50), this amounts to (merely) ignoring the numbering of the particles.

$\mathscr{H}$ is the tensor product of a fermion Fock space and a boson Fock space, and thus the subspace of wave functions in $L^2(\hat{\mathcal{Q}})$ that are anti-symmetric in the fermion coordinates and symmetric in the boson coordinates. Let S denote the appropriate symmetrization operator, i.e., the projection operator $L^2(\hat{\mathcal{Q}}) \to \mathscr{H}$, and s the normalized symmetrizer

$$s\Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m) = \sqrt{n!\, m!}\, S\Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m), \tag{51}$$

i.e., $s = \sqrt{N!\, M!}\, S$ with N and M the fermion and boson number operators, which commute with S and with each other. As in Section 3.10, we denote by π the projection mapping $\hat{\mathcal{Q}} \setminus \Delta \to \mathcal{Q}$, $\pi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m) = (\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}, \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\})$. The configuration PVM $P(B)$ on $\mathcal{Q}$ is multiplication by $\mathbf{1}_{\pi^{-1}(B)}$, which can be understood as acting on $\mathscr{H}$, though it is defined on $L^2(\hat{\mathcal{Q}})$, since it is permutation invariant and thus maps $\mathscr{H}$ to itself. We utilize again the formal kets $|\hat{q}\rangle$ where $\hat{q} \in \hat{\mathcal{Q}} \setminus \Delta$ is a numbered configuration, for which we also write $\hat{q} = (\hat{x}, \hat{y}) = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$. We also use the symmetrized and normalized kets $|s\hat{q}\rangle = s|\hat{q}\rangle$. As in (42), we can write

$$P(dq) = \sum_{\hat{q} \in \pi^{-1}(q)} |\hat{q}\rangle\langle\hat{q}|\, dq = n!\, m!\, S|\hat{q}\rangle\langle\hat{q}|\, dq = |s\hat{q}\rangle\langle s\hat{q}|\, dq \tag{52}$$

with arbitrary $\hat{q} \in \pi^{-1}(q)$. For the probability distribution, we thus have, as in (43),

$$\mathbb{P}(dq) = \sum_{\hat{q} \in \pi^{-1}(q)} |\Psi(\hat{q})|^2\, dq = n!\, m!\, |\Psi(\hat{q})|^2\, dq = |\langle s\hat{q}|\Psi\rangle|^2\, dq \tag{53}$$

with arbitrary $\hat{q} \in \pi^{-1}(q)$.

The free Hamiltonian is the second quantized Schrödinger operator (with zero potential), associated with the free process described in Section 2.6. The interaction Hamiltonian is defined by

$$H_I = \int d^3\boldsymbol{x}\, \psi^\dagger(\boldsymbol{x})\, (a_\varphi^\dagger(\boldsymbol{x}) + a_\varphi(\boldsymbol{x}))\, \psi(\boldsymbol{x}) \tag{54}$$

with $\psi^\dagger(\boldsymbol{x})$ the creation operators (in position representation), acting on the *fermion* Fock space, and $a_\varphi^\dagger(\boldsymbol{x})$ the creation operators (in position representation), acting on the *boson* Fock space, regularized through convolution with an L^2 function $\varphi : \mathbb{R}^3 \to \mathbb{R}$. H_I has a kernel; we will now obtain a formula for it, see (60) below. The $|s\hat{q}\rangle$ are connected to the creation operators according to

$$|s\hat{q}\rangle = \psi^\dagger(\boldsymbol{x}_n) \cdots \psi^\dagger(\boldsymbol{x}_1) a^\dagger(\boldsymbol{y}_m) \cdots a^\dagger(\boldsymbol{y}_1)|0\rangle\,, \tag{55}$$

where $|0\rangle \in \mathscr{H}$ denotes the vacuum state. A relevant fact is that the creation and annihilation operators $\psi^\dagger, \psi, a^\dagger$ and a possess kernels. Using the canonical (anti-)commutation relations for ψ and a, one obtains from (55) the following formulas for the kernels of $\psi(\boldsymbol{r})$ and $a(\boldsymbol{r})$, $\boldsymbol{r} \in \mathbb{R}^3$:

$$\langle s\hat{q}|\psi(\boldsymbol{r})|s\hat{q}'\rangle = \delta_{n,n'-1}\,\delta_{m,m'}\,\delta^{3n'}(x \cup \boldsymbol{r} - x')\,(-1)^{\varrho((\hat{x},\boldsymbol{r}),\hat{x}')}\,\delta^{3m}(y-y') \tag{56}$$

$$\langle s\hat{q}|a(\boldsymbol{r})|s\hat{q}'\rangle = \delta_{n,n'}\,\delta_{m,m'-1}\,\delta^{3n}(x - x')\,(-1)^{\varrho(\hat{x},\hat{x}')}\,\delta^{3m'}(y \cup \boldsymbol{r} - y') \tag{57}$$

where $(x,y) = q = \pi(\hat{q})$, and $\varrho(\hat{x}, \hat{x}')$ denotes the permutation that carries $\hat{x}$ to $\hat{x}'$ given that $x = x'$. The corresponding formulas for $\psi^\dagger$ and $a^\dagger$ can be obtained by exchanging $\hat{q}$ and $\hat{q}'$ on the right hand sides of (56) and (57). For the smeared-out operator $a_\varphi(\boldsymbol{r})$, we obtain

$$\langle s\hat{q}|a_\varphi(\boldsymbol{r})|s\hat{q}'\rangle = \delta_{n,n'}\,\delta_{m,m'-1}\,\delta^{3n}(x - x')\,(-1)^{\varrho(\hat{x},\hat{x}')}\sum_{\boldsymbol{y}'\in y'}\delta^{3m}(y-y'\setminus\boldsymbol{y}')\,\varphi(\boldsymbol{y}'-\boldsymbol{r}) \tag{58}$$

We make use of the resolution of the identity

$$I = \int_{\mathcal{Q}} dq\,|s\hat{q}\rangle\langle s\hat{q}|\,. \tag{59}$$

Inserting (59) twice into (54) and exploiting (56) and (58), we find

$$\langle s\hat{q}|H_I|s\hat{q}'\rangle = \delta_{n,n'}\,\delta_{m-1,m'}\,\delta^{3n}(x - x')\,(-1)^{\varrho(\hat{x},\hat{x}')}\sum_{\boldsymbol{y}\in y}\delta^{3m'}(y\setminus\boldsymbol{y} - y')\sum_{\boldsymbol{x}\in x}\varphi(\boldsymbol{y}-\boldsymbol{x})$$
$$+ \delta_{n,n'}\,\delta_{m'-1,m}\,\delta^{3n}(x - x')\,(-1)^{\varrho(\hat{x},\hat{x}')}\sum_{\boldsymbol{y}'\in y'}\delta^{3m}(y - y'\setminus\boldsymbol{y}')\sum_{\boldsymbol{x}\in x}\varphi(\boldsymbol{y}'-\boldsymbol{x})\,. \tag{60}$$

This is another case of a kernel containing δ functions (see Section 3.4).

By (52), the jump rates (1) are

$$\sigma(q|q') = \frac{\left[\frac{2}{\hbar}\,\mathrm{Im}\,\langle\Psi|s\hat{q}\rangle\langle s\hat{q}|H_I|s\hat{q}'\rangle\langle s\hat{q}'|\Psi\rangle\right]^+}{\langle\Psi|s\hat{q}'\rangle\langle s\hat{q}'|\Psi\rangle}\,. \tag{61}$$

More explicitly, we obtain from (60) the rates

$$\sigma(q|q') = \delta_{nn'}\,\delta_{m-1,m'}\,\delta^{3n}(x - x')\sum_{\boldsymbol{y}\in y}\delta^{3m'}(y\setminus\boldsymbol{y} - y')\,\sigma_{\mathrm{crea}}(q'\cup\boldsymbol{y}|q')$$
$$+ \delta_{nn'}\,\delta_{m,m'-1}\,\delta^{3n}(x - x')\sum_{\boldsymbol{y}'\in y'}\delta^{3m}(y - y'\setminus\boldsymbol{y}')\,\sigma_{\mathrm{ann}}(q'\setminus\boldsymbol{y}'|q') \tag{62}$$

with

$$\sigma_{\mathrm{crea}}(q'\cup\boldsymbol{y}|q') = \frac{2\sqrt{m'+1}}{\hbar}\,\frac{\left[\mathrm{Im}\,\Psi^*(\hat{q})\,(-1)^{\varrho(\hat{x},\hat{x}')}\sum_{\boldsymbol{x}'\in x'}\varphi(\boldsymbol{y}-\boldsymbol{x}')\,\Psi(\hat{q}')\right]^+}{\Psi^*(\hat{q}')\,\Psi(\hat{q}')} \tag{63a}$$

$$\sigma_{\mathrm{ann}}(q'\setminus\boldsymbol{y}'|q') = \frac{2}{\hbar\sqrt{m'}}\,\frac{\left[\mathrm{Im}\,\Psi^*(\hat{q})\,(-1)^{\varrho(\hat{x},\hat{x}')}\sum_{\boldsymbol{x}'\in x'}\varphi(\boldsymbol{y}'-\boldsymbol{x}')\,\Psi(\hat{q}')\right]^+}{\Psi^*(\hat{q}')\,\Psi(\hat{q}')}\,, \tag{63b}$$

for arbitrary $\hat{q}' \in \pi^{-1}(q')$ and $\hat{q} \in \pi^{-1}(q)$ with $q = (x', y' \cup \boldsymbol{y})$ respectively $q = (x', y' \setminus \boldsymbol{y}')$. (Note that a sum sign can be drawn out of the plus function if the terms have disjoint supports.)

Equation (62) is worth looking at closely: One can read off that the only possible jumps are $(x', y') \to (x', y' \cup \boldsymbol{y})$, creation of a boson, and $(x', y') \to (x', y' \setminus \boldsymbol{y}')$, annihilation of a boson. In particular, while one particle is created or annihilated, the other particles do not move. The process that we considered in [11] consists of pieces of Bohmian trajectories interrupted by jumps with rates (62); the process is thus an example of the jump rate formula (1), and an example of combining jumps and Bohmian motion by means of process additivity.

The example shows how, for other QFTs, the jump rates (1) can be applied to relevant interaction Hamiltonians: If H_I is, in the position representation, a polynomial in the creation and annihilation operators, then it possesses a kernel on the relevant configuration space. A cut-off (implemented here by smearing out the creation and annihilation operators) needs to be introduced to make H_I a well-defined operator on L^2.

If, in some QFT, the particle number operator is not conserved, jumps between the sectors of configuration space are inevitable for an equivariant process. And, indeed, when H_I does not commute with the particle number operator (as is usually the case), jumps can occur that change the number of particles. Often, H_I contains *only* off-diagonal terms with respect to the particle number; then every jump will change the particle number. This is precisely what happens in the model of [11].

4   Existence Results

The configuration space $\mathcal{Q}$ is assumed in this paper to be a measurable space, equipped with a σ-algebra $\mathcal{A}$. Every set we consider is assumed to belong to the appropriate σ-algebra: $\mathcal{A}$ on $\mathcal{Q}$ or the product σ-algebra $\mathcal{A} \otimes \mathcal{A}$ on $\mathcal{Q} \times \mathcal{Q}$. If F is a quadratic form, we will usually use the notation $\langle \Phi | F | \Psi \rangle$ rather than $F(\Phi, \Psi)$. If $P(B)\Psi$ and $P(C)\Psi$ lie in the form domain of H, we write $\langle \Psi | P(B) H P(C) | \Psi \rangle$ for $\langle P(B)\Psi | H | P(C)\Psi \rangle$.

4.1   Condition for Finite Rates

For the argument of Section 2.5 to work, it is necessary that (a) the bracket in the numerator of (1) exist as a finite signed measure on $\mathcal{Q} \times \mathcal{Q}$, and (b) the Radon–Nikodým derivative of the numerator with respect to the denominator also be well defined. It turns out that, given (a), (b) is straightforward. However, contrary to what a superficial inspection might suggest, (a) is problematical even when H is bounded. To see this, consider the case $\mathscr{H} = L^2(\mathbb{R})$ with the natural PVM (corresponding to position) on $\mathcal{Q} = \mathbb{R}$, and with H the sum of the Fourier transform on $\mathscr{H}$ and its adjoint, given by the kernel

$$\langle q | H | q' \rangle = \sqrt{\frac{2}{\pi}} \cos(qq') \,.$$

Then, for Ψ real, the bracket in (1) would have to be understood as proportional to

$$\Psi(q)\,\cos(qq')\,\Psi(q')\,,$$

and $\Psi \in \mathscr{H}$ could be so chosen that this does not define a signed measure on $\mathbb{R} \times \mathbb{R}$ because both its positive and negative part have infinite total weight. In fact, Ψ can be so chosen that the resulting $\sigma(\,\cdot\,|q')$ is an infinite measure, $\sigma(\mathcal{Q}|q') = \infty$, for all q', and thus does not define a jump process. Note, however, that for $\Psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\sigma(\,\cdot\,|q')$ is finite for this H.

The following theorem provides a condition under which the argument sketched in Section 2.5 for the equivariance of the jump rates σ, steps 1–5, can be made rigorous.

Theorem 1 *Let $\mathscr{H}$ be a Hilbert space, $\Psi \in \mathscr{H}$ with $\|\Psi\| = 1$, H a self-adjoint operator on $\mathscr{H}$, $\mathcal{Q}$ a standard Borel space,[7] and P a POVM on $\mathcal{Q}$ acting on $\mathscr{H}$. Suppose that for all $B \subseteq \mathcal{Q}$, $P(B)\Psi$ lies in the form domain of H, and there exists a complex measure μ on $\mathcal{Q} \times \mathcal{Q}$ such that for all $B, C \subseteq \mathcal{Q}$,*

$$\mu(B \times C) = \langle \Psi | P(B) H P(C) | \Psi \rangle\,. \tag{64}$$

Then the jump rates (1) are well-defined and finite for $\mathbb{P}$-almost every q', and they are equivariant if, in addition, $\Psi \in \mathrm{domain}(H)$.

Proof. We first show that under the hypotheses of the theorem, the jump rates (1) are well-defined and finite. Then we show that they are equivariant.

To begin with, the measure μ whose existence was assumed in the theorem is conjugate symmetric under the transposition mapping $(q, q')^{\mathrm{tr}} = (q', q)$ on $\mathcal{Q} \times \mathcal{Q}$, i.e., $\mu(A^{\mathrm{tr}}) = \mu(A)^*$. To see this, note that a complex measure on $\mathcal{Q} \times \mathcal{Q}$ is uniquely determined by its values on product sets. $\mu(\,\cdot\,^{\mathrm{tr}})$ and $\mu(\,\cdot\,)^*$ must thus be the same measure since, by the self-adjointness of H, for $A = B \times C$, $\mu(A^{\mathrm{tr}}) = \mu(C \times B) \overset{(64)}{=} \langle \Psi | P(C) H P(B) | \Psi \rangle = \langle \Psi | P(B) H P(C) | \Psi \rangle^* = \mu(A)^*$.

We define a signed measure $\mathbb{J}$ on $\mathcal{Q} \times \mathcal{Q}$ by $\mathbb{J} = \frac{2}{\hbar} \mathrm{Im}\,\mu$. Let $\mathbb{J}^+$ be the positive part of $\mathbb{J}$ (defined by its Hahn–Jordan decomposition, $\mathbb{J} = \mathbb{J}^+ - \mathbb{J}^-$, see e.g. [21, p. 120]). Since μ is a complex measure (and thus assumes only finite values), $\mathbb{J}$ has finite positive and negative parts. Since μ is conjugate symmetric, $\mathbb{J}$ is anti-symmetric.

We now show that for every $B \subseteq \mathcal{Q}$, the measure $\mathbb{J}(B \times \,\cdot\,)$ on $\mathcal{Q}$ is absolutely continuous with respect to $\mathbb{P}(\,\cdot\,)$, the "$|\Psi|^2$" measure defined in (21). If C is a $\mathbb{P}$-null set, that is $\langle \Psi | P(C) | \Psi \rangle = 0$, then $P(C)|\Psi\rangle = 0$: if $P(C)$ is a projection, this is immediate, and if $P(C)$ is just any positive operator, it follows from the spectral theorem—any component of Ψ orthogonal to the eigenspace of $P(C)$ with eigenvalue zero would lie in the positive spectral subspace of $P(C)$ and give a positive contribution to $\langle \Psi | P(C) | \Psi \rangle$.

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[7]A *standard Borel space* is a measurable space isomorphic to a complete separable metric space with its Borel σ-algebra. Basically all spaces that arise in practice are in fact standard Borel spaces, and so are in particular all spaces that we have in mind for $\mathcal{Q}$ (which are countable unions of (separable) Riemannian manifolds). Thus, the condition of being a standard Borel space is not much of a restriction.

From $P(C)\Psi = 0$ it follows that $\langle\Psi|P(B)HP(C)|\Psi\rangle = 0$, so that $\mathbb{J}(B \times C) = 0$, which is what we wanted to show.

Next we show that for every $B \subseteq \mathcal{Q}$, the measure $\mathbb{J}^+(B \times \,\cdot\,)$ is absolutely continuous with respect to $\mathbb{P}(\,\cdot\,)$. Suppose again that $\mathbb{P}(C) = 0$. We have that

$$\mathbb{J}^+(B \times C) \le \mathbb{J}^+(B \times C) + \mathbb{J}^-(B \times C) =$$

$$= |\mathbb{J}|(B \times C) = \sup \sum_{i,j} |\mathbb{J}(B_i \times C_j)|$$

where the sup is taken over all finite partitions $\bigcup_i B_i = B$ of B and $\bigcup_j C_j = C$ of C. Now each $\mathbb{J}(B_i \times C_j) = 0$ because $\mathbb{J}(B_i \times \,\cdot\,) \ll \mathbb{P}(\,\cdot\,)$ and $\mathbb{P}(C_j) \le \mathbb{P}(C) = 0$. Thus $\mathbb{J}^+(B \times C) = 0$.

It follows from the Radon–Nikodým theorem that for every B, $\mathbb{J}^+(B \times \,\cdot\,)$ possesses a density with respect to $\mathbb{P}(\,\cdot\,)$. The density is unique up to changes on $\mathbb{P}$-null sets, and one version of this density is what we will take as $\sigma(B|q')$. We have to make sure, though, that σ is a measure in its dependence on B, and from the Radon–Nikodým theorem alone we do not obtain additivity in B. For this reason, we utilize a standard theorem [27, p. 147] on the existence of regular conditional probabilities, asserting that if $\mathcal{Q}$ (and thus also $\mathcal{Q} \times \mathcal{Q}$) is a standard Borel space, then every probability measure ν on $\mathcal{Q} \times \mathcal{Q}$ possesses regular conditional probabilities, i.e., a function $p(\,\cdot\,|q')$ on $\mathcal{Q}$ with values in the probability measures on $\mathcal{Q} \times \mathcal{Q}$ such that for almost every q', $p(\,\cdot\,|q')$ is concentrated on the set $\mathcal{Q} \times \{q'\} \subseteq \mathcal{Q} \times \mathcal{Q}$, and for every $A \subseteq \mathcal{Q} \times \mathcal{Q}$, $p(A|q')$ is a measurable function of q' with

$$\int\limits_{q' \in \mathcal{Q}} p(A|q')\, \nu(\mathcal{Q} \times dq') = \nu(A). \tag{65}$$

We set $\nu(\,\cdot\,) = \mathbb{J}^+(\,\cdot\,)/\mathbb{J}^+(\mathcal{Q} \times \mathcal{Q})$ and define σ as the corresponding regular conditional probability times a factor that takes into account that (1) involves the density of $\mathbb{J}^+$ relative to $\mathbb{P}$ (rather than to $\nu(\mathcal{Q} \times \,\cdot\,)$ or $\mathbb{J}^+(\mathcal{Q} \times \,\cdot\,)$):

$$\sigma(B|q') := p(B \times \mathcal{Q}|q')\, \frac{d\mathbb{J}^+(\mathcal{Q} \times \,\cdot\,)}{d\mathbb{P}(\,\cdot\,)}(q'). \tag{66}$$

The last factor exists because we have shown above that $\mathbb{J}^+(\mathcal{Q} \times \,\cdot\,) \ll \mathbb{P}(\,\cdot\,)$. $\sigma(\,\cdot\,|q')$ is a (finite) measure because $p(\,\cdot\,|q')$ is. For fixed B, $\sigma(B|q')$ as a function of q' is a version of the Radon–Nikodým derivative $d\mathbb{J}^+(B \times \,\cdot\,)/d\mathbb{P}(\,\cdot\,)$ because

$$\int\limits_{q' \in C} \sigma(B|q')\, \mathbb{P}(dq') \overset{(66)}{=} \int\limits_{q' \in C} p(B \times \mathcal{Q}|q')\, \frac{d\mathbb{J}^+(\mathcal{Q} \times \,\cdot\,)}{d\mathbb{P}(\,\cdot\,)}(q')\, \mathbb{P}(dq') =$$

$$= \mathbb{J}^+(\mathcal{Q} \times \mathcal{Q}) \int\limits_{q' \in \mathcal{Q}} p(B \times C|q')\, \frac{\mathbb{J}^+(\mathcal{Q} \times dq')}{\mathbb{J}^+(\mathcal{Q} \times \mathcal{Q})} \overset{(65)}{=} \mathbb{J}^+(B \times C).$$

According to the theorem on regular conditional probabilities that we used, σ is defined uniquely up to changes on a $\mathbb{P}$-null set of q's.

Now we check the equivariance of the jump rates σ: for any $B \subseteq \mathcal{Q}$,

$$\mathscr{L}_\sigma \mathbb{P}(B) \overset{(16)}{=} \int\limits_{q' \in \mathcal{Q}} \sigma(B|q')\,\mathbb{P}(dq') - \int\limits_{q \in B} \sigma(\mathcal{Q}|q)\,\mathbb{P}(dq) = \mathbb{J}^+(B \times \mathcal{Q}) - \mathbb{J}^+(\mathcal{Q} \times B)\,,$$

using that σ is a version of the Radon–Nikodým derivative of $\mathbb{J}^+$ relative to $\mathbb{P}$. Since $\mathbb{J}$ is anti-symmetric with respect to the permutation mapping $(q,q') \mapsto (q',q)$ on $\mathcal{Q} \times \mathcal{Q}$, we have that $\mathbb{J}^+(C \times B) = \mathbb{J}^-(B \times C)$, and therefore

$$\mathscr{L}_\sigma \mathbb{P}(B) = \mathbb{J}^+(B \times \mathcal{Q}) - \mathbb{J}^-(B \times \mathcal{Q}) = \mathbb{J}(B \times \mathcal{Q}) =$$

$$= \tfrac{2}{\hbar}\operatorname{Im}\mu(B \times \mathcal{Q}) \overset{(64)}{=} \tfrac{2}{\hbar}\operatorname{Im}\langle\Psi|P(B)H|\Psi\rangle.$$

It remains to be shown that $\mathbb{P}_t(B) = \langle e^{-iHt/\hbar}\Psi|P(B)\,e^{-iHt/\hbar}\Psi\rangle$ is differentiable with respect to time at $t = 0$ and has derivative

$$\left.\frac{d\mathbb{P}_t(B)}{dt}\right|_{t=0} = \tfrac{2}{\hbar}\operatorname{Im}\langle\Psi|P(B)H|\Psi\rangle. \tag{67}$$

If Ψ lies in the domain of H, $\Psi_t = e^{-iHt/\hbar}\Psi$ is differentiable with respect to t at $t = 0$ [28, p. 265] and has derivative $\dot\Psi = -\tfrac{i}{\hbar}H\Psi$. Hence

$$\frac{1}{t}\Big(\langle\Psi_t|P(B)\Psi_t\rangle - \langle\Psi_0|P(B)\Psi_0\rangle\Big)$$
$$= \langle\Psi_t|P(B)|(\Psi_t - \Psi_0)/t\,\rangle + \langle(\Psi_t - \Psi_0)/t\,|P(B)\Psi_0\rangle$$

converges, as $t \to 0$, to

$$\langle\Psi|P(B)\dot\Psi\rangle + \langle\dot\Psi|P(B)\Psi\rangle = -\tfrac{i}{\hbar}\langle\Psi|P(B)H\Psi\rangle + \tfrac{i}{\hbar}\langle H\Psi|P(B)\Psi\rangle = \tfrac{2}{\hbar}\operatorname{Im}\langle\Psi|P(B)H\Psi\rangle.$$

It now follows that $\mathscr{L}_\sigma\mathbb{P} = d\mathbb{P}/dt$, which completes the proof. $\qquad\square$

We remark that if, as supposed in Theorem 1, the measure μ exists, it is also unique. This follows from the fact, which we have already mentioned, that a (complex) measure on $\mathcal{Q} \times \mathcal{Q}$ is uniquely determined by its values on the product sets $B \times C$.

Another remark concerns how the (existence) assumption of Theorem 1 can be violated. Since the example Hamiltonian of Section 3.5 leads to infinite jump rates, it also provides an example for which the assumption of Theorem 1 is violated, in fact for *every* nonzero $\Psi \in \mathscr{H}$. To see this directly, note that, while $P(B)\Psi$ lies indeed in the form domain of H (which is $\mathscr{H}$ since H is bounded),

$$\langle\Psi|P(B)HP(C)|\Psi\rangle = i\int\limits_B dq\,\text{P-}\!\int\limits_C dq'\,\frac{\Psi^*(q)\,\Psi(q')}{q - q'}$$

26

where P-$\int$ denotes a principal value integral. For $B \cap C = \emptyset$, P-$\int$ can be replaced by a Lebesgue integral. This, together with (64), would leave for μ only one possibility (up to addition of a complex measure concentrated on the diagonal $\{(q, q) : q \in \mathcal{Q}\}$), namely

$$\mu(dq \times dq') = i\frac{\Psi^*(q)\,\Psi(q')}{q - q'}\,dq\,dq'.$$

But this is not a complex measure for any Ψ since $i\Psi^*(q)\,\Psi(q')/(q - q')$ is not absolutely integrable. This example also nicely illustrates the difference between a complex bi-measure $\nu(B, C)$, i.e., a complex measure in each variable, and a complex measure $\mu(\,\cdot\,)$ on $\mathcal{Q} \times \mathcal{Q}$: $\langle \Psi | P(B) H P(C) | \Psi \rangle$ is here a complex bi-measure and thus defines a finite-valued additive set function on the family of finite unions of product sets $B \times C \subseteq \mathcal{Q} \times \mathcal{Q}$, which, however, cannot be suitably extended to all sets $A \subseteq \mathcal{Q} \times \mathcal{Q}$. The essential reason is that the positive and the negative singularity in $1/(q - q')$ cancel (thanks to the use of principal value integrals) for every product set but do not for some nonproduct sets such as $\{(q, q') : q > q'\}$. In contrast, a (finite) non-negative bi-measure can always be extended to a (finite) measure on the product space; see Section 4.4.

A related remark on the need for the existence assumption of Theorem 1. One might well have imagined that the complex measure μ on $\mathcal{Q} \times \mathcal{Q}$, extending (64) from product sets, can always be constructed, at least when H is bounded, as the quantum expected value of the bounded-operator-valued measure (BOVM) $P \times_H P$ on $\mathcal{Q} \times \mathcal{Q}$, the "$H$-twisted product measure" $P(dq) H P(dq')$ of the POVM P with itself—or, equivalently, the product of the POVM $P(dq)$ and the BOVM $H P(dq')$. Indeed, the nonexistence of μ for the Hamiltonian in the principal-value example that we have just discussed, as well as for the Hamiltonian in the Fourier-transform example at the beginning of this section, implies that $P \times_H P$ does not exist as a BOVM in these cases; if it did, so would μ, for all Ψ. The Fourier-transform example can also easily be adapted to show that the product $P_1 \times P_2$ of two POVMs need not exist as a BOVM, and in fact does not exist when P_1 and P_2 are the most familiar PVMs for quantum mechanics, corresponding respectively to position and momentum. There is, however, an important special case for which the product $P_1 \times P_2$ of two POVMs does exist, in fact as a POVM, namely when P_1 and P_2 mutually commute, i.e., when $[P_1(B), P_2(C)] = 0$ for all B and C. This will be discussed in Section 4.4.

4.2   Integral Operators

In this section we make precise the statement that Hamiltonians with (sufficiently regular) kernels lead to finite jump rates. In particular, we specify a set of wave functions, depending on H, that lead to finite jump rates.

4.2.1   Hilbert–Schmidt Operators

We begin with the simple case in which Ψ is a complex-valued wave function on $\mathcal{Q}$, so that the natural configuration POVM $P(\,\cdot\,)$ is a "nondegenerate" PVM. What first comes

to mind as a class of Hamiltonians possessing a kernel is the class of Hilbert–Schmidt operators; for these, the kernels are in fact square-integrable functions on $\mathcal{Q} \times \mathcal{Q}$.

Corollary 1 *Let $\mathcal{Q}$ be a standard Borel space, $\mathscr{H} = L^2(\mathcal{Q}, \mathbb{C}, dq)$ with respect to a σ-finite nonnegative measure on $\mathcal{Q}$ that we simply denote dq, let $\Psi \in \mathscr{H}$ with $\|\Psi\| = 1$, let H be a self-adjoint operator on $\mathscr{H}$, and let P be the natural PVM on $\mathcal{Q}$ (multiplication by indicator functions) acting on $L^2(\mathcal{Q}, \mathbb{C}, dq)$. Suppose that H is a Hilbert–Schmidt operator. Then, by virtue of Theorem 1, the jump rates given by (1) are well-defined and finite $\mathbb{P}$-almost everywhere, and equivariant. In fact, the jump rates are given by (19) with $\langle q|H|q' \rangle$ the kernel function of H.*

Proof. Since H is a Hilbert–Schmidt operator, it possesses an integral kernel $K(q, q')$ that is a square-integrable function [28, p. 210], i.e., there is a function $K \in L^2(\mathcal{Q} \times \mathcal{Q}, \mathbb{C}, dq\, dq')$ such that for all $\Phi \in \mathscr{H}$,

$$H\Phi(q) = \int_{\mathcal{Q}} K(q, q')\, \Phi(q')\, dq' \, .$$

Thus, for all $\Phi, \Phi' \in \mathscr{H}$,

$$\langle \Phi|H|\Phi' \rangle = \int_{\mathcal{Q}} dq \int_{\mathcal{Q}} dq'\, \Phi^*(q)\, K(q, q')\, \Phi'(q') =$$

(by Fubini's theorem, because the integrand is absolutely integrable)

$$= \int_{\mathcal{Q} \times \mathcal{Q}} dq\, dq'\, \Phi^*(q)\, K(q, q')\, \Phi'(q').$$

It follows that

$$\langle \Psi|P(B)HP(C)|\Psi \rangle = \int_{\mathcal{Q} \times \mathcal{Q}} dq\, dq'\, \mathbf{1}_B(q)\, \Psi^*(q)\, K(q, q')\, \mathbf{1}_C(q')\, \Psi(q') = \tag{68a}$$

$$= \int_{B \times C} dq\, dq'\, \Psi^*(q)\, K(q, q')\, \Psi(q'). \tag{68b}$$

Note that since H is bounded, its form domain is $\mathscr{H}$ and thus contains all $P(B)\Psi$. For $A \subseteq \mathcal{Q} \times \mathcal{Q}$, define

$$\mu(A) = \int_A \Psi^*(q)\, K(q, q')\, \Psi(q')\, dq\, dq' \, .$$

Since

$$\int_{\mathcal{Q} \times \mathcal{Q}} |\Psi(q)|\, |K(q, q')|\, |\Psi(q')|\, dq\, dq' < \infty \, ,$$

28

$\mu(A)$ is always finite, and thus a complex measure. (68) entails that (64) is satisfied, so that Theorem 1 applies. $\qquad\square$

We have already remarked that every Hamiltonian H can be approximated by Hilbert–Schmidt operators H_n. In this context, it is interesting to note that if H is itself a Hilbert–Schmidt operator, and if the H_n converge to H in the Hilbert–Schmidt norm, then the rates σ^{Ψ,H_n} converge to $\sigma^{\Psi,H}$ in the sense that

$$\int\limits_{\mathcal{Q}\times\mathcal{Q}} \left|\sigma^{\Psi,H_n}(dq|q') - \sigma^{\Psi,H}(dq|q')\right| |\Psi(q')|^2 \, dq' \overset{n\to\infty}{\longrightarrow} 0.$$

4.2.2 Complex-Valued Wave Functions

In addition to the case of Hilbert–Schmidt operators, Theorem 1 applies in many other cases, in which the kernel $K(q,q')$ is not square-integrable, nor even a function but instead a measure $K(dq\times dq')$. More precisely, $K(dq\times dq')$ should be a σ-*finite complex measure*, i.e., a product of a complex-valued measurable function $\mathcal{Q}\times\mathcal{Q}\to\mathbb{C}$ and a σ-finite nonnegative measure on $\mathcal{Q}\times\mathcal{Q}$. (Note that this terminology involves a slight abuse of language since a σ-finite complex measure need not be a complex measure.) The complex measure μ assumed to exist in Theorem 1 is then

$$\mu(dq\times dq') = \Psi^*(q)\, K(dq\times dq')\, \Psi(q') . \tag{69}$$

This equation suggests that the minimal amount of regularity that we need to assume on the kernel of H is that it be a σ-finite complex measure. Otherwise, there would be no hope that (69) could be a complex measure for a generic wave function Ψ, that vanishes at most on a set of measure 0. The exact conditions that we need for applying Theorem 1 to a Hamiltonian H with kernel $K(dq\times dq')$ are listed in the following statement:

Corollary 2 *Let $\mathcal{Q}$ be a standard Borel space, $\mathscr{H} = L^2(\mathcal{Q},\mathbb{C},dq)$ with respect to a σ-finite nonnegative measure on $\mathcal{Q}$ that we simply denote dq, let $\Psi\in\mathscr{H}$ with $\|\Psi\|=1$, let H be a self-adjoint operator on $\mathscr{H}$, and let P be the natural PVM on $\mathcal{Q}$ acting on $L^2(\mathcal{Q},\mathbb{C},dq)$. Suppose that H has a kernel $K(dq\times dq')$ for Ψ; i.e., suppose that $K(dq\times dq')$ is a σ-finite complex measure on $\mathcal{Q}\times\mathcal{Q}$, and that some everywhere-defined version $\Psi:\mathcal{Q}\to\mathbb{C}$ of the almost-everywhere-defined function $\Psi\in L^2(\mathcal{Q},\mathbb{C},dq)$ satisfies*

$$\int\limits_{\mathcal{Q}\times\mathcal{Q}} |\Psi(q)|\, |K(dq\times dq')|\, |\Psi(q')| < \infty \tag{70a}$$

$$P(B)\Psi \in \text{form domain}(H) \quad \forall B\subseteq\mathcal{Q} \tag{70b}$$

$$\langle\Psi|P(B)HP(C)|\Psi\rangle = \int\limits_{B\times C} \Psi^*(q)\, K(dq\times dq')\, \Psi(q') \quad \forall B,C\subseteq\mathcal{Q}. \tag{70c}$$

Then, by virtue of Theorem 1, the jump rates given by (1) are well-defined and finite $\mathbb{P}$-almost everywhere, and they are equivariant if $\Psi\in\text{domain}(H)$.

Proof. Set

$$\mu(A) = \int\limits_A \Psi^*(q)\, K(dq \times dq')\, \Psi(q')\,. \tag{71}$$

The integral exists because of (70a) and defines a complex measure μ, which satisfies (64) because of (70c). $\qquad\square$

We remark that the choice of the everywhere-defined version $\Psi : \mathcal{Q} \to \mathbb{C}$ of the almost-everywhere-defined function $\Psi \in L^2(\mathcal{Q}, \mathbb{C}, dq)$ does not affect the jump rates, since the measure μ is uniquely determined by its values on product sets, which are given in (64) in terms of the almost-everywhere-defined function $\Psi \in \mathscr{H}$.

The reader may be surprised that our notion of H having a kernel K seems to depend on Ψ, whereas one may expect that H either has a kernel or does not, independent of Ψ. The reason for our putting it this way is that domain questions are very delicate for such general kernels, and it is a tricky question for which Ψ's the expression $\langle \Psi | P(B) H P(C) | \Psi \rangle$ is actually given by the integral (70c). A discussion of domain questions would only obscure what is actually relevant for having a situation in which Theorem 1 applies, which is (70). Note, though, that if H has kernel $K(dq \times dq')$ for Ψ, then it has kernel K also for every Ψ' from the subspace spanned by $P(B)\Psi$ for all $B \subseteq \mathcal{Q}$.

The conditions (70) become very transparent in the following case: Suppose H is a self-adjoint extension of the integral operator K arising from a kernel $K(q, dq')$ that is a σ-finite complex measure on $\mathcal{Q}$ for every $q \in \mathcal{Q}$ and is such that for every $B \subseteq \mathcal{Q}$, $K(q, B)$ is a measurable function of q. K is defined by

$$K\Phi(q) = \int\limits_{q' \in \mathcal{Q}} K(q, dq')\, \Phi(q') \tag{72}$$

on the domain $\mathscr{D}$ containing the Φ's satisfying

$$\int\limits_{q' \in \mathcal{Q}} |K(q, dq')|\, |\Phi(q')| < \infty \text{ for almost every } q \tag{73}$$

and

$$\int\limits_{\mathcal{Q}} K(q, dq')\, \Phi(q') \text{ is an } L^2 \text{ function of } q. \tag{74}$$

That H is an extension of K means that the domain of H contains $\mathscr{D}$, and $H\Phi = K\Phi$ for all $\Phi \in \mathscr{D}$. Then, for a $\Psi \in \mathscr{D}$ satisfying

$$\int\limits_{q' \in B} K(q, dq')\, \Psi(q') \in L^2(\mathcal{Q}, \mathbb{C}, dq) \quad \forall B \subseteq \mathcal{Q} \tag{75}$$

and

$$\int\limits_{\mathcal{Q} \times \mathcal{Q}} |\Psi(q)|\, |K(q, dq')|\, |\Psi(q')|\, dq < \infty, \tag{76}$$

30

conditions (70) are satisfied with $K(dq \times dq') = K(q, dq') \, dq$, and thus Corollary 2 applies. The jump rates (1) can still be written as in (19), understood as a measure in q.

Corollary 2 defines a set of good Ψ's, for which the jump rates are finite, for the examples of Sections 3.1, 3.4, and for (38a).

4.2.3 Vector-Valued Wave Functions

We now consider wave functions with spin, i.e., with values in $\mathbb{C}^k$. In this case, let $\Psi^*(q)$ denote, as before, the adjoint spinor, and $\Phi^*(q) \, \Psi(q)$ the inner product in $\mathbb{C}^k$. Corollary 2 remains true if we replace $\mathbb{C}$ by $\mathbb{C}^k$ everywhere and understand $K(dq \times dq')$ as matrix-valued, i.e., as the product of a matrix-valued function and a σ-finite nonnegative measure. The proof goes through without changes.

Let us now be a bit more general and allow the value space of the wave function to vary with q; we reformulate Corollary 2 for wave functions that are cross-sections of a vector bundle E over $\mathcal{Q}$. The kernel is then matrix valued in the sense that $\langle q|H|q' \rangle$ is a linear mapping $E_{q'} \to E_q$.

Corollary 3 *Let $\mathcal{Q} = \bigcup_n \mathcal{Q}^{(n)}$ be an (at most) countable union of (separable) Riemannian manifolds, and $E = \bigcup_n E^{(n)}$ the union of vector bundles $E^{(n)}$ over $\mathcal{Q}^{(n)}$, where the fiber spaces E_q are endowed with Hermitian inner products, which we denote by $\Phi^*(q) \, \Psi(q)$. Let $\mathscr{H} = L^2(E, dq)$ be the space of square-integrable (with respect to the Riemannian volume measure that we denote dq) cross-sections of E, let $\Psi \in \mathscr{H}$ with $\|\Psi\| = 1$, let H be a self-adjoint operator on $\mathscr{H}$, and let P be the natural PVM on $\mathcal{Q}$ acting on $\mathscr{H}$. Suppose that $K(dq \times dq')$, the product of a σ-finite nonnegative measure on $\mathcal{Q} \times \mathcal{Q}$ and a section of the bundle $\bigcup_{q,q'} E_q \otimes E_{q'}^*$ over $\mathcal{Q} \times \mathcal{Q}$, is a kernel of H for Ψ; i.e., suppose that some everywhere-defined version Ψ of the almost-everywhere-defined cross-section $\Psi \in L^2(E, dq)$ satisfies (70a)-(70c) (where the integrand on the right hand side of (70c) should now be understood as involving the Hermitian inner product of E_q, and (70a) as involving the operator norm of $K(dq \times dq')$). Then, by virtue of Theorem 1, the jump rates given by (1) are well-defined and finite $\mathbb{P}$-almost everywhere, and they are equivariant if $\Psi \in \mathrm{domain}(H)$.*

The proof of Corollary 2 applies here without changes. (19) remains valid if suitably interpreted. Corollary 3 defines a set of good Ψ's, for which the jump rates are finite, for the examples of Sections 3.2, 3.3, 3.10, 3.12, and for (39) in case the sum over i is always finite.

4.2.4 POVMs

We now proceed to the fully general case of an arbitrary POVM. First, we provide two important mathematical tools for dealing with POVMs.

- Any POVM corresponds to a PVM on a larger Hilbert space, according to the following theorem of Naimark [7, p. 142]: *If P is a POVM on the standard Borel*

space $\mathcal{Q}$ acting on the Hilbert space $\mathcal{H}$, then there is a Hilbert space $\mathcal{H}_{\text{ext}} \supseteq \mathcal{H}$ and a PVM P_{ext} on $\mathcal{Q}$ acting on $\mathcal{H}_{\text{ext}}$ such that $P(\,\cdot\,) = P_+ P_{\text{ext}}(\,\cdot\,)I$ with $P_+ :$ $\mathcal{H}_{\text{ext}} \to \mathcal{H}$ the projection and $I : \mathcal{H} \to \mathcal{H}_{\text{ext}}$ the inclusion, and $\mathcal{H}_{\text{ext}}$ is the closed linear hull of $\{P_{\text{ext}}(B)\mathcal{H} : B \subseteq \mathcal{Q}\}$. The pair $\mathcal{H}_{\text{ext}}, P_{\text{ext}}$ is unique in the sense that if $\mathcal{H}'_{\text{ext}}, P'_{\text{ext}}$ is another such pair then there is a unitary isomorphism between $\mathcal{H}_{\text{ext}}$ and $\mathcal{H}'_{\text{ext}}$ fixing $\mathcal{H}$ and carrying P_{ext} to P'_{ext}.

We call $\mathcal{H}_{\text{ext}}$ and P_{ext} the *Naimark extension* of $\mathcal{H}$ and P. We recall that for the Hilbert space of positive energy solutions of the Dirac equation and the corresponding POVM introduced earlier, the Naimark extension is given by $L^2(\mathbb{R}^3, \mathbb{C}^4)$ and its natural PVM; this example indicates that the Naimark extension may be, in practice, something natural to consider.

- In Corollaries 2 and 3, we were considering, for $\mathcal{H}$ and P, L^2 spaces with their *natural* PVMs. But when we are given an *arbitrary* PVM on a Hilbert space, the situation is not genuinely more general, since it can be viewed as the natural PVM of an L^2 space. We call this the *naturalization* of the PVM. It is based on the following version of the spectral theorem (which can be obtained from the representation theory of abelian operator algebras, see, e.g., [9]): *If P is a PVM on the standard Borel space $\mathcal{Q}$ acting on the Hilbert space $\mathcal{H}$, then there is a measurable field of Hilbert spaces[8] $\mathcal{H}_q$ over $\mathcal{Q}$, a σ-finite nonnegative measure dq on $\mathcal{Q}$, and a unitary isomorphism $U : \mathcal{H} \to \int^{\oplus} \mathcal{H}_q \, dq$ to the direct integral[9] of $\mathcal{H}_q$ that carries P to the natural PVM on $\mathcal{Q}$ acting on $\int^{\oplus} \mathcal{H}_q \, dq$. The naturalization is unique in the sense that if $\{\mathcal{H}'_q\}, (dq)', U'$ is another such triple, then there is a measurable function $f : \mathcal{Q} \to (0, \infty)$ such that $(dq)' = f(q)\,dq$ and a measurable field of unitary isomorphisms $U_q : \mathcal{H}_q \to \mathcal{H}'_q$ such that $U'\Psi(q) = f(q)^{-1/2}U_q U \Psi(q)$.*

A naturalized PVM is similar to a vector bundle in that with every $q \in \mathcal{Q}$ there is associated a value space $\mathcal{H}_q$, which however may be infinite-dimensional, and $\Psi \in \mathcal{H}$ can be understood as a function on $\mathcal{Q}$ such that $\Psi(q) \in \mathcal{H}_q$. Of course, instead of the differentiable structure of a vector bundle the naturalization of a PVM leads merely to a measurable structure.

Thus, the situation with a general POVM is not much different from the situation with a vector bundle, as treated in Corollary 3.

For Hilbert–Schmidt operators, the kernel is so well-behaved that no further conditions on Ψ are necessary:

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[8]A *measurable field of Hilbert spaces* on $\mathcal{Q}$ is a family of Hilbert spaces $\mathcal{H}_q$ with scalar products $\langle\!\langle \,\cdot\, | \,\cdot\, \rangle\!\rangle_q$, endowed with a measurable structure that can be defined by specifying a family of cross-sections $\Phi_i(q)$ such that for all i, i' the functions $q \mapsto \langle\!\langle \Phi_i(q)|\Phi_{i'}(q) \rangle\!\rangle_q$ are measurable and for every q the family $\Phi_i(q)$ is total in $\mathcal{H}_q$ [18].

[9]This is the Hilbert space of square-integrable measurable cross-sections of the field $\{\mathcal{H}_q\}$, i.e., cross-sections $\Phi(q)$ such that all functions $q \mapsto \langle\!\langle \Phi_i(q)|\Phi(q) \rangle\!\rangle_q$ are measurable and $\int \langle\!\langle \Phi(q)|\Phi(q) \rangle\!\rangle_q \, dq < \infty$ [18].

Corollary 4 *Let $\mathscr{H}$ be a Hilbert space, $\Psi \in \mathscr{H}$ with $\|\Psi\| = 1$, H a self-adjoint operator on $\mathscr{H}$, $\mathcal{Q}$ a standard Borel space, and let P be a POVM on $\mathcal{Q}$ acting on $\mathscr{H}$. Suppose that H is a Hilbert–Schmidt operator. Then, by virtue of Theorem 1, the jump rates given by (1) are well-defined and finite $\mathbb{P}$-almost everywhere, and they are equivariant.*

Proof. Let P_{ext} be the Naimark extension PVM of P acting on $\mathscr{H}_{\mathrm{ext}} \supseteq \mathscr{H}$ with P_+ the projection $\mathscr{H}_{\mathrm{ext}} \to \mathscr{H}$, and let $U : \mathscr{H}_{\mathrm{ext}} \to \int^{\oplus} \mathscr{H}_q \, dq$ be a naturalization of P_{ext}. For every $q \in \mathcal{Q}$, pick an orthonormal basis $\mathscr{I}_q = \{|q, i\rangle\}$ of $\mathscr{H}_q$, with measurable dependence on q. When each set $\mathscr{I}_q$ is thought of as equipped with the counting measure, then from dq we obtain a measure on $\mathscr{I} = \bigcup_q \mathscr{I}_q$, and $\int^{\oplus} \mathscr{H}_q \, dq$ is naturally identified with $L^2(\mathscr{I}, \mathbb{C})$. Since H is a Hilbert–Schmidt operator, so is $H_{\mathrm{ext}} = IHP_+$, which thus possesses a kernel function $K \in L^2(\mathscr{I} \times \mathscr{I}, \mathbb{C})$ such that for all $\Phi \in \mathscr{H}_{\mathrm{ext}}$

$$UH_{\mathrm{ext}}\Phi(q, i) = \int_{\mathcal{Q}} dq' \sum_{i' \in \mathscr{I}_{q'}} K(q, i, q', i') \, U\Phi(q', i').$$

Since

$$\langle \Psi | P(B) H P(C) | \Psi \rangle = \langle \Psi | P_{\mathrm{ext}}(B) H_{\mathrm{ext}} P_{\mathrm{ext}}(C) | \Psi \rangle \, ,$$

we have, for the same reasons as in the proof of Corollary 1, that

$$\langle \Psi | P(B) H P(C) | \Psi \rangle = \int_{B \times C} dq \, dq' \sum_{i \in \mathscr{I}_q} \sum_{i' \in \mathscr{I}_{q'}} U\Psi^*(q, i) \, K(q, i, q', i') \, U\Psi(q', i') \, . \qquad (77)$$

For $A \subseteq \mathcal{Q} \times \mathcal{Q}$, set

$$\mu(A) = \int_{A} dq \, dq' \sum_{i \in \mathscr{I}_q} \sum_{i' \in \mathscr{I}_{q'}} U\Psi^*(q, i) \, K(q, i, q', i') \, U\Psi(q', i').$$

Since $U\Psi^*(q, i) \, K(q, i, q', i') \, U\Psi(q', i')$ is absolutely summable and integrable over q, i, q', and i', $\mu(A)$ is finite, and thus a complex measure. (77) entails that (64) is satisfied. Thus Theorem 1 applies. $\qquad \square$

We now provide the most general version of our statement about jump rates for Hamiltonians with kernel measures. Let $\mathscr{B}(\mathscr{H}_{q'}, \mathscr{H}_q)$ denote the space of bounded linear operators $\mathscr{H}_{q'} \to \mathscr{H}_q$ with the operator norm

$$|O| = \sup_{\Phi \in \mathscr{H}_{q'}, \Phi \neq 0} \frac{\|O\Phi\|}{\|\Phi\|}.$$

For the norm of $\Psi(q)$ in $\mathscr{H}_q$, $\langle\!\langle \Psi(q) | \Psi(q) \rangle\!\rangle_q^{1/2}$, we also write $|\Psi(q)|$.

Corollary 5 *Let $\mathscr{H}$ be a Hilbert space, $\Psi \in \mathscr{H}$ with $\|\Psi\| = 1$, H a self-adjoint operator on $\mathscr{H}$, $\mathcal{Q}$ a standard Borel space, and P a POVM on $\mathcal{Q}$ acting on $\mathscr{H}$. Let P_{ext} be the Naimark extension PVM of P acting on $\mathscr{H}_{\mathrm{ext}} \supseteq \mathscr{H}$, and $U : \mathscr{H}_{\mathrm{ext}} \to \int^{\oplus} \mathscr{H}_q \, dq$ the*

naturalization of P_{ext}. Suppose that H has a kernel $K(dq \times dq')$ for Ψ in the position representation defined by P; i.e., suppose that $K(dq \times dq')$ is the product of a σ-finite nonnegative measure on $\mathcal{Q} \times \mathcal{Q}$ and a measurable cross-section of the field $\mathscr{B}(\mathscr{H}_{q'}, \mathscr{H}_q)$ over $\mathcal{Q} \times \mathcal{Q}$, that Ψ satisfies (70b), and that some everywhere-defined version $\Psi(q)$ of the almost-everywhere-defined cross-section $U\Psi \in \int^{\oplus} \mathscr{H}_q \, dq$ satisfies (70a) and (70c) (where the integrand on the right hand side of (70c) is understood as involving the inner product of $\mathscr{H}_q$). Then, by virtue of Theorem 1, the jump rates given by (1) are well-defined and finite $\mathbb{P}$-almost everywhere, and they are equivariant if $\Psi \in \text{domain}(H)$.

The proof of Corollary 2 applies here without changes if one understands $\Psi^*(q) \, K(dq \times dq') \, \Psi(q)$ as meaning $\langle\!\langle \Psi(q) | K(dq \times dq') \, \Psi(q') \rangle\!\rangle_q$. Corollary 5 defines a set of good Ψ's, for which the jump rates are finite, for the examples of Sections 3.7, 3.8, and 3.9.

4.3 Global Existence Question

The rates σ_t and velocities v_t, together with $\mathbb{P}_t$, define the process Q_t associated with H, P, and Ψ, which can be constructed along the lines of Section 2.3. However, the rigorous existence of this process, like the global existence of solutions for an ordinary differential equation, is no trivial matter. In order to establish the global existence of the process (see [15] for an example), a variety of aspects must be controlled, including the following: (i) One has to show that for a sufficiently large set of initial state vectors, the relevant conditions for finiteness of the jump rates, see Sections 4.1 and 4.2, are satisfied at all times. (ii) One has to show that there is probability zero that infinitely many jumps accumulate in finite time. (iii) One has to show that there is probability zero that the process runs into a configuration where σ is ill defined (e.g., where the denominator of (19) vanishes, if that equation is appropriate).

4.4 Extensions of Bi-Measures

We have pointed out in the next-to-last paragraph of Section 4.1 that a complex bi-measure need not possess an extension to a complex measure on the product space, a fact relevant to the conditions for finite rates. In this section we show, see Theorem 2 below, that nonnegative real bi-measures always possess such an extension.

A useful corollary of Theorem 2, see Corollary 7 below, asserts that one can form the tensor product of any two POVMs. This is a special case of the more general statement, see Corollary 6 below, asserting that one can form the product of any two POVMs that commute with each other; this statement can be regarded as the generalization from PVMs to POVMs of the fact that two commuting observables can be measured simultaneously; it is also related to the discussion in the last paragraph of Section 4.1.

Though we could not find the explicit statement of Corollary 6 in this form in the literature, it does follow from a part of a proof given by Halmos [20, p. 72]. Below, however, we give a somewhat different proof, using Theorem 2 instead of the lemma of

von Neumann [33, p. 167] that Halmos uses. It is also presumably possible to derive Corollary 6 from Lemma 2.1 or Theorem 2.2 of [7].

Theorem 2 *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be standard Borel spaces with σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, and let $\nu(\,\cdot\,,\,\cdot\,)$ be a finite nonnegative bi-measure, i.e., a mapping $\nu : \mathcal{A}_1 \times \mathcal{A}_2 \to [0,a]$, $a > 0$, that is a measure in each variable when the other variable is a fixed set. Then ν can be extended to a measure μ on $\mathcal{Q}_1 \times \mathcal{Q}_2$: there exists a unique finite nonnegative measure $\mu : \mathcal{A}_1 \otimes \mathcal{A}_2 \to [0,a]$ such that for all $B_1 \in \mathcal{A}_1$ and $B_2 \in \mathcal{A}_2$,*

$$\mu(B_1 \times B_2) = \nu(B_1, B_2). \tag{78}$$

Proof. Suppose first that $\mathcal{Q}_1$ is finite or countably infinite. Then every set $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is an (at most) countable union of product sets,

$$A = \bigcup_{q_1 \in \mathcal{Q}_1} \{q_1\} \times B_{q_1},$$

where every $B_{q_1} \in \mathcal{A}_2$. Therefore, the unique way of extending ν is by setting

$$\mu(A) := \sum_{q_1 \in \mathcal{Q}_1} \nu(\{q_1\}, B_{q_1}). \tag{79}$$

One easily checks that (79) indeed defines a finite measure satisfying (78), noting first that the sum is always finite because $\sum \nu(\{q_1\}, B_{q_1}) \leq \sum \nu(\{q_1\}, \mathcal{Q}_2) = \nu(\mathcal{Q}_1, \mathcal{Q}_2)$. The same argument can of course be applied if $\mathcal{Q}_2$ is finite or countably infinite.

Suppose now that neither $\mathcal{Q}_1$ nor $\mathcal{Q}_2$ is finite or countable. Every uncountable standard Borel space $\mathcal{Q}$ is isomorphic, as a measurable space, to the space of binary sequences $\{0,1\}^{\mathbb{N}}$ (equipped with the σ-algebra generated by the family $\mathcal{B}$ of sets that depend on only finitely many terms of the sequence), i.e., there exists a bijection $\varphi : \mathcal{Q} \to \{0,1\}^{\mathbb{N}}$ that is measurable in both directions, see [24, p. 138] and [23, p. 358]. We may thus assume, without loss of generality, that $\mathcal{Q}_1 = \{0,1\}^{\{-1,-2,-3,\dots\}}$ and $\mathcal{Q}_2 = \{0,1\}^{\{0,1,2,\dots\}}$, with $\mathcal{B}_i$ defined accordingly. $\mathcal{Q}_1 \times \mathcal{Q}_2$ can then be canonically identified with $\{0,1\}^{\mathbb{Z}}$.

From the restriction of ν to sets $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, one easily obtains a consistent family of finite-dimensional distributions, and hence, by the Kolmogorov extension theorem, e.g. [6, p. 24], a unique measure μ on $\mathcal{Q}_1 \times \mathcal{Q}_2$ obeying (78) for all $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

It remains to establish (78) for all $B_1 \in \mathcal{A}_1$ and $B_2 \in \mathcal{A}_2$. First fix B_2 in $\mathcal{B}_2$. Then $\mu(\,\cdot\, \times B_2)$ and $\nu(\,\cdot\,, B_2)$ are measures on $\mathcal{A}_1$ that agree on $\mathcal{B}_1$. Hence they agree on $\mathcal{A}_1$. Thus, fixing B_1 in $\mathcal{A}_1$, we have that $\mu(B_1 \times \,\cdot\,)$ and $\nu(B_1, \,\cdot\,)$ are measures on $\mathcal{A}_2$ that agree on $\mathcal{B}_2$, and hence on all of $\mathcal{A}_2$, completing the proof. $\qquad\square$

In the following, we will again write $B_1 \subseteq \mathcal{Q}_1$ instead of $B_1 \in \mathcal{A}_1$.

Corollary 6 *Let $\mathscr{H}$ be a Hilbert space, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ standard Borel spaces, and P_1 and P_2 POVMs on $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively, acting on $\mathscr{H}$. If $[P_1(B_1), P_2(B_2)] = 0$ for all*

$B_1 \subseteq \mathcal{Q}_1$ and $B_2 \subseteq \mathcal{Q}_2$, then there exists a unique POVM P on $\mathcal{Q}_1 \times \mathcal{Q}_2$ acting on $\mathcal{H}$ such that for all $B_1 \subseteq \mathcal{Q}_1$ and $B_2 \subseteq \mathcal{Q}_2$,

$$P(B_1 \times B_2) = P_1(B_1)P_2(B_2). \tag{80}$$

Proof. (We largely follow [20, p. 72].) For $\Psi \in \mathcal{H}$ we define a bi-measure ν_Ψ by setting $\nu_\Psi(B_1, B_2) := \langle \Psi | P_1(B_1)P_2(B_2) | \Psi \rangle$. ν_Ψ is obviously a complex bi-measure, and it takes values only in the nonnegative reals because $P_1(B_1)P_2(B_2)$ is a positive operator (since the two positive operators $P_1(B_1)$ and $P_2(B_2)$ can be simultaneously diagonalized). The values of ν_Ψ are bounded by $\|\Psi\|^2$. By Lemma 2, ν_Ψ can be extended to a measure μ_Ψ on $\mathcal{Q}_1 \times \mathcal{Q}_2$.

We now define complex measures $\mu_{\Phi,\Psi}$ on $\mathcal{Q}_1 \times \mathcal{Q}_2$ by "polarization": for every $A \subseteq \mathcal{Q}_1 \times \mathcal{Q}_2$ and for every pair of vectors Φ, Ψ we write

$$\mu_{\Phi,\Psi}(A) := \mu_{\frac{1}{2}\Phi + \frac{1}{2}\Psi}(A) - \mu_{\frac{1}{2}\Phi - \frac{1}{2}\Psi}(A) + i\mu_{\frac{1}{2}\Phi - \frac{i}{2}\Psi}(A) - i\mu_{\frac{1}{2}\Phi + \frac{i}{2}\Psi}(A). \tag{81}$$

We assert that $\mu_{\Phi,\Psi}(A)$ is, for each fixed set A, a symmetric bilinear functional. This assertion is proved by noting that (i) it is true if $A = B_1 \times B_2$, and (ii) the class of all sets for which it is true is closed under the formation of complements and countable unions. To see (ii), note that $\mu_{\Phi,\Psi}(A^c) = \mu_{\Phi,\Psi}(\mathcal{Q}_1 \times \mathcal{Q}_2) - \mu_{\Phi,\Psi}(A)$ and $\mu_{\Phi,\Psi}(\bigcup_{k=1}^\infty A_k) = \lim_{n\to\infty} \sum_{k=1}^n \mu_{\Phi,\Psi}(A_k)$.

Since $\mu_{\Psi,\Psi}(A) = \mu_\Psi(A) \leq \|\Psi\|^2$ for every $A \subseteq \mathcal{Q}_1 \times \mathcal{Q}_2$, the bilinear functional $\mu_{\Phi,\Psi}(A)$ is bounded and has, in fact, a norm ≤ 1. Therefore, there is a bounded operator $P(A)$ such that $\mu_{\Phi,\Psi}(A) = \langle \Phi | P(A) | \Psi \rangle$. $P(A)$ is positive since $\mu_{\Psi,\Psi}(A) \geq 0$ for every Ψ. $P(\cdot)$ is countably additive in the weak operator topology because $\mu_{\Phi,\Psi}(\cdot)$ is countably additive. $P(\cdot)$ satisfies (80), and thus $P(\mathcal{Q}_1 \times \mathcal{Q}_2) = I$. $\qquad\square$

Note that P_1 need not be a commuting POVM, i.e., possibly $[P_1(B_1), P_1(C_1)] \neq 0$, and correspondingly for P_2.

An immediate consequence of Corollary 6, which we use in several places of [12], is

Corollary 7 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ standard Borel spaces, and P_1 and P_2 POVMs on $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively, acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then there exists a unique POVM P on $\mathcal{Q}_1 \times \mathcal{Q}_2$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that for all $B_1 \subseteq \mathcal{Q}_1$ and $B_2 \subseteq \mathcal{Q}_2$,*

$$P(B_1 \times B_2) = P_1(B_1) \otimes P_2(B_2). \tag{82}$$

5 Minimality

In this section we explain in what sense the minimal jump rates (1)—or (19) or (38a)—are minimal. In so doing, we will also explain the significance of the quantity $\mathbb{J}$ defined in (25), and clarify the meaning of the steps taken in Sections 2.4 and 2.5 to arrive at the jump rate formulas.

Given a Markov process Q_t on $\mathcal{Q}$, we define the *net probability current* j_t at time t between sets B and B' by

$$j_t(B, B') = \lim_{\Delta t \searrow 0} \frac{1}{\Delta t} \left[\text{Prob}\{Q_t \in B', Q_{t+\Delta t} \in B\} - \right. \tag{83}$$

$$\left. -\text{Prob}\{Q_t \in B, Q_{t+\Delta t} \in B'\} \right].$$

This is the amount of probability that flows, per unit time, from B' to B minus the amount from B to B'. For a pure jump process, we have that

$$j_t(B, B') = \int_{q' \in B'} \sigma_t(B|q') \, \rho_t(dq') - \int_{q \in B} \sigma_t(B'|q) \, \rho_t(dq) \,, \tag{84}$$

so that
$$j_t(B, B') = j_{\sigma, \rho}(B \times B') \tag{85}$$

where $j_{\sigma, \rho}$ is the signed measure, on $\mathcal{Q} \times \mathcal{Q}$, given by the integrand of (16),

$$j_{\sigma, \rho}(dq \times dq') = \sigma(dq|q') \, \rho(dq') - \sigma(dq'|q) \, \rho(dq) \,. \tag{86}$$

For minimal jump rates σ, defined by (1) or (19) or (38a) (and with the probabilities ρ given by (21), $\rho = \mathbb{P}$), this agrees with (25), as was noted earlier,

$$j_{\sigma, \rho} = \mathbb{J}_{\Psi, H, P} \,, \tag{87}$$

where we have made explicit the fact that $\mathbb{J}$ is defined in terms of the quantum entities Ψ, H, and P. Note that both $\mathbb{J}$ and the net current j are anti-symmetric, $\mathbb{J}^{\text{tr}} = -\mathbb{J}$ and $j^{\text{tr}} = -j$, the latter by construction and the former because H is Hermitian. (Here tr indicates the action on measures of the transposition $(q, q') \mapsto (q', q)$ on $\mathcal{Q} \times \mathcal{Q}$.) The property (87) is stronger than the equivariance of the rates σ, $\mathcal{L}_\sigma \mathbb{P}_t = d\mathbb{P}_t/dt$: Since, by (16),

$$(\mathcal{L}_\sigma \rho)(dq) = j_{\sigma, \rho}(dq \times \mathcal{Q}), \tag{88}$$

and, by (25),

$$\frac{d\mathbb{P}}{dt}(dq) = \mathbb{J}(dq \times \mathcal{Q}), \tag{89}$$

the equivariance of the jump rates σ amounts to the condition that the marginals of both sides of (87) agree,

$$j_{\sigma, \rho}(dq \times \mathcal{Q}) = \mathbb{J}(dq \times \mathcal{Q}) \,. \tag{90}$$

In other words, what is special about processes with rates satisfying (87) is that not only the single-time *distribution* but also the *current* is given by a standard quantum theoretical expression in terms of H, Ψ, and P. That is why we call (87) the *standard-current property*—defining *standard-current rates* and *standard-current processes*.

Though the standard-current property is stronger than equivariance, it alone does not determine the jump rates, as already remarked in [2, 29]. This can perhaps be best

appreciated as follows: Note that (86) expresses $j_{\sigma,\rho}$ as twice the anti-symmetric part of the (nonnegative) measure

$$C(dq \times dq') = \sigma(dq|q')\, \rho(dq') \tag{91}$$

on $\mathcal{Q} \times \mathcal{Q}$ whose right marginal $C(\mathcal{Q} \times dq')$ is absolutely continuous with respect to ρ. Conversely, from any such measure C the jump rates σ can be recovered by forming the Radon–Nikodým derivative

$$\sigma(dq|q') = \frac{C(dq \times dq')}{\rho(dq')}\,. \tag{92}$$

Thus, given ρ, specifying σ is equivalent to specifying such a measure C.

In terms of C, the standard-current property becomes (with $\rho = \mathbb{P}$)

$$2\,\mathrm{Anti}\,C = \mathbb{J}. \tag{93}$$

Since (recalling that $\mathbb{J} = \mathbb{J}^+ - \mathbb{J}^-$ is anti-symmetric)

$$\mathbb{J} = 2\,\mathrm{Anti}\,\mathbb{J}^+, \tag{94}$$

an obvious solution to (93) is

$$C = \mathbb{J}^+,$$

corresponding to the minimal jump rates. However, (87) fixes only the anti-symmetric part of C. The general solution to (93) is of the form

$$C = \mathbb{J}^+ + S \tag{95}$$

where $S(dq \times dq')$ is symmetric, since any two solutions to (93) have the same anti-symmetric part, and $S \geq 0$, since $S = C \wedge C^{\mathrm{tr}}$, because $\mathbb{J}^+ \wedge (\mathbb{J}^+)^{\mathrm{tr}} = 0$.

In particular, for any standard-current rates, we have that

$$C \geq \mathbb{J}^+, \quad \text{or} \quad \sigma(dq|q') \geq \frac{\mathbb{J}^+(dq \times dq')}{\mathbb{P}(dq')}. \tag{96}$$

Thus, among all jump rates consistent with the standard-current property, one choice, distinguished by equality in (96), has the least frequent jumps, or the smallest amount of stochasticity: the minimal rates (1).

6 Remarks

6.1 Symmetries

Quantum theories, and in particular QFTs, often have important symmetries. To name a few examples: space translations, rotations and inversion, time translations and reversal, Galilean or Lorentz boosts, global change of phase $\Psi \to e^{i\theta}\Psi$, and gauge transformations.

This gives rise to the question whether the process Q_t of the corresponding Bell-type QFT respects these symmetries as well. Except for Lorentz invariance, which is difficult in that Lorentz boosts fail to map equal-time configurations into equal-time configurations, the answer is yes; a discussion is given in [12, Sec. 6.1]. An essential ingredient of this result is the manifest fact that the minimal jump rates (1) inherit the symmetries of the Hamiltonian (under which the POVM transforms covariantly).

6.2   Homogeneity of the Rates

The minimal jump rates (1) define a homogeneous function of degree 0 in Ψ, i.e., $\sigma^{\lambda\Psi} = \sigma^{\Psi}$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. This property is noteworthy since it forms the essential mathematical basis for a number of desirable properties of theories using such jump rates (such as that of [11]): (i) that (when P is a product PVM) unentangled and decoupled subsystems behave independently and follow the same laws as the entire system, (ii) that "collapsed-away," i.e., sufficiently distant, parts of the wave function do not influence the future behaviour of the configuration Q_t, (iii) invariance under a global change of phase $\Psi \to e^{i\theta}\Psi$, (iv) invariance under the replacement $\Psi \to e^{-iEt/\hbar}\Psi$ for some constant E, which corresponds to adding E to the total Hamiltonian, (v) invariance under relabeling of the particles (which may cause a replacement $\Psi \to -\Psi$ due to the Pauli principle).

6.3 $H + E$

Adding a constant E to the interaction Hamiltonian will not change the jump rates (1) provided P is a PVM. This is because $\langle\Psi|P(B)EP(C)|\Psi\rangle = E\langle\Psi|P(B \cap C)|\Psi\rangle$ has vanishing imaginary part. For a POVM, however, this need not be true.

6.4   Nondegenerate Eigenstates

As mentioned earlier, after (19), it is a consequence of the minimal jump rate formula (1), in fact of the very minimality, that at each time t either $\sigma(q|q')$ or $\sigma(q'|q)$ is zero. It follows that for a time-reversible Hamiltonian H and POVM P, all jump rates vanish if Ψ is a nondegenerate eigenstate of H. This is because, in the simplest cases, $\langle q|H|q'\rangle$ is real, and the coefficients $\langle q|\Psi\rangle$ can also be chosen real, or, more generally and more to the point, because in this case the process must coincide with its time reverse, which implies that the current from q to q' is as large as the one from q' to q, so that minimality requires both to vanish.

6.5   Left or Right Continuity

From what we have said so far, there remains an ambiguity as to whether Q_t at the jump times should be the point of departure or the destination, in other words, whether the realization $t \mapsto Q_t$ should be chosen to be left or to be right continuous. Although we think there is not much physical content to this question, we should point out that demanding either left or right continuity will destroy time-reversal invariance (cf. Section

6.1). A prescription that preserves time-reversal invariance can, however, be devised provided the possible jumps can be divided into two classes, A and B, in such a way that the time reverse of a class-A jump necessarily belongs to class B and vice versa. Then class-A jumps can be chosen left continuous and class-B as right continuous. An example is provided by the model of [11]: since at every jump the number of particles either increases or decreases, the jumps naturally form two classes ("creation" and "annihilation"), and the time reverse of a creation is an annihilation. The prescription could be that if a particle is created (annihilated) at time t, then Q_t already (still) contains the additional particle. But the opposite rule would be just as consistent with time-reversal symmetry, and we can see no compelling reason to prefer one rule over the other.

7 Conclusions

We have investigated the possibility of understanding QFT as a theory about moving particles, an idea pioneered, in the realm of nonrelativistic quantum mechanics, by de Broglie and Bohm. The models proposed by Bell [3] and ourselves [11] turn out to be rather universal; that is, their construction can be transferred to a variety of situations, involving different Hamiltonians and configuration spaces, and invoking formulas of a canonical character.

One ingredient of the construction is the use of stochastic jumps whose rates are determined by the quantum state vector (and the Hamiltonian). These rates can be specified through an explicit formula (1) that has a status similar to the velocity formula in Bohmian mechanics. We have provided a version of this jump rate formula that is more general than any previous one. Indeed, it seems to be the most general version possible: we need assume merely that the configuration space $\mathcal{Q}$ is a measurable space (the weakest notion of "space" available in mathematics), that the Hamiltonian is well-defined, and that $\mathcal{Q}$ and the Hilbert space are related through a generalized position observable (a positive-operator-valued measure, or POVM, the most general notion available in quantum theory of how a vector in Hilbert space may define a probability distribution). We have shown that these jump rates are well-defined and finite if the interaction Hamiltonian possesses a sufficiently regular kernel in the position representation defined by the POVM.

We have also indicated that in a Bell-type QFT, the different contributions to the Hamiltonian correspond to different contributions to the motion of the configuration Q_t. The relevant fact is process additivity, i.e., that the generator of the Markov process Q_t is additive in the Hamiltonian. The free process usually consists of continuous trajectories, Bohmian or similar, an observation already made in [11] for the model considered there. Exploiting process additivity, we obtain that Q_t is piecewise deterministic, the pieces being Bohm-type trajectories, interrupted by stochastic jumps. Given a Hamiltonian and POVM, our prescription determines the Markov process Q_t. As an example, we have described the process explicitly for a simple QFT.

The essential point of this paper is that there is a direct and natural way—a canonical way—of devising a Bell-type version of any QFT.

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