LMU Munich
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Prof. Dr. Sabine Jansen, Dr. Robert Helling
S. Das, S. Aristarkhov

Exercises on Mathematical Statistical Physics
Sheet 10

Problem 1 For the CCR algebra

$$W(f)W(g) = e^{\frac{i}{2}\mathrm{Im}\langle f,g\rangle} W(f+g)$$

the usual Fock space is recovered as the GNS-representation $(\mathcal{H}, \pi, \Omega)$ for

$$\omega_F(W(f)) = e^{-\frac{||f||}{4}}.$$

a) Show that this is a regular representation, i.e. $t \mapsto \omega(W(tf))$ is continuous at $t = 0$.

b) Define the operator Φ on $\mathcal{H}$ via

$$\Phi(f)\psi := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \pi(W(tf))\psi$$

for those ψ where the limit expsts, as well as

$$a(f) := \frac{1}{\sqrt{2}}\big(\Phi(f) + i\,\Phi(if)\big). \tag{1}$$

Show that $[a(f), a(g)^\star] = \langle g, f\rangle$, and $a(f)\Omega = 0$ for all $f,\ g \in\ h$.

Problem 2 Let $\Lambda \subset \mathbb{R}^d$ be a connected bounded open region of $\mathbb{R}^d$, $3 \le d \in \mathbb{N}$, and define $\Lambda_L = \{x \in \mathbb{R}^d \,|\, x/L \in \Lambda\}$ for any $L > 0$. Let $h_L = -\Delta$ be the self-adjoint Hamiltonian with Derichlet boundary condition on $\partial\Lambda_L$, i.e., the closure of the Laplacian on $\{f \in \mathbb{C}^\infty(\Lambda_L) \,|\, f_{|\partial\Lambda_L} = 0\}$, where the boundary $\partial\Lambda_L$ is assumed to be sufficiently smooth. Then h_L enjoys a purely discrete spectrum, and the asymptotic distribution of eigenvalues λ_i enjoys the Weyl law,

$$\lim_{\lambda\to\infty} \lambda^{-d/2} N(\lambda) = \text{const}.$$

Here, $N(\lambda)$ stands for the number of eigenvalues that do not exceed λ, and the constant depends on the volume of Λ_L. For $\beta > 0$ and $0 < z \le 1$, define

$$\rho_L(z,\beta) = \frac{1}{|\Lambda_L|}\mathrm{Tr}\left[\frac{z\,e^{-\beta h_L}}{1 - z\,e^{-\beta h_L}}\right].$$

Assuming $\lambda_1 < \lambda_2$, $\bar\rho > 0$, let $z_L := e^{\beta\mu_L}$ be the unique solution to

$$\bar\rho = \rho_L(z,\beta)$$

with $\mu_L < h_L$. Employing the expansion

$$\rho_L(z_L,\beta) = \sum_{n=1}^{\infty} \rho_L^{(n)}(z_L,\beta),$$

with

$$\rho_L^{(n)}(z_L,\beta) = \frac{1}{|\Lambda_L|} \left\langle \psi_L^{(n)} \middle| \frac{z_L\, e^{-\beta h_L}}{1 - z_L\, e^{-\beta h_L}} \middle| \psi_L^{(n)} \right\rangle,$$

and $\psi_L^{(n)}$ being the n^{th} eigenfunction of h_L, $n = 1,2\ldots$

a) Prove that for $n > 1$

$$\lim_{L\to\infty} \rho_L^{(n)}(z_L,\beta) = 0.$$

b) Show that

$$\lim_{L\to\infty} \sum_{n=2}^{\infty} \rho_L^{(n)}(z_L,\beta) = \text{const.},$$

where the constant is finite, and independent of $\bar{\rho}$.

The solutions to these exercises will be discussed on Monday, 25.06.