LMU Munich
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Exercises on Mathematical Statistical Physics
Math Sheet 8

Problem 1 (CS inequality for states) Let $\mathcal{A}$ be a C$^\star-$algebra and ω a state. Show that for all a, $b \in \mathcal{A}$, we have

$$\omega(a^\star) = \overline{\omega(a)},$$

and

$$|\omega(a^\star b)| \leq \sqrt{\omega(a^\star a)\,\omega(b^\star b)}.$$

Use this, to show that $\mathcal{J} := \{a \in \mathcal{A} \,|\, \omega(a^\star a) = 0\}$ is a right ideal, i.e., for $a \in \mathcal{A}$, $j \in \mathcal{J}$, we have $aj \in \mathcal{J}$ and thus $\mathcal{A}/\mathcal{J}$ has a well defined left-$\mathcal{A}$-action.

Hint: Consider positivity of ω for elements of the form $a + \lambda\, b$, $\lambda \in \mathbb{C}$ and extremize over λ.

Problem 2 Consider the C$^\star-$algebra $\mathcal{A} = \mathrm{Mat}(n \times n, \mathbb{C})$ and a self-adjoint $H \in \mathcal{A}$. Define a time evolution

$$\alpha_t(a) := e^{-itH} a\, e^{itH},$$

and take $\beta > 0$. show that there is a unique state ω that fulfills

$$\omega(ab) = \omega\big(\alpha_{i\beta}(b)a\big)$$

for all a, $b \in \mathcal{A}$, namely

$$\omega(a) = \frac{\mathrm{tr}\big(e^{-\beta H} a\big)}{\mathrm{tr}\big(e^{-\beta H}\big)}.$$

Furthermore, let $\tilde{\omega}$ be a faithful state of $\mathcal{A}$, i.e., $\tilde{\omega}(a^\star a) = 0 \Leftrightarrow 0$. Show that there is a unique Hamiltonian $\tilde{H} \in \mathcal{A}$ such that ω is a β-KMS-state for the time evolution induced by $\tilde{H}$ (this is the finite dimensional version of the Tomita-Takesaki theorem).

Problem 3 Let $(\mathcal{A}, \alpha_t)$ be a C*-dynamical system and δ its generator. For $a \in \mathcal{A}$ and $m \in \mathbb{N}$, define

$$a_m := \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \alpha_t(a)\, e^{-mt^2}\, \mathrm{d}t\,.$$

Show: a_m is analytic for α_t and δ, and the $\star-$subalgebra $\mathcal{A}_\tau := \{a_m \,|\, a \in \mathcal{A},\, m \in \mathbb{N}\}$ is dense in $\mathcal{A}$.

The solutions to these exercises will be discussed on Monday, 11.06.