LMU Munich
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Exercises on Mathematical Statistical Physics
Math Sheet 7

Problem 1 If $\mathcal{A}$ is a C^* algebra without unit element, show that by setting $\mathcal{A}_I = \mathcal{A} \oplus \mathbb{C}$ (as linear spaces) and introducing the product $(a \oplus \alpha)(b \oplus \beta) = ab + \alpha b + \beta a, \alpha\beta$, the involution $(a \oplus \alpha)^* = a^* \oplus \bar{\alpha}$, and the norm $||a \oplus \alpha||_{\mathcal{A}_I} = \sup\{||ab + \alpha b||_{\mathcal{A}} \mid b \in \mathcal{A}, ||b||_{\mathcal{A}} \leq 1\}$ (for all $\alpha, \beta \in \mathbb{C}$, $a, b \in \mathcal{A}$), a C^* algebra with unit $I = 0 \oplus 1$ is defined. In particular, check that by $|| \cdot ||_{\mathcal{A}_I}$ indeed a norm in $\mathcal{A}_I$ is provided.
(Remark: This construction of $\mathcal{A}_I$ implies that $\mathcal{A}$ is isometrically $\star$ isomorphic to C^* subalgebra of codimension one of $\mathcal{A}_I$.)

Problem 2 Let CCR(h) a Weyl-algebra. Show that for every $f \in h, f \neq 0$

$$||W(f) - \mathbb{1}|| = 2$$

and the map $t \mapsto W(tf)$ for $t \in \mathbb{R}$ is not continuous.
(Hint: First show that the spectrum of $W(f)$ is the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.)

Problem 3 (Polymer representation of quantum mechanics) Consider CCR($\mathbb{C}$) with generators $W(t)$ for $z \in \mathbb{C}$.

a) For $\Psi : \mathbb{R} \to \mathbb{C}$ show that for $a, b \in \mathbb{R}$ the operators

$$(\pi(W(a + ib))\Psi)(x) := e^{ia(b-x)}\Psi(x - b)$$

obey the canonical commutation relations in Weyl form.

b) Show that for $s \in \mathbb{R}$

$$d_s(W(z)) := W(is)W(z)W(-is)$$

is a $\star$-automorphism of CCR($\mathbb{C}$), which can be implemented by $U(s)\Psi)(x) := \Psi(x+s)$

c) Find the smallest Hilbert space $(H, < \cdot, \cdot >)$ of functions $\mathbb{R} \to \mathbb{C}$ that

(i) $\exists \Psi \in H, \Psi \neq 0 : \forall s \in \mathbb{R} \quad U(s)\Psi = \Psi$.

(ii) π : CCR($\mathbb{C}$) $\to B(H)$ is a representation as defined in a) (in particular find $< \cdot, \cdot >$ such that $\pi(W(z))$ is unitary).

d) Find $z \in \mathbb{C}$ such that $t \mapsto \pi(W(tz))$ is not strongly continuous (i.e. π is not a regular representation) and conclude that π is not unitary equivalent to the Schrödinger representation.

The solutions to these exercises will be discussed on Monday, 04.06.