LMU Munich
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Exercises on Mathematical Statistical Physics
Math Sheet 6

Problem 1 (1D Ising model, $\beta = 1$ and external field h) We identify the following objects:

Functions $f : \{\pm 1\} \to \mathbb{R}$ and column vectors

$$f \equiv \begin{pmatrix} f(+1) \\ f(-1) \end{pmatrix}$$

Measures ν on E and row vectors

$$\nu \equiv \begin{pmatrix} \nu(+1) & \nu(-1) \end{pmatrix}$$

Maps $M : \{\pm 1\}^2 \to \mathbb{R}$ and 2×2 matrices

$$M \equiv \begin{pmatrix} M(+1,+1) & M(+1,-1) \\ M(-1,+1) & M(-1,-1) \end{pmatrix}$$

a) Find a symmetric 2×2 matrix T such that

$$\mu_{\Lambda_n|\sigma_{\Lambda_n^c}}(\sigma_1, \cdots, \sigma_n) = \frac{T(\sigma_0, \sigma_1)T(\sigma_1, \sigma_2) \cdots T(\sigma_n, \sigma_{n+1})}{T^{n+1}(\sigma_0, \sigma_{n+1})}, \qquad \Lambda_n = \{1, \cdots, n\}$$

b) Let $\lambda > 0$ and f be the principle eigenvalue and eigenvector of T, respectively. We choose $f > 0$ and $f(1)^2 + f(-1)^2 = 1$. Define

$$P(\sigma_1, \sigma_2) := \frac{1}{\lambda f(\sigma_1)} T(\sigma_1, \sigma_2) f(\sigma_2)$$

check that P is a stochastic matrix (positive entries and row sums = 1). Determine the invariant measure ν for P, and check that the detailed balance condition is satisfied.

c) Let μ be a probability measure on $\{\pm 1\}^{\mathbb{Z}}$. Consider the following statements

(i) For all $\Lambda = \{x, x+1, \cdots, y\}$ and all $\sigma_{x-1}, \cdots, \sigma_{y+1} \in \{\pm 1\}$

$$\mu\left(\{\omega : \omega_x = \sigma_x, \cdots, \omega_y = \sigma_y\} | \{\omega : \omega_{x-1} = \sigma_{x-1}, \omega_{y+1} = \sigma_{y+1}\}\right)$$

(i.e. μ is a Gibbs measure).

(ii) μ is the unique measure on $\{\pm 1\}^{\mathbb{Z}}$ with finite dimensional distributions

$$\mu\left(\{\omega : \omega_x = \sigma_x, \cdots, \omega_y = \sigma_y\}\right) = \nu(\sigma_x) P(\sigma_x, \sigma_{x+1}) \cdots P(\sigma_{y-1}, \sigma_y)$$

for all $x, y \in \mathbb{Z}$ with $x < y$.

Prove that (i) $\Leftrightarrow$ (ii).

Problem 2 Let Φ be an absolutely summable potential, $\mu_{\Lambda|\sigma^c_\Lambda}$ Gibbs measure with b.c., $\mathscr{G}(\Phi)$ is the collection of infinite volume Gibbs measures. Let $\mathcal{R}$ be a collection of finite sets $\Lambda \subset \mathbb{Z}^d$ such that for every finite $\Delta \subset \mathbb{Z}^d$, there exists a $\Lambda \in \mathcal{R}$ with $\Delta \subset \Lambda$.

a) Give an example of such a collection $\mathcal{R}$.

b) Let μ be a probability measure on $\{\pm 1\}^{\mathbb{Z}^d} = \Omega$. Show that $\mu \in \mathscr{G}(\Phi)$ if and only if: $\forall \Lambda \in \mathcal{R}$, for all measurable $f : \Omega \to \mathbb{R}_+$

$$\int f \,\mathrm{d}\mu = \int_\Omega \Big(\sum_{\sigma_\Lambda \in \{\pm 1\}^\Lambda} f(\sigma_\Lambda, \tau_{\Lambda^c})\, \mu_{\Lambda|\tau_{\Lambda^c}}(\sigma_\Lambda) \Big) \mathrm{d}\mu(\tau).$$

Problem 3 Consider a $C^\star$ algebra $\mathcal{A}$ with unit element I, and the spectrum of a normal element $a \in \mathcal{A}$, defined by

$$\sigma(a) := \{\lambda \in \mathbb{C} | \lambda I - a \text{ has no inverse in } \mathcal{A}\}.$$

Prove that for all $a \in \mathcal{A}$ the limit $\lim_{n\to\infty} ||a^n||^{1/n}$ exists and equals the spectral radius

$$\varrho(a) := \mathrm{sum}_{\lambda \in \sigma(a)} |\lambda|$$

of a. In addition, show that $\varrho(a) = ||a||$.

Hint: Prove the complementary inequalities $\varrho(a) \leq \liminf_n ||a^n||^{1/n}$ and $\varrho(a) \geq \limsup_n ||a^n||^{1/n}$. To derive the first inequality, study the series $\lambda^{-1} \sum_k (a/\lambda)^k$; for the second inequality, investigate the radius of convergence of this series. To establish $\varrho(a) = ||a||$, consider $||a^{2^k}||^2$.

This sheet will be discussed on 28.05